10 Basic aspects of CFT

An important break-through occurred in 1984 when Belavin, Polyakov and Zamolodchikov [BPZ84] applied ideas of conformal invariance to classify the possible types of critical behaviour in two dimensions. These ideas had emerged earlier in string theory and mathematics, and in fact go back to earlier (1970) work of Polyakov [Po70] in which global conformal invariance is used to constrain the form of correlation functions in d-dimensional theories. It is however only by imposing *local* conformal invariance in $d = 2$ that this approach becomes really powerful. In particular, it immediately permitted a full classification of an infinite family of conformally invariant theories (the so-called "minimal models") having a finite number of fundamental ("primary") fields, and the exact computation of the corresponding critical exponents. In the aftermath of these developments, conformal field theory (CFT) became for some years one of the most hectic research fields of theoretical physics, and indeed has remained a very active area up to this date.

This chapter focusses on the basic aspects of CFT, with a special emphasis on the ingredients which will allow us to tackle the geometrically defined loop models via the so-called Coulomb Gas (CG) approach. The CG technique will be exposed in the following chapter. The aim is to make the presentation self-contained while remaining rather brief; the reader interested in more details should turn to the comprehensive textbook [DMS87] or the Les Houches volume [LH89].

10.1 Global conformal invariance

A conformal transformation in d dimensions is an invertible mapping $\mathbf{x} \to \mathbf{x}'$ which multiplies the metric tensor $g_{\mu\nu}(\mathbf{x})$ by a space-dependent scale factor:

$$g'_{\mu\nu}(\mathbf{x}') = \Lambda(\mathbf{x}) g_{\mu\nu}(\mathbf{x}). \tag{10.1}$$

Note that such a mapping preserves angles. Therefore, just as Wilson [Wi69] suggested using *global* scale invariance as the starting point for investigating a system at its critical point, Polyakov [Po70] proposed imposing the *local* scale invariance (10.1) as the fundamental requirement for studying a critical system in which the microscopic interactions are short ranged.

A priori, a geometrical model of self-avoiding objects such as loops does not seem to be governed by short-range interactions. However, we have

already seen in (8.30)–(8.31) how to transform it into a vertex model with local interactions (albeit the complex Boltzmann weights still point to its non-local origin). We shall see later that the critical exponents of the Potts model can indeed be derived by CFT and CG techniques.

10.1.1 The conformal group

We first investigate the consequences of (10.1) for an infinitesimal transformation of the form[29]

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(\mathbf{x})\,. \tag{10.2}$$

To first order in ϵ the change in metric is given by

$$\begin{aligned} g'_{\mu\nu} &= \frac{\partial x^\alpha}{\partial x'^\mu}\,\frac{\partial x^\beta}{\partial x'^\nu}\,g_{\alpha\beta} \\ &= (\partial^\alpha_\mu - \partial_\mu\epsilon^\alpha)(\partial^\beta_\nu - \partial_\nu\epsilon^\beta)g_{\alpha\beta} \\ &= g_{\mu\nu} - (\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu)\,. \end{aligned} \tag{10.3}$$

The requirement (10.1) means that

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = f(\mathbf{x})g_{\mu\nu}\,, \tag{10.4}$$

where the factor $f(\mathbf{x})$ can be determined by taking traces on both sides of (10.4):

$$f(\mathbf{x}) = \frac{2}{d}\,\partial_\rho\epsilon^\rho\,. \tag{10.5}$$

We can assume that the conformal transformation amounts to an infinitesimal deformation of the standard Cartesian metric $g_{\mu\nu} = \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the d-dimensional identity matrix. By differentiating (10.4), permuting indices and forming a linear combination one establishes

$$2\partial_\mu\partial_\nu\epsilon_\rho = \eta_{\mu\rho}\partial_\nu f + \eta_{\nu\rho}\partial_\mu f - \eta_{\mu\nu}\partial_\rho f \tag{10.6}$$

and contracting this with $\eta^{\mu\nu}$ we arrive at

$$2\partial^2\epsilon_\mu = (2-d)\partial_\mu f\,. \tag{10.7}$$

On the other hand, applying ∂_ν to (10.7) and ∂^2 to (10.4) gives

$$(2-d)\partial_\mu\partial_\nu f = \eta_{\mu\nu}\partial^2 f\,, \tag{10.8}$$

[29] Below we use the summation convention on repeated indices.

and contracting this with $\eta^{\mu\nu}$ leads to

$$(d-1)\partial^2 f = 0\,. \tag{10.9}$$

We are now ready to draw some important conclusions from (10.8)–(10.9) and valid in arbitrary dimension d. The case $d = 1$ is somewhat particular, since no constraints on f are implied: any smooth transformation is conformal. On the other hand, we are not likely to need CFT to solve simple short-ranged one-dimensional models! The case $d = 2$ is where CFT has the most to offer, and we shall discuss it in detail later.

For the moment we thus concentrate on the case $d \geq 3$. Eqs. (10.8)–(10.9) imply that $\partial_\mu \partial_\nu f = 0$, whence f is at most linear in the coordinates. Using (10.6) this means that $\partial_\mu \partial_\nu \epsilon_\rho$ is constant, whence

$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho \quad \text{with } c_{\mu\nu\rho} = c_{\mu\rho\nu}\,. \tag{10.10}$$

Since the above discussion holds for all $\mathbf{x}$, we may treat each power of the coordinates separately. The constant term

$$\epsilon_\mu = a_\mu \tag{10.11}$$

corresponds obviously to translations. For the linear term it is useful to distinguish between the diagonal and off-diagonal parts. The former

$$\epsilon_\mu = \lambda x^\nu \tag{10.12}$$

corresponds to dilatations, while the latter

$$\epsilon_\mu = \omega_{\mu\nu} x^\nu\,, \tag{10.13}$$

with $\omega_{\mu\nu} = -\omega_{\nu\mu}$ an antisymmetric tensor, corresponds to rotations.

The important new ingredient comes from the quadratic term which corresponds to the *special conformal transformation* (SCT). It can be written as (after some work)

$$x'^\mu = \frac{x^\mu - b^\mu \mathbf{x}^2}{1 - 2\mathbf{b}\cdot\mathbf{x} + b^2\mathbf{x}^2}\,, \tag{10.14}$$

or equivalently as a translation, preceded and followed by an inversion $x^\mu \to x'^\mu = x^\mu/\mathbf{x}^2$, viz.

$$\frac{x'^\mu}{\mathbf{x}'^2} = \frac{x^\mu}{\mathbf{x}^2} - b^\mu\,. \tag{10.15}$$

The infinitesimal form of the SCT is found by developing (10.14) to linear order in b^μ:

$$x'^\mu = x^\mu + 2(\mathbf{x}\cdot\mathbf{b})x^\mu - b^\mu \mathbf{x}^2\,. \tag{10.16}$$

The corresponding scale factor is determined by

$$\left|\frac{\partial \mathbf{x}'}{\partial \mathbf{x}}\right| = \frac{1}{(1-2\mathbf{b}\cdot\mathbf{x}+b^2\mathbf{x}^2)^d}\,. \tag{10.17}$$

In particular the distance separating two points $\mathbf{x}_i$ and $\mathbf{x}_j$ scales like

$$|\mathbf{x}'_i - \mathbf{x}'_j| = \frac{|\mathbf{x}_i - \mathbf{x}_j|}{(1-2\mathbf{b}\cdot\mathbf{x}_i+b^2\mathbf{x}_i^2)^{1/2}(1-2\mathbf{b}\cdot\mathbf{x}_j+b^2\mathbf{x}_j^2)^{1/2}}\,. \tag{10.18}$$

One can now write down the generators of infinitesimal conformal transformations and study their commutation relations. In this way one establishes that the conformal group is isomorphic to the pseudo-orthogonal group $\mathrm{SO}(d+1,1)$ with $\frac{1}{2}(d+1)(d+2)$ real parameters.

10.1.2 Correlation function of quasi-primary fields

The connection between a statistical mechanics model and quantum field theory is made as usual by writing the partition function and correlation functions of the former as functional integrals in the latter:

$$Z = \int \mathcal{D}\Phi\, \mathrm{e}^{-S[\Phi]} \tag{10.19}$$

$$\langle \phi_1(\mathbf{x}_1)\dots\phi_k(\mathbf{x}_k)\rangle = Z^{-1}\int \mathcal{D}\Phi\, \phi_1(\mathbf{x}_1)\dots\phi_k(\mathbf{x}_k)\mathrm{e}^{-S[\Phi]} \tag{10.20}$$

Here $S[\Phi]$ is the euclidean action, Φ the collection of fields, and $\phi_i \in \Phi$. In other words, $Z^{-1}\mathrm{e}^{-S[\Phi]}\mathcal{D}\Phi$ is the Gibbs measure in the continuum limit. Paradoxically, in many cases the hypothesis of conformal invariance may permit one to classify and precisely characterise the possible continuum theories without ever having to write down explicitly the action $S[\Phi]$.

