Algèbres, Intégrabilité et Modèles Exactement Solubles
Written exam, 11 June 2015 from 1.30 pm to 4.30 pm

Instructions. The use of all AIMES related material (lecture notes, problem sheets and personal notes) is allowed. All other resources (books, electronic devices, etc) are prohibited.

The two exercises are independent. It is not necessary to provide the complete solution of both exercises in order to obtain the maximum grade.

1 Dimer coverings of the square lattice

Let $\mathcal{G}$ be a square lattice with M rows and N columns, with periodic boundary conditions in both directions. Consider the partition function

$$Z = \sum_{\mathcal{C}} x^h y^v z^m \,, \tag{1}$$

where $\mathcal{C}$ denotes all configurations in which $\mathcal{G}$ is covered by h horizontal dimers, v vertical dimers and m monomers, such that each vertex is covered by precisely one object.

Question 1: Rewrite Z so that the variable v does not appear any longer.

On each vertical edge, we place an up-pointing arrow (spin $+$) if the edge is covered by a dimer, and a down-pointing arrow (spin $-$) if it is not. We recall the expressions of the Pauli matrices:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

as well as $\sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y)$. If a represents any one of the symbols $x, y, z, +, -$, we denote by σ^a_k the operator that acts as the Pauli matrix σ^a on the k'th spin in a row of N vertical edges, and as the identity matrix on all other spins in the row.

Question 2: Define

$$V_1 = \prod_{k=1}^{N} \sigma^x_k \,. \tag{3}$$

Show that the operator that generates exactly m monomers on the row is

$$\frac{1}{m!} \left(\sum_{k=1}^{N} \sigma^-_k \right)^m V_1 \,. \tag{4}$$

Define also

$$V_2 = \exp\left(\beta \sum_{k=1}^{N} \sigma^-_k \right) \,. \tag{5}$$

Deduce that the operator $V_2 V_1$ generates an arbitrary number of monomers on the row with the weight β^m, while correctly taking into account the interaction with vertical dimers below and above that row.

Question 3: Show that the row-to-row transfer matrix of the monomer-dimer problem defined by (1) is

$$V = V_3 V_2 V_1 \,, \tag{6}$$

where

$$V_3 = \exp\left(\alpha \sum_{k=1}^{N} \sigma_k^- \sigma_{k+1}^-\right) , \tag{7}$$

and we have set $\sigma_{N+1}^- \equiv \sigma_1^-$. Give the expression of Z in terms of V.

Question 4: Show that the two-row transfer matrix is

$$V^2 = V_3 V_2 \bar{V}_3 \bar{V}_2 \,, \tag{8}$$

where

$$\bar{V}_2 = \exp\left(\beta \sum_{k=1}^{N} \sigma_k^+\right) , \qquad \bar{V}_3 = \exp\left(\alpha \sum_{k=1}^{N} \sigma_k^+ \sigma_{k+1}^+\right) . \tag{9}$$

Supposing now M even, express Z in terms of the eigenvalues $\lambda_j(N)$ of V^2.

We now specialise to the pure dimer problem, i.e., we set $\beta = 0$ so that $V^2 = V_3 \bar{V}_3$.

Question 5: We introduce the operators

$$c_j \equiv (-1)^{j-1} \left(\prod_{k=1}^{j-1} \sigma_k^z\right) \sigma_j^- \,, \qquad c_j^\dagger \equiv (-1)^{j-1} \left(\prod_{k=1}^{j-1} \sigma_k^z\right) \sigma_j^+ \,. \tag{10}$$

Use the relations $\{\sigma^a, \sigma^b\} = 2\delta_{ab}\mathbf{1}$ to prove that the operators c_j satisfy the canonical anti-commutation rules for fermions:

$$\{c_i, c_j^\dagger\} = \delta_{ij} \,, \qquad \{c_i, c_j\} = 0 \,, \qquad \{c_i^\dagger, c_j^\dagger\} = 0 \,. \tag{11}$$

Question 6: Compute the interaction terms $\sigma_k^- \sigma_{k+1}^-$ and $\sigma_k^+ \sigma_{k+1}^+$ appearing in V_3 and $\bar{V}_3$ for *all* $k = 1, 2, \ldots, N$ in terms of the fermion operators $c_j, c_j^\dagger$ and the number operator $\mathcal{N} = \sum_{j=1}^{N} c_j^\dagger c_j$. Show that $\left[(-1)^{\mathcal{N}}, V^2\right] = 0$.

