**Algèbres, Intégrabilité et Modèles Exactement Solubles**
**Written exam, 4 April 2019 from 1.30 pm to 4.30 pm**

**Instructions.** The use of all AIMES related material (lecture notes, problem sheets and personal notes) is allowed. All other resources (books, electronic devices, etc) are prohibited.

# Hecke algebra and spider webs

We begin with an $R$-matrix of the form

$$R_i(u) = \sin(\gamma - u)I + \sin(u)e_i \tag{1}$$

that acts at sites $i$ and $i+1$ on some tensor product of representations that we do not specify yet. Here $u = u_i - u_{i+1}$ denotes the usual difference of spectral parameters, $\gamma$ is the crossing parameter, $I$ is the identity operator, and $e_i$ is a non-trivial operator that we do not specify yet.

**Question 1:** Recall the spectral parameter dependent Yang-Baxter relation that $R_i(u)$ and $R_{i+1}(v)$ must satisfy in order for the corresponding model to be integrable.

**Question 2:** Show that the Yang-Baxter relation is satisfied provided that we impose the following relations

$$e_i^2 = ne_i\,, \tag{2a}$$

$$e_ie_{i+1}e_i - e_i = e_{i+1}e_ie_{i+1} - e_{i+1}\,, \tag{2b}$$

$$e_ie_j = e_je_i \text{ if } |i-j| > 1\,. \tag{2c}$$

Give the expression for $n$ in terms of the crossing parameter.

We now set $n = q + q^{-1}$ and define the operators $g_i = qe_i - I$. The algebra generated by the $g_i$ (with $i = 1, 2, \ldots, M-1$) and $I$ is called the *Hecke algebra* $H_M(q)$.

**Question 3:** Give the relations satisfied by the Hecke algebra. Show that this is a $q$-deformation of the symmetric group $S_M$.

We wish to study certain quotients of the Hecke algebra. More specifically this is done by ensuring that the generators $e_i$ commute with the quantum group $SU(N)_q$ for some integer $N = 1, 2, \ldots$. We here accept without proof that this commutation property is obtained by imposing $A_N(q) = 0$, where $A_N(q)$ denotes the $q$-deformed antisymmetriser on $N+1$ spins. The antisymmetriser is defined as follows:

First set $q = \mathrm{e}^{-i\gamma}$. Then define $X_i = 2i \lim_{u \to i\infty} \mathrm{e}^{iu} R_i(u)$. Moreover let $S_N$ denote the symmetric group of permutations of $N$ objects. Any element $\sigma \in S_N$ can be written as a product

$$\sigma = \prod_{i \in I_\sigma} \tau_{i,i+1}\,, \tag{3}$$

where $\tau_{i,i+1}$ is the transposition of objects $i$ and $i+1$ (note that only nearest neighbour transpositions occur); let $|I_\sigma|$ denote the number of factors in the product. Correspondingly set $X_\sigma = \prod_{i \in I_\sigma} X_i$. Finally we define

$$A_N(q) = \sum_{\sigma \in S_{N+1}} (-q)^{|I_\sigma|} X_\sigma \,. \tag{4}$$

**Question 4:** Express $X_i$ in terms of $q$ and $e_i$.

**Question 5:** Compute $A_N(q)$ for $N = 1$. Describe the quotient of the Hecke algebra obtained by setting $A_1(q) = 0$ and give its dimension for a system on $M$ sites. Is this an interesting object to study?

We now study the case $N = 2$ in details.

**Question 6:** Write the elements of $S_3$ in the form (3). Exhibit *two* mechanisms by which this writing is not unique. Show that the antisymmetriser $A_N(q)$ is nevertheless well defined by (4).

**Question 7:** Compute $A_2(q)$ explicitly. Show that the quotient of the Hecke algebra obtained by setting $A_2(q) = 0$ can be identified with the Temperley-Lieb algebra $TL_M(n)$. Give its dimension for a system on $M$ sites.

We next move to the case $N = 3$. The straightforward computation of the quotient $A_3(q) = 0$ is more cumbersome in this case, so we admit here without proof that it is given by the following defining relations:

$$
\begin{aligned}
(e_i)^2 &= [2]e_i \,, &\text{(5a)}\\
(f_i)^2 &= [3]f_i \,, &\text{(5b)}\\
e_i e_{i+1} e_i &= e_i + [2] f_i \,, &\text{(5c)}\\
e_{i+1} e_i e_{i+1} &= e_{i+1} + [2] f_i \,, &\text{(5d)}\\
f_i f_{i+1} f_i &= f_i \,, &\text{(5e)}\\
f_{i+1} f_i f_{i+1} &= f_{i+1} \,, &\text{(5f)}\\
e_i e_j &= e_j e_i \text{ if } |i-j| > 1 \,, &\text{(5g)}
\end{aligned}
$$

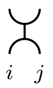
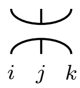
where we have introduced the $q$-deformed numbers

$$[k] := \frac{q^k - q^{-k}}{q - q^{-1}} \,. \tag{6}$$

**Question 8:** What is the commutation property analogous of (5g) obeyed by the $f_i$ generators?

**Question 9:** Verify that (5) is indeed a quotient of the Hecke algebra. Give its dimension for systems on $M = 3$ and $M = 4$ sites. Compare these dimensions with those of the corresponding $q$-deformed symmetric groups.

The algebra defined by (5) admits a representation—the so-called fundamental representation—in which $e_i$, $f_i$ and the identity operator $1_i$ acting at position $i$ are represented by the following diagrams:

$e_i = [2] \times$  , $\quad f_i =$  , $\quad 1_i =$  , (7)

where we have abbreviated the neighbouring sites as $j = i + 1$ and $k = i + 2$. Any site not shown in a given diagram is understood to be acted upon by the identity operator.

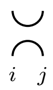
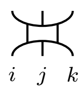
**Question 10:** Show that this indeed provides a representation of (5), provided we admit a set of three diagrammatic rules.[1] Specifically, one can:

1. replace a tadpole (a bubble on a strand) by a certain number;
2. replace a loop by a certain number;
3. resolve internal cycles of degree four (squares) in a certain way.

State these diagrammatic rules precisely.

**Question 11:** For the algebra on $M$ sites, the diagrams generated by the representation (8) of the algebra (5) are known as *spider webs*. Show that these spider webs are in fact bipartite cubic graphs inside a rectangle with $M$ points on the top and $M$ points on the bottom side, in which all internal cycles are polygons with an even number of sides $\geq 6$.

The exists another type of spider webs—the so-called alternating representation—in which $e_i$, $f_i$ and $1_i$ are instead represented by the following diagrams:

$e_i =$  , $\quad f_i =$  , $\quad 1_i =$  . (8)

This representation is no longer obtained by setting $A_3(q) = 0$, and in particular the relations (5) cannot be taken for granted. Instead, one imposes the same diagrammatic rules found in Question 10.

**Question 12:** Derive the relations that replace (5) for these alternating spider webs.

**Question 13:** Give the dimension of this algebra for $M = 3$ and $M = 4$ sites.

**Question 14:** Despite of its different construction, is the algebra of alternating spider webs nevertheless a quotient of the Hecke algebra?

[1] In the Temperley-Lieb case ($N = 2$) there was just one rule: replace any loop by the number [2].