A field $\phi(\mathbf{x})$, here supposed spinless for simplicity, is called *quasi-primary* provided it transforms covariantly under the conformal transformation (10.1):

$$\phi(\mathbf{x}) \to \phi'(\mathbf{x}') = \left|\frac{\partial \mathbf{x}'}{\partial \mathbf{x}}\right|^{-\Delta/d} \phi(\mathbf{x}). \tag{10.21}$$

The number $\Delta = \Delta_\phi$ is a property of the field and is called its *scaling dimension*. Using this, conformal invariance completely fixes [Po70] the form of the two- and three-point correlation functions, as we shall now see.

The assumption of quasi-primarity implies the following covariance condition for a general two-point function

$$\langle \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)\rangle = \left|\frac{\partial \mathbf{x}'}{\partial \mathbf{x}}\right|^{\Delta_1/d}_{\mathbf{x}=\mathbf{x}_1} \left|\frac{\partial \mathbf{x}'}{\partial \mathbf{x}}\right|^{\Delta_2/d}_{\mathbf{x}=\mathbf{x}_2} \langle \phi_1(\mathbf{x}'_1)\phi_2(\mathbf{x}'_2)\rangle \,. \tag{10.22}$$

Rotation and translation invariance imply that

$$\langle \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)\rangle = f(|\mathbf{x}_1 - \mathbf{x}_2|)\,, \tag{10.23}$$

and covariance under a scale transformation $\mathbf{x} \to \lambda\mathbf{x}$ fixes $f(\mathbf{x}) = \lambda^{\Delta_1+\Delta_2} f(\lambda\mathbf{x})$. Therefore

$$\langle \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)\rangle = \frac{C_{12}}{|\mathbf{x}_1 - \mathbf{x}_2|^{\Delta_1+\Delta_2}} \tag{10.24}$$

for some constant C_{12}. Inserting now this into (10.22) and using the SCT with scale factor (10.17) we obtain

$$\frac{C_{12}}{|\mathbf{x}_1 - \mathbf{x}_2|^{\Delta_1+\Delta_2}} = \frac{C_{12}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}} \frac{(\gamma_1\gamma_2)^{(\Delta_1+\Delta_2)/2}}{|\mathbf{x}_1 - \mathbf{x}_2|^{\Delta_1+\Delta_2}}\,, \tag{10.25}$$

with

$$\gamma_i = 1 - 2\mathbf{b}\cdot\mathbf{x}_i + b^2\mathbf{x}_i^2\,. \tag{10.26}$$

Equating powers of γ_i in (10.25) gives $2\Delta_1 = 2\Delta_2 = \Delta_1 + \Delta_2$ with the unique solution $\Delta_1 = \Delta_2$. This means that the two-point function vanishes unless the two fields have the same scaling dimension. Moreover it is conventional to normalise the fields so that $C_{12} = 1$. In conclusion

$$\langle \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)\rangle = \frac{\delta_{\Delta_1,\Delta_2}}{x_{12}^{2\Delta_1}}\,, \tag{10.27}$$

where we have set $x_{ij} = |\mathbf{x}_i - \mathbf{x}_j|$.

We next discuss the case of a three-point function. Covariance under rotations, translations and dilations imply that it must be of the form[30]

$$\langle \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)\phi_3(\mathbf{x}_3)\rangle = \frac{C_{123}}{x_{12}^a x_{23}^b x_{13}^c} \tag{10.28}$$

[30] A priory the right-hand side of (10.28) may be replaced with a sum over several terms satisfying (10.29), but see below.

with

$$a+b+c=\Delta_1+\Delta_2+\Delta_3\,. \tag{10.29}$$

Covariance under SCT implies that

$$\frac{C_{123}}{x_{12}^a x_{23}^b x_{13}^c}=\frac{C_{123}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}\gamma_3^{\Delta_3}}\,\frac{(\gamma_1\gamma_2)^{a/2}(\gamma_2\gamma_3)^{b/2}(\gamma_1\gamma_3)^{c/2}}{x_{12}^a x_{23}^b x_{13}^c}\,, \tag{10.30}$$

so that

$$a+c=2\Delta_1\,,\qquad a+b=2\Delta_2\,,\qquad b+c=2\Delta_3\,.$$

This system has the unique solution

$$\langle\phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)\phi_3(\mathbf{x}_3)\rangle=\frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{23}^{\Delta_2+\Delta_3-\Delta_1}x_{31}^{\Delta_3+\Delta_1-\Delta_2}}\,. \tag{10.31}$$

The constants C_{123} are non-trivial parameters, which will reappear below as structure constants in the operator product expansion.

The complete determination (up to C_{123}) of two- and three-point functions is a consequence of the fact that (10.18) does not allow us to construct conformal invariants of two or three points. For $N\geq 4$ points one can however construct $N(N-3)/2$ independent invariants, known as *anharmonic ratios* or *cross-ratios*. For instance, the four-point function takes the form

$$\langle\phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)\phi_3(\mathbf{x}_3)\phi_4(\mathbf{x}_4)\rangle=f\left(\frac{x_{12}x_{34}}{x_{13}x_{24}}\,,\frac{x_{12}x_{34}}{x_{23}x_{14}}\right)\prod_{i<j}^{4}x_{ij}^{\Delta/3-\Delta_i-\Delta_j} \tag{10.32}$$

with $\Delta=\sum_{i=1}^4\Delta_i$. We stress that the function f is not fixed solely by *global* conformal invariance.

10.1.3 Stress tensor and global Ward identity

The stress tensor $T^{\mu\nu}$ is the conserved Noether current associated with the conformal symmetry. It can be defined[31] as the response of the partition function to a local change in the metric:

$$T^{\mu\nu}(\mathbf{x})=-\frac{1}{(2\pi)^{d-1}}\,\frac{\delta\log Z}{\delta g_{\mu\nu}(\mathbf{x})} \tag{10.33}$$

[31] Note the analogy with the theory of integrable systems, where the conserved charges are obtained as derivatives of the transfer matrix with respect to the anisotropy (spectral parameter).

The power of 2π is conventional and will lead to convenient simplifications later.

Because of (10.19) this can be written equivalently as the variation of the local action $S[\phi]$ under the transformation (10.2):

$$\delta S = \frac{1}{(2\pi)^{d-1}} \int \mathrm{d}^d x \, T^{\mu\nu}(x) \, \partial_\mu \epsilon_\nu(x) \, . \tag{10.34}$$

This point of view will be useful later when we consider the response of the action to transformations that are only conformal in some parts of space. But for truly (global) conformal transformations we have obviously $\delta S = 0$. This immediately entails some important symmetry properties of $T^{\mu\nu}$.

For the translational invariance (10.11), $\epsilon_\nu(x) = a_\nu$, one has of course $\partial_\mu \epsilon_\nu(x) = 0$, whence $\delta S = 0$ as expected. But performing instead an integration by parts in (10.34), and using that a_ν is arbitrary, we obtain the conservation law

$$\partial_\mu T^{\mu\nu}(x) = 0 \, . \tag{10.35}$$

Thus $T^{\mu\nu}(x)$ is indeed equivalent to the usual Noether current.

Regarding the rotational invariance (10.13), for the integral (10.34) to vanish, the stress tensor must be symmetric:

$$T^{\mu\nu}(x) = T^{\nu\mu}(x) \, . \tag{10.36}$$

And finally the dilatation invariance (10.12) has $\partial_\mu \epsilon_\nu(x) = \delta^\nu_\mu$, so the stress tensor is traceless:

$$T^\mu_\mu(x) = 0 \, . \tag{10.37}$$

The stress tensor also satisfies a very important constraint known as the Ward identity. This identity is most powerful in the case of local conformal invariance in $d = 2$ (see below), but the starting point is a global identity valid in any dimension that we derive now.

