11 Coulomb gas construction

It has been known since the 1970's [LP75, Ka78, KB79, Kn81] that the critical point of many two-dimensional models of statistical physics can be identified with a Gaussian free-field theory. A general framework for the computation of critical exponents was first given in 1977 by José et al. in the so-called spin wave picture [Jo77]. This was further elaborated in the early 1980's by den Nijs [Ni83, Ni84] and Nienhuis [Nh84] into what has become known as the Coulomb gas (CG) construction. These developments have been reviewed by Nienhuis [Nh87].

The CG approach is particularly suited to deal with the continuum limit of lattice models of closed self-avoiding loops, in which each loop carries a Boltzmann weight n. There are two prototype models which can be represented in terms of such loops. The Potts model has already been discussed at length in chapter 8. Another useful example is the O(n) model, which can be reformulated elegantly as a loop model on the hexagonal lattice [Nh82].

The marriage between the CG and conformal field theory (CFT) happened in 1986–87, when Di Francesco, Saleur and Zuber [DSZ87a, DSZ87b] made the loop model $\leftrightarrow$ CG correspondence more precise and showed how the ideas of modular invariance [Ca86, IZ86] can be put to good use in the study of loop models. At the same time, Duplantier and Saleur developed a range of applications to self-avoiding walks and polygons (see in particular [DS87]).

Any model of oriented self-avoiding loops is equivalent to a height model on the dual lattice. It is the continuum limit of this height which acts as the conformally invariant free field. The underlying lattice model implies that this height field is compactified, thus making contact with the modular invariance results of section 10.7. The naive free field action however needs to be modified with extra terms, traditionally known as background and screening electric charges [Nh87]. The resulting CFT, known as a Liouville field theory, is written down in section 11.2.

The requirement that the Liouville potential be RG marginal determines the coupling constant of the free field as a function of n, as first pointed out by Kondev [Ko97]. This is an important ingredient, since otherwise one would have to rely on an independent exact solution to fix the coupling. We discuss these developments in section 11.3.

11.1 From loops to a compactified boson

In chapter 8 we have seen how to transform the Potts model into a model of oriented loops on the medial lattice. The weight

$$n = \sqrt{Q} = 2\cos\gamma \tag{11.1}$$

per loop was transformed into local complex weights at the vertices.

The oriented loop model can easily be turned into a *height model*. For this, assign a scalar variable $h(\mathbf{x})$ to each lattice face $\mathbf{x}$ (i.e., to each vertex of the lattice *dual* to the one on which the loop model has been defined), so that h increases (resp. decreases) by a each time one traverses a left-going (resp. right-going) edge. This definition of the height h is consistent, since each vertex is incident on as many ingoing as outgoing edges. Since this defines only height differences, one may imagine fixing h completely by arbitrarily fixing $h(\mathbf{0}) = 0$.

In the continuum limit, we expect the local height field h to converge to a free bosonic field $\phi(\mathbf{x})$, whose entropic fluctuations are described by an action of the form (10.95), with coupling $g = g(n)$ which is a monotonically increasing function of n. In particular, for $n \to \infty$ the lattice model is dominated by the configuration where loops of the minimal possible length cover the lattice densely; the height field is then flat, $\phi(\mathbf{x}) =$ constant, and the correlation length ξ is of the order of the lattice spacing. For finite but large n, ϕ will start fluctuating, loop lengths will be exponentially distributed, and ξ will be of the order of the linear size of the largest loop. When $n \to n_{\mathrm{c}}^{+}$, for some critical n_{c} (we shall see that $n_{\mathrm{c}} = 2$), this size will diverge, and for $n \leq n_{\mathrm{c}}$ the loop model will be conformally invariant with critical exponents that depend on $g(n)$. The interface described by $\phi(\mathbf{x})$ is then in a *rough* phase. The remainder of this section is devoted to making this intuitive picture more precise, and to refine the free bosonic description of the critical phase.

As a first step towards greater precision, we now argue that $\phi(\mathbf{x})$ is in fact a *compactified boson*, cf. section 10.7. To see this, it is convenient to consider the *oriented* loop configurations that give rise to a maximally flat microscopic height h; following Henley and Kondev [KH95] we shall refer to them as *ideal states*. For the Potts model (11.4), an ideal state is a dense packing of length-four loops, all having the same orientation. There are four such states, corresponding to two choices of orientation and two choices of the sublattice of lattice faces surrounded by the loops.