Question 7: Show that the following transformation from c_j to η_q preserves the fermionic commutation relations:

$$c_j = \frac{\mathrm{e}^{-i\pi/4}}{N^{1/2}} \sum_q \mathrm{e}^{iqj} \eta_q \,. \tag{12}$$

Show that V^2 can be written in the form

$$V^2 = \prod_{0 \le q \le \pi} A_q \,, \tag{13}$$

and give an explicit expression for A_q in terms of η_q. We henceforth suppose N even. What are the allowed values of the wave number q, for each parity of $\mathcal{N}$?

Question 8: We define the following four basis states:

$$\Phi_0 = |0\rangle \,, \quad \Phi_q = \eta_q^\dagger |0\rangle \,, \quad \Phi_{-q} = \eta_{-q}^\dagger |0\rangle \,, \quad \Phi_{-qq} = \eta_{-q}^\dagger \eta_q^\dagger |0\rangle \,, \tag{14}$$

where $|0\rangle$ denotes the fermionic vacuum. Suppose $\mathcal{N}$ even. Write the action of A_q in this basis and diagonalise it.

Question 9: Write the free energy in the thermodynamic limit,

$$f \equiv \lim_{M,N\to\infty} \frac{1}{MN} \log Z \,, \tag{15}$$

in the form of an integral.

2 Energy-momentum tensor in WZW theory

We consider a conformal field theory which includes a finite family of fields $\{J^a(z)\}_{a=1\ldots\dim(\mathbf{g})}$ obeying the following OPE (current algebra):

$$J^a(z)J^b(w) \sim \frac{k\delta_{ab}}{(z-w)^2} + \sum_c i f_{abc} \frac{J^c(w)}{z-w} \,, \tag{16}$$

where k is an integer parameter, and the f_{abc} are the structure constants of a Lie algebra $\mathbf{g}$ with generators t^a:

$$[t^a, t^b] = \sum_c i f_{abc} \; t^c \,. \tag{17}$$

In particular, the symbol f_{abc} is antisymmetric under the permuation of indices, and satisfies the identity

$$\sum_{b,c} f_{abc} f_{dbc} = 2g\delta_{ad} \,, \tag{18}$$

where g is the dual Coxeter number. We define the operator $T = \gamma \sum_a : J^a J^a :$ with a constant γ to be determined. The goal of this problem is to show that $T(z)$ satisfies the defining properties of an energy-momentum tensor.

Question 1: The modes of the current operators J^a are defined as

$$J^a_n = \frac{1}{2i\pi} \oint_C J^a(z) z^n dz \,, \qquad \text{for} \quad n \in \mathbb{Z} \,, \tag{19}$$

where C is a counter-clockwise contour enclosing the origin. Compute the commutator $[J^a_n, J^b_m]$ for any a, b and $m, n \in \mathbb{Z}$.

Question 2: The operator T is defined as $T = \gamma \sum_a : J^a J^a :$ where the normal ordering is understood in the following sense:

$$: A(w)B(w) := \frac{1}{2i\pi} \oint_{C_w} A(z)B(w) \frac{dz}{z-w} \,, \tag{20}$$

where C_w is a counter-clockwise contour enclosing the point w. Using this definition and the OPE (16), compute the contraction:

$$\overbracket{J^a(z) : J^b(w)} J^b(w) : \quad = \quad \frac{1}{2i\pi} \oint_{C_w} \frac{dx}{x-w} \left[\overbracket{J^a(z) J^b}(x) J^b(w) + J^b(x) \overbracket{J^a(z) J^b}(w) \right] ,$$

for the indices a, b fixed.

Question 3: Find the singular terms in the OPE $T(z)J^a(w)$. Fix the normalisation constant γ through the relation $L_{-1}J^a(w) = \partial J^a(w)$.

Question 4: What is the conformal weight of $J^a(w)$? Is it a primary or a descendent field with respect to the Virasoro algebra?

Question 5: Show that the singular terms in the OPE $T(z)T(w)$ are of the form

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(z)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \,, \tag{21}$$

and compute the central charge c.

Question 6: The Virasoro generators L_n are defined as

$$L_n = \frac{1}{2i\pi} \oint_C T(z) z^{n+1} dz \,, \qquad \text{for} \quad n \in \mathbb{Z} \,, \tag{22}$$

where C is a counter-clockwise contour enclosing the origin. Compute the commutator $[L_n, J^a_m]$. Express L_n in terms of the modes J^a_m.