Consider the correlation function of a product of local fields $\phi_i(\mathbf{x}_i)$ that we denote for simplicity as

$$X = \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2) \cdots \phi_n(\mathbf{x}_n) \, . \tag{10.38}$$

The correlation function $\langle X \rangle$ is a physical observable and does not change under an infinitesimal coordinate transformation (10.2). We have thus $\delta \langle X \rangle =$

0. A non-trivial identity however results from decomposing the various changes that add up to zero. There is an explicit variation of the fields,

$$\phi_i(\mathbf{x}_i) \to \phi_i'(\mathbf{x}_i) = \phi_i(\mathbf{x}_i) + \delta\phi_i(\mathbf{x}_i) \tag{10.39}$$

and the action pick up a variation δS given by (10.34). But since the correlation function is defined by the functional integral (10.20) there will be three types of changes to $\langle X \rangle$ induced by 1) the explicit field variation $\delta\phi_i(\mathbf{x}_i)$, 2) the variation δS in the correlation functional integral, and 3) the variation δS in the normalisation Z^{-1}. Summing these up leads to

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \langle \phi_1(\mathbf{x}_1) \cdots \delta\phi_i(\mathbf{x}_i) \cdots \phi_n(\mathbf{x}_n) \rangle \\ &- \frac{1}{(2\pi)^{d-1}} \int \mathrm{d}^d x \langle T^{\mu\nu}(x) X \rangle \partial_\mu \epsilon_\nu(x) \\ &- \left[\frac{1}{(2\pi)^{d-1}} \int \mathrm{d}^d x \langle T^{\mu\nu}(x) \rangle \partial_\mu \epsilon_\nu(x) \right] \langle X \rangle \,. \end{aligned} \tag{10.40}$$

For a theory at its critical point only the identity operator has a non-zero one-point function. In particular $\langle T^{\mu\nu}(x) \rangle = 0$. The global Ward identity therefore takes the form

$$\sum_{i=1}^{n} \langle \phi_1(\mathbf{x}_1) \cdots \delta\phi_i(\mathbf{x}_i) \cdots \phi_n(\mathbf{x}_n) \rangle = \frac{1}{(2\pi)^{d-1}} \int \mathrm{d}^d x \langle T^{\mu\nu}(x) X \rangle \partial_\mu \epsilon_\nu(x) \tag{10.41}$$

10.2 Two dimensions and local conformal invariance

Conformal invariance is especially powerful in two dimensions for reasons that we shall expose presently. For the moment, we work in the geometry of the Riemann sphere, i.e., the plane with a point at infinity, and we shall write the coordinates as $\mathbf{x} = (x^1, x^2)$. Under a general coordinate transformation $x^\mu \to x'^\mu = w^\mu(x^1, x^2)$ application of (10.1) implies the Cauchy-Riemann equations

$$\begin{aligned} \frac{\partial w^2}{\partial x^1} &= \pm \frac{\partial w^1}{\partial x^2} \,, \\ \frac{\partial w^1}{\partial x^1} &= \mp \frac{\partial w^2}{\partial x^2} \,, \end{aligned} \tag{10.42}$$

i.e., $\mathbf{w}(\mathbf{x})$ is either a holomorphic or an antiholomorphic function. Important simplifications will therefore result upon introducing the complex coordinates

$$z \equiv x^1 + ix^2\,, \qquad \bar{z} \equiv x^1 - ix^2\,. \tag{10.43}$$

A conformal mapping then reads simply $z \to z' = w(z)$.

It is convenient to consider $(x^1, x^2) \in \mathbb{C}^2$, so that z and $\bar{z}$ can be considered independent complex variables, not linked by complex conjugation. For that reason we can often concentrate on the transformations of z alone, bearing in mind that $\bar{z}$ satisfies the same properties. Ultimately the relationship between the two—i.e., $\bar{z}$ indeed *is* the complex conjugate of $\bar{z}$—will be enforced through the constraint of modular invariance (see section 10.7).

The identification of two-dimensional conformal transformations with analytic maps $w(z)$ could have been anticipated from the well-known fact that the latter are angle-preserving. It should be noted that an analytic map is defined (via its Laurent series) by an *infinite* number of parameters. This does not contradict the result of section 10.1 that the set of global conformal transformations is defined by only $\frac{1}{2}(d+1)(d+2) = 6$ real parameters, since analytic maps are not necessarily invertible and defined in the whole complex plane.

Global conformal transformations in $d = 2$ take the form of the projective transformations

$$w(z) = \frac{a_{11}z + a_{12}}{a_{21}z + a_{22}} \tag{10.44}$$

with $a_{ij} \in \mathbb{C}$ and a normalisation constraint that we can take as $\det a_{ij} = 1$.

It is straightforward to verify that the composition of two projective transformations is again projective, with parameters $\{a_{ij}\}$ that correspond to multiplying those of the individual transformations as 2×2 matrices. In other words, $d = 2$ global conformal transformations form the group $\mathrm{SL}(2, \mathbb{C}) \simeq \mathrm{SO}(3, 1)$.

We can sketch an argument why the projective transformations (10.44) are the only globally defined invertible holomorphic mappings $f(z)$. First, for f to be single-valued it cannot have branch points. Second, for f to be invertible it cannot have essential singularities. Therefore $f(z) = P(z)/Q(z)$ must be a ratio of polynomials without common zeros. For the inverse image of zero to exist, $P(z)$ can only have a single zero. This cannot be a multiple

zero, since otherwise f would not be invertible. Therefore $P(z) = a_{11}z + a_{12}$. The same argument with zero replaced by infinity implies that $Q(z) = a_{21}z + a_{22}$.

In complex coordinates, the transformation law (10.21) becomes

$$\phi'(w,\bar{w}) = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-h} \left(\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}}\right)^{-\bar{h}} \phi(z,\bar{z}) \tag{10.45}$$

where the *real* parameters $(h, \bar{h})$ are called the *conformal weights*. The combinations $\Delta = h + \bar{h}$ and $s = h - \bar{h}$ are called respectively the scaling dimension and the spin of ϕ. A field ϕ satisfying (10.45) for any projective transformation (resp. any analytic map) $w(z)$ is called *quasi-primary* (resp. *primary*). An example of a quasi-primary field which is not primary is furnished by the stress tensor (see below).

The expressions (10.27)–(10.31) for the two- and three-point correlation functions still hold true with the obvious modification that the dependence in $z_{ij} \equiv z_i - z_j$ (resp. in $\bar{z}_{ij}$) goes with the conformal weights h (resp. $\bar{h}$).

10.3 Stress tensor and local Ward identity

The change to complex coordinates implies that the conservation laws of $T^{\mu\nu}$ need some rewriting. Directly from (10.43) the corresponding derivatives read

$$\begin{aligned} \partial_z &= \frac{1}{2}(\partial_1 - i\partial_2)\,, \\ \partial_{\bar{z}} &= \frac{1}{2}(\partial_1 + i\partial_2) \end{aligned} \tag{10.46}$$

with inverses

$$\begin{aligned} \partial_1 &= \partial_z + \partial_{\bar{z}}\,, \\ \partial_2 &= i(\partial_z - \partial_{\bar{z}})\,. \end{aligned} \tag{10.47}$$

The elements of the complex metric can be read off from the obvious rewriting of the line element in Euclidean $d = 2$ space:

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 = \mathrm{d}z\,\mathrm{d}\bar{z}\,. \tag{10.48}$$

This leads to $g_{zz} = g_{\bar{z}\bar{z}} = 0$ and $g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}$. In particular the components of the stress tensor read now in complex coordinates:

$$\begin{aligned} T_{zz} &\equiv T(z,\bar{z}) = \frac{1}{4}(T_{11} - T_{22} + 2iT_{12})\,, \\ T_{\bar{z}\bar{z}} &\equiv \bar{T}(z,\bar{z}) = \frac{1}{4}(T_{11} - T_{22} - 2iT_{12}) \\ T_{z\bar{z}} &= T_{\bar{z}z} = \frac{1}{4}(T_{11} + T_{22}) = \frac{1}{4}T^{\mu}_{\mu}\,. \end{aligned} \tag{10.49}$$

We can likewise rewrite the conservation law (10.35) in complex coordinates:

$$\begin{aligned} \partial_{\bar{z}} T(z,\bar{z}) + \frac{1}{4}\partial_z T^{\mu}_{\mu} &= 0\,, \\ \partial_z \bar{T}(z,\bar{z}) + \frac{1}{4}\partial_{\bar{z}} T^{\mu}_{\mu} &= 0\,. \end{aligned} \tag{10.50}$$

Recall that scale invariance further implies the tracelessness $T^{\mu}_{\mu} = 0$ from (10.37); in general the trace would be proportional to the beta function, which vanishes at a renormalisation group fixed point. At the fixed point we thus have

$$\partial_{\bar{z}} T(z,\bar{z}) = \partial_z \bar{T}(z,\bar{z}) = 0\,. \tag{10.51}$$

This means that T depends only on z, hence is an holomorphic function, and that $\bar{T}$ depends only on $\bar{z}$, hence is an anti-holomorphic function. This is a very important element in the solvability of two-dimensional CFT.