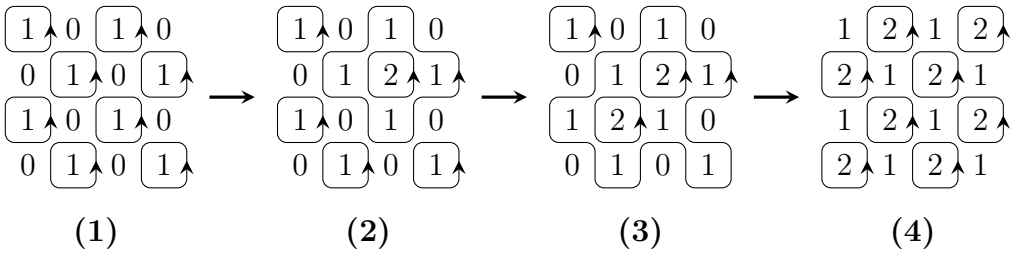

Figure 25: An ideal state (panel 1) in the oriented loop model is gradually changed into another (panel 4). The large loop created in panels 2 and 3 is annihilated via the periodic boundary conditions to obtain panel 4, which is a different ideal state. In the process the mean height changes from ϕ to $\phi + a$ (with $a = 1$ in the figure).

An ideal state can be gradually changed into another by means of $\sim N$ local changes of the transition system and/or the edge orientations. This is shown in Fig. 25. As a result, the mean height will change, $\phi \to \phi \pm a$. Repeating this, one sees that one may return to the initial ideal state whilst having $\phi \to \phi \pm 2a$. For consistency, we must therefore require $\phi(\mathbf{x}) \in \mathbb{R}/(2a\mathbb{Z})$, i.e., the field is compactified with radius $R = 1/\pi$, cf. (10.97).

In section 10.7 we have seen in detail that the normalisation constant a drops out from the final physical results. We shall therefore follow standard conventions and set $a = \pi$ in what follows.

11.2 Liouville field theory

The essence of the above discussion is that the critical properties of the loop model under consideration can be described by a continuum-limit partition function that takes the form of a functional integral

$$Z = \int \mathcal{D}\phi(\mathbf{x}) \, \exp\left(-S[\phi(\mathbf{x})]\right) \, . \tag{11.2}$$

Here $S[\phi(\mathbf{x})]$ is the Euclidean action of the compactified scalar field $\phi(\mathbf{x}) \in \mathbb{R}/(2\pi\mathbb{Z})$. The hypothesis that the critical phase is described by bounded elastic fluctuations around the ideal states means that S must contain a term

$$S_{\mathrm{E}} = \frac{g}{4\pi} \int \mathrm{d}^2\mathbf{x} \, (\nabla\phi)^2 \tag{11.3}$$

with coupling constant $g > 0$. Higher derivative terms that one may think of adding to (11.3) can be ruled out by the $\phi \to -\phi$ symmetry, or by arguing *a posteriori* that they are RG irrelevant in the full field theory that we are about to construct.

Note that the partition function (11.2) does not purport to coincide with that of the critical Potts model

$$Z = Q^{|V|/2} \sum_{E' \subseteq E} n^{l(E')} \tag{11.4}$$

on the scale of the lattice constant. (A similar remark holds true for the correlation functions that one may similarly write down.) We do however claim that their long-distance properties are the same. In that sense, the CG approach is an exact, albeit by no means rigorous, method for computing critical exponents and related quantities. A more precise equivalence between discrete and continuum-limit partition functions can however be achieved on a torus [DSZ87b] or on an annulus (see chapter 13).

The action (11.3) coincides with (10.95) for the compactified boson. To obtain the full physics of the loop model one however needs to add two more terms to the action, as we now shall see.

We consider the underlying lattice model as being defined on a cylinder, $\mathbf{x} = (x, t)$. This has the advantage of making direct contact with the radial quantisation formalism of section 10.5. The boundary conditions are thus periodic in the space direction, $x = x + L$, and free in the time (t) direction. Ultimately, the results obtained on the cylinder can always be transformed into other geometries by means of a conformal mapping.

We have seen in section 8.8 that with this geometry, in order to obtain the correct weighting of non-contractible loops, the corresponding six-vertex model must be twisted across a seam that runs along the cylinder. Consider now adding a term

$$S_{\mathrm{B}} = \frac{ie_0}{4\pi} \int \mathrm{d}^2\mathbf{x}\, \phi(\mathbf{x}) \mathcal{R}(\mathbf{x}) \tag{11.5}$$

to the effective action S, where $\mathcal{R}$ is the scalar curvature[32] of the space $\mathbf{x}$. The parameter e_0 is known in CG language as the *background electric charge*. On the cylinder, one has simply $S_{\mathrm{B}} = ie_0 \left(\phi(x, \infty) - \phi(x, -\infty)\right)$, meaning that in

[32] We consider the scalar curvature in a generalised sense, so that delta function contributions may be located at the boundaries. Implicitly, we are just applying the Gauss-Bonnet theorem.

the partition function (11.2) an oriented loop with winding number $q = 0, \pm 1$ (all other winding numbers are forbidden by the self-avoidance of the loops) can equivalently be assigned an extra weight of $\exp(i\pi q e_0)$. This leaves the weight n of non-winding loops unchanged, while winding loops get a modified weight

$$n_1 = 2\cos\gamma_1\,, \qquad \text{with } \gamma_1 = \pi e_0\,, \tag{11.6}$$

as in (8.37). The choice $\gamma_1 = \gamma$ will thus lead to $n_1 = n$. Note however that the possibility of having $n_1 \neq n$ is useful in some applications of the CG technique.