To emphasize this crucial result we henceforth denote the two non-vanishing components of the stress tensor $T(z)$ and $\bar{T}(\bar{z})$, viz.

$$T(z) \equiv T_{zz}\,, \qquad \bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}\,. \tag{10.52}$$

Following Fateev and Zamolodchikov [FZ87] it is even possible to go (much) further: CFT's in which the conformal symmetry is enhanced with other, so-called extended, symmetries (superconformal, parafermionic, W algebra,...) can be constructed by requiring more analytic currents and making them coexist with $T(z)$ by imposing certain associativity requirements.

We now come back to the Ward identity (10.41) for a product of local operators $\phi_i(z_i, \bar{z}_i)$. In $d = 2$, if we suppose that these operators are *primary* and that the infinitesimal transformation is only *locally* conformal, we will get a much stronger local form of the Ward identity.

Let C be a circle centered at the origin of radius sufficiently large so as to surround all the points $(z_i, \bar{z}_i)$ with $i = 1, 2, \ldots, n$. Denote by n^μ its outgoing normal vector. We shall suppose that the infinitesimal transformation, $z' = z + \epsilon(z)$ and $\bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z})$, is conformal only inside C, whereas on the outside it is merely a differentiable function that tends to zero sufficiently fast at infinity.

Consider first the right-hand side of (10.41). We can perform an integration by parts and invoke the conservation law (10.35) to get rid of the bulk part of this integral. Only remains the boundary terms. The boundary term at infinity vanishes due to the hypothesis that $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$ tend to zero sufficiently fast at infinity. The boundary term at C can be written as

$$\frac{1}{2\pi}\int_C \mathrm{d}\Sigma\, n_\mu \epsilon_\mu \langle T^{\mu\nu} X\rangle = \frac{1}{2\pi i}\oint_C \mathrm{d}z\, \epsilon(z)\langle T(z)X\rangle - \frac{1}{2\pi i}\oint_C \mathrm{d}\bar{z}\, \bar{\epsilon}(\bar{z})\langle \bar{T}(\bar{z})X\rangle\,, \tag{10.53}$$

where Σ denotes the surface (actually line) element of the circle C and we recall that $\epsilon(z) = \epsilon^1 + i\epsilon^2$ and $\bar{\epsilon}(\bar{z}) = \epsilon^1 - i\epsilon^2$.

Consider next the left-hand side of (10.41). The transformation law (10.45) for the primary field $\phi_i(z_i, \bar{z}_i)$ can be written as

$$\phi_i'(z_i', \bar{z}_i')\,(\mathrm{d}z')^{h_i}\,(\mathrm{d}\bar{z}')^{\bar{h}_i} = \phi_i(z, \bar{z})\,(\mathrm{d}z)^{h_i}\,(\mathrm{d}\bar{z})^{\bar{h}_i}\,, \tag{10.54}$$

where $(h_i, \bar{h}_i)$ are the corresponding conformal weights. Developping the infinitesimal transformation to first order this reads

$$\delta\phi_i \equiv \phi(z_i, \bar{z}_i) - \phi'(z_i', \bar{z}_i') = \left[(h_i\partial_i\epsilon + \epsilon\partial_i) + (\bar{h}_i\bar{\partial}_i\bar{\epsilon} + \bar{\epsilon}\bar{\partial}_i)\right]\phi_i(z_i, \bar{z}_i)\,. \tag{10.55}$$

Assembling these ingredients, and using the independence of the analytic and antianalytic parts of the expressions, we arrive at

$$\frac{1}{2\pi i}\oint_C \mathrm{d}z\, \epsilon(z)\left[\sum_{i=1}^n \left(\frac{h_i}{(z-z_i)^2} + \frac{\partial_i}{z-z_i}\right)\langle X\rangle - \langle T(z)X\rangle\right] = 0 \tag{10.56}$$

We have here used the Cauchy theorem. There is a corresponding expression with bars. Since $\epsilon(z)$ is arbitrary the integrand must in fact vanish:

$$\langle T(z)X\rangle = \sum_{i=1}^n \left(\frac{h_i}{(z-z_i)^2} + \frac{\partial_i}{z-z_i}\right)\langle X\rangle\,. \tag{10.57}$$

This is the desired *conformal Ward identity*. On the right-hand side we see manifestly the singularities in each of the coordinates z_i of the primary fields

$\phi_i(z_i, \bar{z}_i)$ entering the product X. These are the expected short-distance singularities whenever $T(z)$ approaches any of the primary fields:

$$T(z)\phi_j(z_j, \bar{z}_j) = \frac{h_j}{(z - z_j)^2}\phi_j(z_j, \bar{z}_j) + \frac{1}{z - z_j}\,\partial_{z_j}\phi(z_j, \bar{z}_j) + \mathcal{O}(1)\,. \quad (10.58)$$

The conformal Ward identity written in this local form is our first example of an *operator product expansion* (OPE), i.e., a formal power series in the coordinate difference that expresses the effect of bringing close together two operators.

Several remarks are in order:

1. It is tacitly understood that OPE's only have a sense when placed between the brackets $\langle \cdots \rangle$ of a correlation function.

2. We generically expect singularities to arise when approaching two local operators in a quantum field theory; in particular the average of a field over some small volume will have a variance that diverges when that volume is taken to zero.

3. An OPE should be considered an exact identity (valid in a finite domain of the field coordinates) rather than an approximation, provided the formal expansion is written out to arbitrarily high order. In our example, (10.57) only determines the first two terms in the OPE (10.58).

4. Contracting any field ϕ with $T(z)$ and comparing with (10.58) is actually a useful practical means of determining its primarity and its conformal dimension h_ϕ.

It is not difficult to see from (10.33) that on dimensional grounds T itself is a quasi-primary field of conformal dimension $h = 2$, since the partition function Z is dimensionless. However, the average $\langle T(z_1)T(z_2)\rangle \sim (z_1 - z_2)^{-4}$ has no reason to vanish, and so the OPE of T with itself takes the form

$$T(z_1)T(z_2) = \frac{c/2}{(z_1 - z_2)^4} + \frac{2T(z_2)}{(z_1 - z_2)^2} + \frac{\partial T(z_2)}{z_1 - z_2} + \mathcal{O}(1)\,. \quad (10.59)$$

In particular, T is *not* primary. The constant c appearing in (10.59) is called the *central charge*. Considering two non-interacting CFT's as a whole, one has from (10.33) that their stress tensors, and hence their central charges, add up, and so c can be considered as a measure of the number of quantum degrees of liberty in the CFT.

It is straightforward (but somewhat lengthy) exercise to establish that $c = \frac{1}{2}$ for a free fermion and $c = 1$ for a free boson. Details can be found in section 5.3 of [DMS87].

As T is not primary, it cannot transform like (10.45) under a finite conformal transformation $z \to w(z)$. We can always write the modified transformation law as

$$T'(w) = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-2} \left[T(z) - \frac{c}{12}\{w; z\}\right]. \tag{10.60}$$

To determine what $\{w; z\}$ represents, we use the constraint due to two successive applications of (10.60) and the fact that $\{w; z\} = 0$ for projective conformal transformations, since T is quasi-primary. The result is that $\{w; z\}$ is the Schwarzian derivative

$$\{w; z\} = \frac{\mathrm{d}^3 w/\mathrm{d}z^3}{\mathrm{d}w/\mathrm{d}z} - \frac{3}{2}\left(\frac{\mathrm{d}^2 w/\mathrm{d}z^2}{\mathrm{d}w/\mathrm{d}z}\right)^2. \tag{10.61}$$

10.4 Finite-size scaling on a cylinder

The central charge c is ubiquitous in situations where the CFT is placed in a finite geometry, i.e., interacts with some boundary condition. An important example is furnished by conformally mapping the plane to a cylinder of circumference L by means of the transformation

$$w(z) = \frac{L}{2\pi} \log z\,. \tag{10.62}$$

This transformation can be visualised by viewing the cylinder in perspective, with one rim contracting to the origin and the other expanding to form the point at infinity. Taking the expectation value of (10.60), and using the fact that $\langle T(z)\rangle = 0$ in the plane on symmetry grounds, one finds that $\langle T(w)\rangle = -\pi^2 c/6L^2$ on the cylinder. Applying (10.33) then implies that the free energy per unit area $f_0(L)$ satisfies [BCN86, Af86]

$$f_0(L) = f_0(\infty) - \frac{\pi c}{6L^2} + o(L^{-2})\,. \tag{10.63}$$

This is a very useful result for obtaining c for a concrete statistical model, since $f(L)$ can usually be determined from the corresponding transfer matrix,

either numerically for small L by using exact diagonalisation techniques, or analytically in the Bethe Ansatz context by using the Euler-Maclaurin formula.

One may note the clear analogy between (10.63) and the Casimir effect between two uncharged metallic plates. According to quantum electrodynamics, the vanishing of the wave function on the plates induces a force between them. This force may be attracting or repelling depending on the specific arrangement of the plates.

It is also of interest to study such finite-size effects on the level of the two-point correlation function of a primary field ϕ. Again using the mapping (10.62), the covariance property (10.45) and the form (10.27) of the correlator in the plane can be used to deduce its form on the cylinder. Assuming for simplicity $h = \bar{h} = \Delta/2$, and writing the coordinates on the cylinder as $w = t + ix$, with $t \in \mathbb{R}$ and $x \in [0, L)$, one arrives at

$$\langle \phi(t_1, x_1)\phi(t_2, x_2)\rangle = \left(\frac{2\pi}{L}\right)^{2\Delta} \left[2\cosh\left(\frac{2\pi t_{12}}{L}\right) - 2\cos\left(\frac{2\pi x_{12}}{L}\right)\right]^{-\Delta}, \tag{10.64}$$

where $t_{12} = t_1 - t_2$ and $x_{12} = x_1 - x_2$. In the limit of a large separation of the fields, $t_{12} \to \infty$, this decays like $\mathrm{e}^{-t_{12}/\xi}$ with correlation length $\xi = L/2\pi\Delta$. But this decay can also be written $(\Lambda_\phi/\Lambda_0)^{-t_{12}}$, where Λ_0 is the largest eigenvalue of the transfer matrix, and Λ_ϕ is the largest eigenvalue compatible with the constraint that an operator ϕ has been inserted at each extremity $t = \pm\infty$ of the cylinder. Denoting the corresponding free energies per unit area $f(L) = -L^{-1}\log\Lambda$, we conclude that [Ca84a]

$$f_\phi(L) - f_0(L) = \frac{2\pi\Delta}{L^2} + o(L^{-2})\,. \tag{10.65}$$

This is as useful as (10.63) in (numerical or analytical) transfer matrix studies, since the constraint imposed by ϕ can usually be related explicitly to properties of the transfer matrix spectrum.

10.5 Virasoro algebra and its representation theory

Up to this point, we have worked in a setup where the fields were seen as functionals of the complex coordinates z, $\bar{z}$. To obtain an operator formalism, one must impose a quantisation scheme, i.e., single out a time and a

space direction. In this formalism a crucial role will be played by the mode operators of the stress tensor, defined by

$$L_n = \frac{1}{2\pi i}\oint_C z^{n+1}T(z)\,\mathrm{d}z, \qquad \bar{L}_n = \frac{1}{2\pi i}\oint_C \bar{z}^{n+1}\bar{T}(\bar{z})\,\mathrm{d}\bar{z}\,. \tag{10.66}$$

The transfer matrix then propagates the system from one time slice to the following and is written as the exponential of the Hamiltonian $\mathcal{H}$, i.e., the energy operator on a fixed-time surface. In the continuum limit, one may freely choose the time direction. In CFT this is most conveniently done by giving full honours to the scale invariance of the theory, viz., by using for $\mathcal{H}$ the dilation operator

$$\mathcal{D} = \frac{1}{2\pi i}\oint_C z\,T(z)\,\mathrm{d}z + \frac{1}{2\pi i}\oint_C \bar{z}\,\bar{T}(\bar{z})\,\mathrm{d}\bar{z} = L_0 + \bar{L}_0\,, \tag{10.67}$$

where C is a counterclockwise contour enclosing the origin. The following choice of additive and multiplicative normalisations defines $\mathcal{H}$ precisely:

$$\mathcal{H} = (2\pi/L)(L_0 + \bar{L}_0 - c/12)\,. \tag{10.68}$$

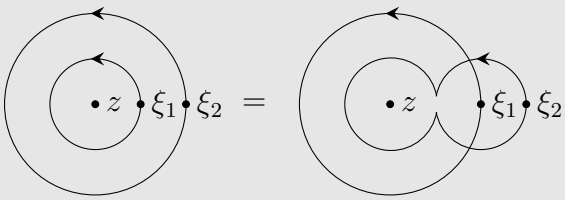
This is called the *radial quantisation* scheme: the constant-time surfaces are concentric circles around the origin. Under the map (10.62) the time becomes simply the coordinate along the cylinder axis. The usual time ordering of operators then becomes a prescription of radial ordering.

Using the radial ordering, the OPE (10.59) can be turned into a commutation relation $[L_n, L_m]$.

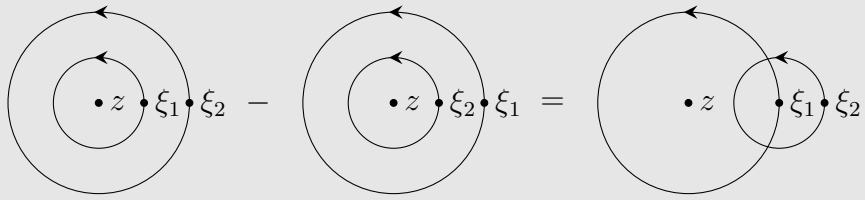
Consider first the action of L_nL_m on an operator $\Phi(z)$. By (10.66) we have

$$L_nL_m\Phi(z) = \frac{1}{(2\pi i)^2}\oint_{C_{z,\xi_1}}\mathrm{d}\xi_2\oint_{C_z}\mathrm{d}\xi_1\,(\xi_2-z)^{n+1}(\xi_1-z)^{m+1}T(\xi_2)T(\xi_1)\Phi(z)\,,$$

where the first integration contour C_z encircles only z, whereas the second C_{z,ξ_1} encircles both z and ξ_1. We can deform the latter contour as follows:



so that



This implies that

$$[L_n, L_m]\Phi(z) = \frac{1}{(2\pi i)^2}\oint_{C_z} \mathrm{d}\xi_1 (\xi_1 - z)^{m+1} \oint_{C_{\xi_1}} \mathrm{d}\xi_2 (\xi_2 - z)^{n+1} T(\xi_2)T(\xi_1)\Phi(z)\,.$$

But in the innermost integral—the one over C_{ξ_1}—the points ξ_1 and ξ_2 can be taken arbitrarily close, so that it is appropriate to replace $T(\xi_2)T(\xi_1)$ by the OPE (10.59). The innermost integral therefore reads

$$\begin{aligned} &\frac{1}{2\pi i}\oint_{C_{\xi_1}} \mathrm{d}\xi_2 (\xi_2 - z)^{n+1}\left[\frac{c/2}{(\xi_2-\xi_1)^4} + \frac{2T(\xi_1)}{(\xi_2-\xi_1)^2} + \frac{\partial T(\xi_1)}{\xi_2-\xi_1} + \mathcal{O}(1)\right] \\ = \;& \frac{c}{2}\frac{n+1}{3}\frac{n}{2}\frac{n-1}{1}(\xi_1 - z)^{n-2} + 2(n+1)(\xi_1 - z)^n T(\xi_1) + (\xi_1 - z)^{n+1}\partial T(\xi_1)\,, \end{aligned}$$

where we have used the Cauchy theorem on each of the three singular terms. Performing now the outermost integral—the one over C_z—gives us back the mode operators (10.66):

$$[L_n, L_m] = \frac{c}{12} n(n^2-1)\delta_{n+m,0} + 2(n+1)L_{m+n} + \frac{1}{2\pi i}\oint_{C_z} \mathrm{d}\xi_1 (\xi_1 - z)^{n+m+2}\partial T(\xi_1)\,.$$

In this expression, the remaining integral can be found by partial integration:

$$-\frac{1}{2\pi i}\oint_{C_z} \mathrm{d}\xi_1 (n+m+2)(\xi_1 - z)^{n+m+1} T(\xi_1) = -(n+m+2)L_{m+n}\,.$$

Inserting this gives us the final form of the commutation relations:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}\,. \tag{10.69}$$

A similar expression holds for $[\bar{L}_n, \bar{L}_m]$, whereas $[L_n, \bar{L}_m] = 0$. The algebra defined by (10.69) is called the *Virasoro algebra*. Importantly, the decoupling

into two isomorphic Virasoro algebras, one for L_n and another for $\bar{L}_n$, means that in the geometry chosen we can focus exclusively on L_n. It should be stressed that in the geometry of a torus, the two algebras couple non-trivially, in a way that is revealed by imposing modular invariance (see section 10.7 below).

We now describe the structure of the Hilbert space in radial quantisation. The vacuum state $|0\rangle$ must be invariant under projective transformations, whence $L_{\pm 1}|0\rangle = 0$, and we fix the ground state energy by $L_0|0\rangle = 0$. Non-trivial eigenstates of $\mathcal{H}$ are created by action with a primary field, $|h,\bar{h}\rangle = \phi(0,0)|0\rangle$. Translating (10.58) into operator language implies then in particular $L_0|h,\bar{h}\rangle = h|h,\bar{h}\rangle$. We must also impose the highest-weight condition $L_n|h,\bar{h}\rangle = \bar{L}_n|h,\bar{h}\rangle = 0$ for $n > 0$. Excited states with respect to the primary ϕ then read

$$\phi^{\{n,\bar{n}\}} \equiv L_{-n_1} L_{-n_2} \cdots L_{-n_k}\, \bar{L}_{-\bar{n}_1} \bar{L}_{-\bar{n}_2} \cdots \bar{L}_{-\bar{n}_{\bar{k}}} |h,\bar{h}\rangle \tag{10.70}$$

with $1 \le n_1 \le n_2 \le \cdots \le n_k$ and similarly for $\{\bar{n}\}$. These states are called the *descendents* of ϕ at level $\{N,\bar{N}\}$, where $N = \sum_{i=1}^{k} n_i$. A primary state and its descendents form a highest weight representation (or Verma module) of the Virasoro algebra.

Correlation functions of descendent fields can be obtained by acting with appropriate differential operators on the correlation functions of the corresponding primary fields. To see this, consider first for $n \ge 1$ the descendent $\big(L_{-n}\phi\big)(w)$ of the primary field $\phi(w)$, and let $X = \prod_j \phi_j(w_j)$ be an arbitrary product of other primaries as in the conformal Ward identity (10.57). Using (10.66) and (10.58) we have then

$$\begin{aligned} \big\langle \big(L_{-n}\phi\big)(w) X \big\rangle &= \frac{1}{2\pi i} \oint_{C_z} \mathrm{d}z\, (z-w)^{1-n} \,\langle T(z)\phi(w)X\rangle \qquad (10.71)\\ &= -\frac{1}{2\pi i} \oint_{C_{\{w_j\}}} \mathrm{d}z\, (z-w)^{1-n} \times \\ &\qquad \sum_j \left\{ \frac{\partial_{w_j}}{z-w_j} + \frac{h_j}{(z-w_j)^2} \right\} \langle \phi(w) X\rangle \ , \end{aligned}$$

where the minus sign comes from turning the integration contour inside out, so that it surrounds all the points $\{w_j\}$. In other words, a descendent in a correlation function may be replaced by the corresponding primary

$$\big\langle \big(L_{-n}\phi\big)(w) X \big\rangle = \mathcal{L}_{-n} \,\langle \phi(w) X\rangle \tag{10.72}$$

provided that we act instead on the correlator with the linear differential operator

$$\mathcal{L}_{-n} \equiv \sum_j \left\{ \frac{(n-1)h_j}{(w_j - w)^n} - \frac{\partial_{w_j}}{(w_j - w)^{n-1}} \right\} \tag{10.73}$$

It is readily seen that a general descendent (10.70) is similarly dealt with by replacing each factor L_{-n_i} by the corresponding factor of $\mathcal{L}_{-n_i}$ in (10.72).

We can now write the general form of the OPE of two primary fields ϕ_1 and ϕ_2. It reads

$$\phi_1(z,\bar{z})\phi_2(0,0) = \sum_p C_{12p} \sum_{\{n,\bar{n}\}\cup\{\emptyset,\emptyset\}} C_{12p}^{\{n,\bar{n}\}} z^{h_p - h_1 - h_2 + N} \bar{z}^{\bar{h}_p - \bar{h}_1 - \bar{h}_2 + \bar{N}} \phi_p^{\{n,\bar{n}\}}(0,0)\,, \tag{10.74}$$

where the summation is over a certain set of primaries $\phi_p \equiv \phi_p^{\{\emptyset,\emptyset\}}$ as well as their descendents. The coefficients $C_{12p}^{\{n,\bar{n}\}}$ (we have set $C_{12p}^{\{\emptyset,\emptyset\}} = 1$) can be determined by acting with all combinations of positive-index mode operators on both sides of (10.74) and solving the resulting set of linear equations.

In view of (10.69) it actually suffices to act with L_1 and L_2. Determining the $C_{12p}^{\{n,\bar{n}\}}$ is then a nice exercise of contour integration. (The answer can be found in Appendix B of [BPZ84].)

In contradistinction, the coefficients C_{12p} are fundamental quantities. Contracting both sides of (10.74) with ϕ_p and using the orthogonality of two-point functions (10.27) we see that the coefficients C_{12p} coincide with those appearing in the three-point functions (10.31).

The C_{12p} can be computed by the so-called conformal bootstrap method, i.e., by assuming crossing symmetry of the four-point functions. In concrete terms, this amounts to writing a well-chosen four-point function, mapping three of its points to 0, 1, ∞ by means of a projective transformation (10.44), and comparing all possible limits of the remaining point z. When computing those limits, one successively uses (10.74).

10.6 Minimal models

Denote by $\mathcal{V}(c,h)$ the highest weight representation (Verma module) generated by the mode operators $\{L_n\}$ acting on a highest weight state $|h\rangle$ in a

CFT of central charge c. The Hilbert space of the CFT can then be written

$$\bigoplus_{h,\bar{h}} n_{h,\bar{h}} \mathcal{V}(c,h) \otimes \mathcal{V}(c,\bar{h})\,, \tag{10.75}$$

where the multiplicities $n_{h,\bar{h}}$ indicate the number of distinct primaries of conformal weights $(h,\bar{h})$ that are present in the theory. A *minimal model* is a CFT for which the sum in (10.75) is finite.

The Hermitian conjugate of a mode operator is defined by $L_n^\dagger = L_{-n}$; this induces an inner product on the Verma module. The *character* $\chi_{(c,h)}$ of the module $\mathcal{V}(c,h)$ can then be defined as

$$\chi_{(c,h)}(\tau) = \mathrm{Tr}\, q^{L_0 - c/24}\,, \tag{10.76}$$

where $\tau \in \mathbb{C}$ is the so-called *modular parameter* (see section 10.7 below) and $q = \mathrm{e}^{2\pi i \tau}$. Since the number of descendents of $|h\rangle$ at level N is just the number $p(N)$ of integer partitions of N, cf. (10.70), we have simply

$$\chi_{(c,h)}(\tau) = \frac{q^{h-c/24}}{P(q)}\,, \tag{10.77}$$

where

$$\frac{1}{P(q)} \equiv \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} p(n) q^n \tag{10.78}$$

is the generating function of partition numbers; this is also often expressed in terms of the Dedekind function

$$\eta(\tau) = q^{1/24} P(q)\,. \tag{10.79}$$

However, the generic Verma module is not necessarily irreducible, so further work is needed.

For certain values of h, it may happen that a specific linear combination $|\chi\rangle$ of the descendents of $|h\rangle$ at level N is itself primary, i.e., $L_n|\chi\rangle = 0$ for $n > 0$. In other words, $|\chi\rangle$ is primary and descendent at the same time, and it generates its own Verma module $\mathcal{V}_\chi(c,h) \subset \mathcal{V}(c,h)$.

The states in $\mathcal{V}_\chi(c,h)$ are orthogonal to those in $\mathcal{V}(c,h)$,

$$\langle \chi | L_{-n_1} L_{-n_2} \cdots L_{-n_k} | h \rangle = \langle h | L_{n_k} \cdots L_{n_2} L_{n_1} | \chi \rangle^* = 0\,, \tag{10.80}$$

and so in particular they have zero norm. A Verma module $\mathcal{V}(c,h)$ containing one or more such *null fields* $|\chi\rangle$ is called reducible, and can be turned into

an irreducible Verma module $\mathcal{M}(c,h)$ by quotienting out the the null fields, i.e., by setting $|\chi\rangle = 0$. The Hilbert space is then given by (10.75) with $\mathcal{V}$ replaced by $\mathcal{M}$; since it contains fewer states the corresponding characters (10.76) are *not* given by the simple result (10.77).

The concept of null states is instrumental in constructing *unitary* representations of the Virasoro algebra (10.69), i.e., representations in which no state of *negative* norm occurs. An important first step is the calculation of determinant of the Gram matrix of inner products between descendents at level N. This is known as the the Kac determinant $\det M^{(N)}$. Its roots can be expressed through the following parameterisation:

$$\begin{aligned} c(m) &= 1 - \frac{6}{m(m+1)} \\ h(m) &= h_{r,s}(m) \equiv \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)} \end{aligned} \tag{10.81}$$

where $r, s \geq 1$ are integers with $rs \leq N$. The condition for unitarity of models with $c < 1$, first found by Friedan, Qiu and Shenker [FQS84] reads: $m, r, s \in \mathbb{Z}$ with $m \geq 2$, and (r, s) must satisfy $1 \leq r < m$ and $1 \leq s \leq m$.

To get an idea of the origin of (10.81) it is instructive to compute the Kac determinant at the first few levels. For instance, at level $N = 1$ the only state is $L_{-1}|h\rangle$, while at level $N = 2$ there are two states: $L_{-1}^2|h\rangle$ and $L_{-2}|h\rangle$. The Kac determinants read

$$\begin{aligned} \det M^{(1)} &= 2h\,, \\ \det M^{(2)} &= 32(h - h_{1,1})(h - h_{1,2})(h - h_{2,1})\,. \end{aligned} \tag{10.82}$$

The general result is

$$\det M^{(N)} = \alpha_N \prod_{r,s \geq 1}^{rs \leq N} [h - h_{r,s}(c)]^{p(N-rs)}\,, \tag{10.83}$$

where $p(n)$ was defined in (10.78) and $\alpha_N > 0$ is independent of h and c.

According to (10.72) the presence of a descendent field in a correlation function can be replaced by the action of a differential operator (10.73). Now

let

$$\chi(w) = \sum_{Y,|Y|=N} \alpha_Y L_{-Y} \phi(w) \tag{10.84}$$

be an arbitrary null state. Here, α_Y are some coefficients, and we have introduced the abbreviations

$$\begin{aligned} Y &= \{r_1, r_2, \ldots, r_k\}\,, \\ |Y| &= r_1 + r_2 + \ldots + r_k\,, \\ L_{-Y} &= L_{-r_1} L_{-r_2} \cdots L_{-r_k} \end{aligned} \tag{10.85}$$

with $1 \leq r_1 \leq r_2 \leq \cdots \leq r_k$. A correlation function involving χ must vanish (since we have in fact set $\chi = 0$), and so

$$\langle \chi(w) X \rangle = \sum_{Y,|Y|=N} \alpha_Y \mathcal{L}_{-Y}(w) \langle \phi(w) X \rangle = 0\,. \tag{10.86}$$

Solving this Nth order linear differential equation is a very useful practical means of computing the four-point correlation functions of a given CFT, provided that the level of degeneracy N is not too large. Indeed, since the coordinate dependence is through a single anharmonic ratio η, one has simply an ordinary linear differential equation.

Moreover, requiring consistency with (10.74) places restrictions on the primaries that can occur on the right-hand side of the OPE. One can then study the conditions under which this so-called *fusion algebra* closes over a finite number of primaries. The end result is that the minimal models are given by

$$\begin{aligned} c &= 1 - \frac{6(m-m')^2}{mm'} \\ h_{r,s} &= \frac{(mr - m's)^2 - (m-m')^2}{4mm'} \end{aligned} \tag{10.87}$$

with $m, m', r, s \in \mathbb{Z}$, and the allowed values of (r, s) are restricted by $1 \leq r < m'$ and $1 \leq s < m$. The corresponding $h_{r,s}$ are referred to as the *Kac table* of conformal weights. The corresponding fusion algebra reads (for clarity we omit scaling factors, structure constants, and descendents):

$$\phi_{(r_1,s_1)} \phi_{(r_2,s_2)} = \sum_{r,s} \phi_{(r,s)}\,, \tag{10.88}$$

where r runs from $1+|r_1-r_2|$ to $\min(r_1+r_2-1, 2m'-1-r_1-r_2)$ in steps of 2, and s runs from $1+|s_1-s_2|$ to $\min(s_1+s_2-1, 2m-1-s_1-s_2)$ in steps of 2.

The Kac table (10.87) is the starting point for elucidating the structure of the reducible Verma modules $\mathcal{V}_{r,s}$ for minimal models, and for constructing the proper irreducible modules $\mathcal{M}_{r,s}$. The fundamental observation is that

$$h_{r,s} + rs = h_{r,-s}\,. \tag{10.89}$$

This equation holds true for any value of c. It means that for $r, s \in \mathbb{Z}$ the Verma module $\mathcal{V}_{r,s}$ contains a singular vector at level rs that generates the submodule $\mathcal{V}_{r,-s}$. Quotienting out this submodule, we get an irreducible representation with character

$$K_{r,s}(\tau) = \frac{q^{-c/24}}{P(q)}\left(q^{h_{r,s}} - q^{h_{r,-s}}\right)\,. \tag{10.90}$$

For $r, s \in \mathbb{Z}$ this replaces the generic character $\chi_{c,h}(\tau)$ defined in (10.77).

The case of minimal models is however different. Using the symmetry property $h_{r,s} = h_{m'-r,m-s}$ and the periodicity property $h_{r,s} = h_{r+m',s+m}$ it is seen that $h_{r,s}+rs = h_{m'+r,m-s}$ and that $h_{r,s}+(m'-r)(m-s) = h_{r,2m-s}$. This means that $\mathcal{V}_{r,s}$ contains *two* submodules, $\mathcal{V}_{m'+r,m-s}$ and $\mathcal{V}_{r,2m-s}$, at levels rs and $(m'-r)(m-s)$ respectively, and these must correspond to null vectors. To construct the irreducible module $\mathcal{M}_{r,s}$ one might at first think that it suffices to quotient out these two submodules. However, iterating the above observations, the two submodules are seen to share two sub-submodules, and so on. So $\mathcal{M}_{r,s}$ is constructed from $\mathcal{V}_{r,s}$ by an infinite series of inclusions-exclusions of pairs of submodules. This allows us in particular to compute the irreducible characters of minimal models as

$$\chi_{(r,s)}(\tau) = K_{r,s}^{(m,m')}(q) - K_{r,-s}^{(m,m')}(q)\,, \tag{10.91}$$

where the infinite addition-subtraction scheme has been tucked away in the functions

$$K_{r,s}^{(m,m')}(q) = \frac{q^{-1/24}}{P(q)}\sum_{n\in\mathbb{Z}} q^{(2mm'n+mr-m's)^2/4mm'}\,. \tag{10.92}$$

This should be compared with the generic character (10.77) and with (10.90). Note also the similarity between (10.89) and (10.91) on the level of the indices.

It is truly remarkable that the above classification of minimal models has been achieved without ever writing down the action S appearing in (10.19). In fact, an effective Landau-Ginzburg Lagrangian description for the unitary minimal models ($m' = m+1$) has been suggested *a posteori* by Zamolodchikov [Za86]. It suggests that the minimal models can be interpreted physically as an infinite series of multicritical versions of the Ising model. Indeed, the Ising model can be identified with the first non-trivial member in the series, $m = 3$, and the following, $m = 4$, with the tricritical Ising model.

To finish this section, we comment on the relation with self-avoiding walks and polygons. In section 11 we shall see that these (to be precise, the dilute O($n \to 0$) model) can be identified with the minimal model $m = 2$, $m' = 3$. Note that this is *not* a unitary theory. The central charge is $c = 0$, and the only field in the Kac table—modulo the symmetry property given after (10.89)—is the identity operator with conformal weight $h_{1,1} = 0$. Seemingly we have learnt nothing more than the trivial statement $Z = 1$. However, the operators of interest are of a *non-local* nature, and it is a pleasant surprise to find that their dimensions fit perfectly well into the Kac formula, although they are situated *outside* the "allowed" range of (r, s) values, and sometimes require the indices r, s to be half-integer. So the Kac formula, and the surrounding theoretical framework, is still a most useful tool for investigating these types of models.

10.7 Modular invariance

In section 10.3 we have seen that conformal symmetry makes the stress tensor decouple into its holomorphic and antiholomorphic components, $T(z)$ and $\bar{T}(\bar{z})$, implying in particular that the corresponding mode operators, L_n and $\bar{L}_n$, form two non-interacting Virasoro algebras (10.69). As a consequence, the key results of section 10.6 could be derived by considering only the holomorphic sector of the CFT. There are however constraints on the ways in which the two sectors may ultimately couple, the diagonal coupling (10.75) being just the simplest example in the context of minimal models. As first pointed out by Cardy [Ca86], a powerful tool for examining which couplings are allowed—and for placing constraints on the operator content and the conformal weights—is obtained by defining the CFT on a torus and imposing the constraint of *modular invariance*.

In this section we expose the principles of modular invariance and apply them to a CFT known as the *compactified boson*, which is going to play a central role in the Coulomb gas approach of section 11. Many other applications, including a detailed study of the minimal models, can be found in Ref. [DMS87].

Let $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ such that $\tau \equiv \omega_2/\omega_1 \notin \mathbb{R}$. A torus is then defined as $\mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})$, i.e., by identifying points in the complex plane that differ by an element in the lattice spanned by ω_1, ω_2. The numbers ω_1, ω_2 are called the *periods* of the lattice, and τ the *modular parameter*. Without loss of generality we can assume $\omega_1 \in \mathbb{R}$ and $\Im\tau > 0$.

Instead of using the radial quantisation scheme of section 10.5 we now define the time (resp. space) direction to be the imaginary (resp. real) axis in $\mathbb{C}$. The partition function on the torus may then be written $Z(\tau) = \mathrm{Tr}\, \exp\left[-(\Im\omega_2)\mathcal{H} - (\Re\omega_2)\mathcal{P}\right]$, where $\mathcal{H} = (2\pi/\omega_1)(L_0 + \bar{L}_0 - c/12)$ is the Hamiltonian and $\mathcal{P} = (2\pi/i\omega_1)(L_0 - \bar{L}_0 - c/12)$ the momentum operator. This gives

$$Z(\tau) = \mathrm{Tr}\, \left(q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} \right) , \tag{10.93}$$

where we have defined $q = \exp(2\pi i\tau)$. Comparing with (10.75)–(10.76) we have also

$$Z(\tau) = \sum_{h,\bar{h}} n_{h,\bar{h}}\, \chi_{(c,h)}(\tau) \bar{\chi}_{(c,\bar{h})}(\tau) \, . \tag{10.94}$$

An explicit computation of $Z(\tau)$ will therefore give information on the coupling $n_{h,\bar{h}}$ between the holomorphic and antiholomorphic sectors. In many cases, but not all, the coupling turns out to be simply diagonal, $n_{h,\bar{h}} = \delta_{h,\bar{h}}$.

The fundamental remark is now that $Z(\tau)$ is invariant upon making a different choice ω_1', ω_2' of the periods, inasmuch as they span the same lattice as ω_1, ω_2. Any two set of equivalent periods must therefore be related by $\omega_i' = \sum_j a_{ij}\omega_j$, where $\{a_{ij}\} \in \mathrm{Mat}(2, \mathbb{Z})$ with $\det a_{ij} = 1$. Moreover, an overall sign change, $a_{ij} \to -a_{ij}$ is immaterial, so the relevant symmetry group is the so-called *modular group* $\mathrm{SL}(2, \mathbb{Z})/\mathbb{Z}_2 \simeq \mathrm{PSL}(2, \mathbb{Z})$.

The remainder of this section is concerned with the the construction of modular invariant partition functions for certain bosonic systems on the torus. As a warmup we consider the free boson, defined by the action

$$S[\phi] = \frac{g}{2} \int \mathrm{d}^2\mathbf{x}\, (\nabla\phi)^2 \tag{10.95}$$

and $\phi(\mathbf{x}) \in \mathbb{R}$. Comparing (10.93) with (10.76)–(10.78), and bearing in mind that $c = 1$, we would expect the corresponding partition function to be of the form $Z_0(\tau) \propto 1/|\eta(\tau)|^2$. Fixing the proportionality constant is somewhat tricky [IZ86]. In a first step, ϕ is decomposed on the normalised eigenfunctions of the Laplacian, and $Z_0(\tau)$ is expressed as a product over the eigenvalues. This product however diverges, due to the presence of a zero-mode, and must be regularised. A sensible result is obtained by a shrewd analytic continuation, the so-called ζ-function regularisation technique [IZ86]:

$$Z_0(\tau) = \frac{\sqrt{4\pi g}}{\sqrt{\Im\tau}\,|\eta(\tau)|^2}\,. \tag{10.96}$$

The CFT which is of main interest for the CG technique is the so-called *compactified boson* in which $\phi(\mathbf{x}) \in \mathbb{R}/(2\pi aR\mathbb{Z})$. In other words, the field lives on a circle of radius aR (the reason for the appearance of *two* parameters, a and R, will become clear shortly). In this context, suitable periodic boundary conditions are specified by a pair of numbers, $m, m' \in a\mathbb{Z}$, so that for any $k, k' \in \mathbb{Z}$

$$\phi(z + k\omega_1 + k'\omega_2) = \phi(z) + 2\pi R(km + k'm')\,. \tag{10.97}$$

It is convenient to decompose $\phi = \phi_{m,m'} + \phi_0$, where

$$\phi_{m,m'} = \frac{2\pi R}{\bar\tau - \tau}\left[\frac{z}{\omega_1}(m\bar\tau - m') - \frac{\bar z}{\bar\omega_1}(m\tau - m')\right] \tag{10.98}$$

is the classical solution satisfying the topological constraint, and ϕ_0 represents the quantum fluctuations, i.e., is a standard free boson satisfying standard periodic boundary conditions.

Integrating over ϕ_0 as before, and keeping m, m' fixed, gives the partition function

$$Z_{m,m'}(\tau) = Z_0(\tau)\,\exp\left(-2\pi^2 gR^2\frac{|m\tau - m'|^2}{\Im\tau}\right)\,. \tag{10.99}$$

It is easy to see that this is not modular invariant. A modular invariant is however obtained by summing over all possible values of m, m':

$$Z(\tau) \equiv \frac{R}{\sqrt{2}}\,Z_0(\tau)\sum_{m,m'\in a\mathbb{Z}}\exp\left(-2\pi^2 gR^2\frac{|m\tau - m'|^2}{\Im\tau}\right) \tag{10.100}$$

The prefactor $R/\sqrt{2}$ is again a subtle effect of the zero-mode integration. It is actually most easily justified *a posteriori* by requiring the correct normalisation of the identity operator in (10.101) below.

A more useful, and more physically revealing, form of (10.100) is obtained by using the Poisson resummation formula to replace the sum over $m' \in a\mathbb{Z}$ by a sum over the dual variable $e \in \mathbb{Z}/a$. The result is

$$Z(\tau) = \frac{1}{|\eta(\tau)|^2} \sum_{e\in\mathbb{Z}/a,\ m\in a\mathbb{Z}} q^{h_{e,m}} \bar{q}^{\bar{h}_{e,m}}\,, \tag{10.101}$$

with

$$h_{e,m} = \frac{1}{2}\left(\frac{e}{R\sqrt{4\pi g}} + \frac{mR}{2}\sqrt{4\pi g}\right)^2, \qquad \bar{h}_{e,m} = \frac{1}{2}\left(\frac{e}{R\sqrt{4\pi g}} - \frac{mR}{2}\sqrt{4\pi g}\right)^2. \tag{10.102}$$

Comparing now with (10.93) and (10.76)–(10.78) we see that (10.102) is nothing else than the conformal weights of the CFT at hand.

The requirement of modular invariance has therefore completely specified the operator content of the compactified boson system. An operator is characterised by two numbers, $e \in \mathbb{Z}/a$ and $m \in a\mathbb{Z}$, living on mutually dual lattices. A physical interpretation will be furnished by the CG formalism of section 11: e is the "electric" charge of a vertex operator (spin wave), and m is the "magnetic" charge of a topological defect (screw dislocation in the field ϕ). Let us write for later reference the corresponding scaling dimension and spin:

$$\Delta_{e,m} = \frac{e^2}{4\pi g R^2} + m^2 \pi g R^2, \qquad s_{e,m} = em \tag{10.103}$$

Observe in particular that the spin is integer, as expected for a bosonic system.

The reader will notice that the three constants R, a and g are related by the fact that they always appear in the dimensionless combination R^2a^2g. Field-theoretic literature often makes the choice $a = 1$ and $g = 1/4\pi$ in order to simplify formulae such as (10.102). In the CG approach—the subject of section 11—one starts from a geometrical construction (mapping to a height model) in which a convention for a must be chosen. The compactification radius aR then follows from a "geometrical" computation (identification of the ideal state lattice), and the correct coupling constant g is only fixed in the

end by a field-theoretic argument (marginality requirement of the Liouville potential). Needless to say, the results, such as (10.103) for the dimensions of physical operators, need (and will) be independent of the initial choice made for a.

To conclude, note that the roles of e and m in (10.102) are interchanged under the transformation $Ra\sqrt{2\pi g} \to (Ra\sqrt{2\pi g})^{-1}$, which leaves (10.101) invariant. This is another manifestation of the electro-magnetic duality. Ultimately, the distinction between e and m comes down to the choice of transfer direction. In the geometry of the torus this choice is immaterial, of course. In sections 11.3–12.3 we shall compare the geometries of the cylinder and the annulus; these are related by interchanging the space and time directions, and accordingly the electric and magnetic charges switch role when going from one to the other.