Verify the above argument for more than one non-contractible loop, bearing in mind that the orientation of each loop has to be summed over independently!

The object $\mathrm{e}^{ie\phi}$ (or more precisely, its normal ordered product $:\mathrm{e}^{ie\phi}:$) is known in CFT as a *vertex operator* of (electric) charge e. The boundary term (11.5) thus corresponds to the insertion of two oppositely charged vertex operators at either end of the cylinder (and more generally at the root vertices of section 8.8).

At this stage two problems remain: the field theory does not yet take account of the weight n of contractible loops, and the coupling constant g has not yet been determined. These two problems are closely linked, and allow [Ko97] us to fix exactly $g = g(n)$. The idea is to add a further *Liouville term*

$$S_{\mathrm{L}} = \int \mathrm{d}^2\mathbf{x}\, w[\phi(\mathbf{x})] \tag{11.7}$$

to the action, which then reads in full

$$S[\phi(\mathbf{x})] = S_{\mathrm{E}} + S_{\mathrm{B}} + S_{\mathrm{L}}\,. \tag{11.8}$$

In (11.7), $\mathrm{e}^{-w[\phi(\mathbf{x})]}$ is the scaling limit of the microscopic vertex weights w_i that we now identify.

Due to the compactification, $S_L[\phi]$ is a periodic functional of the field, and as such it can be developed as a Fourier sum over vertex operators

$$w[\phi] = \sum_{e\in\mathcal{L}_w} \tilde{w}_e\, \mathrm{e}^{ie\phi}\,, \tag{11.9}$$

where $\mathcal{L}_w$ is some sublattice of $\mathcal{L}_0 \equiv \mathbb{Z}$. Note that $\mathcal{L}_w$ may be a proper sublattice of $\mathcal{L}_0$ if $w[\phi]$ has a higher periodicity than that trivially conferred by the compactification of ϕ. By inspecting Fig. 25 we see that this is indeed the case here: the (geometric) averages of the miscroscopic weights coincide on panels 1 and 4, indicating that the correct choice is $\mathcal{L}_w = 2\mathcal{L}_0$. This intuitive derivation of $\mathcal{L}_w$ demonstrates the utility of the ideal state construction.

Some important properties of the compactified boson with action S_{E} have already been derived in section 10.7. In particular, its central charge is $c = 1$ and the dimension $\Delta_{e,m}$ of an operator with electromagnetic charge (e, m) is given by (10.103). Having now identified the electric charge e with that of the vertex operator $\mathrm{e}^{ie\phi}$, one could alternatively rederive (10.103) by computing the two-point function $\left\langle \mathrm{e}^{ie\phi(\mathbf{x})}\mathrm{e}^{-ie\phi(\mathbf{y})} \right\rangle$ by standard Gaussian integration.

The physical interpretation of the magnetic charge m is already obvious from (10.97): it corresponds to dislocations in the height field ϕ due to the presence of defect lines. In section 11.4 we shall see how such defects are used in the computation of critical exponents.

It remains to assess how the properties of the compactified boson are modified by the inclusion of the term S_{B}. Physical reasoning consists in arguing that the vertex operators $\mathrm{e}^{\pm ie_0\phi}$ will create a "floating" electric charge of magnitude $2e_0$ that "screens" that of the other fields in any given correlation function. We infer that (10.103) must be changed into

$$\Delta_{e,m} = \frac{1}{2}\left[\frac{e(e-2e_0)}{g} + gm^2\right] . \tag{11.10}$$

Note that to obtain (11.10) we have changed our normalisation so that both e and m are integers. This is consistent with the normalisation (11.3) of the coupling constant, rather than (10.95), which is the standard choice in the CG literature.

11.3 Marginality requirement

Following Kondev [Ko97] we now claim that the Liouville potential S_L must be exactly marginal. This follows from the fact that all loops carry the same weight n, independently of their size, and so the term S_L in the action that enforces the loop weight must not renormalise under a scale transformation. The most relevant vertex operator appearing in (11.9) has charge $e_w = 2\pi/a = 2$, and so $\Delta_{e_w,0} = 2$. Using (11.10), this fixes the coupling

constant as $g = 1 - e_0$. In other words, the loop weight has been related to the CG coupling as

$$n = \sqrt{Q} = -2\cos(\pi g)\,, \tag{11.11}$$

with $0 < g \leq 1$ for the critical Potts model.

The term S_B shifts the ground state energy with respect to the $c = 1$ theory described by S_E alone. The corrected central charge is then $c = 1+12\Delta_{e_0,0}$, where the factor of 12 comes from comparing (10.63) and (10.65). This gives

$$c = 1 - \frac{6(1-g)^2}{g}\,. \tag{11.12}$$

It should be noted that the choice $e_w = 2$ is not the only one possible. Namely, the coefficient $\tilde{w}_{e_w}$ of the corresponding vertex operator in (11.9) may be made to vanish, for instance by driving the Potts model to tricriticality via the introduction of a carefully tuned coupling to non-magnetic vacancies. The next-most relevant choice is then $\tilde{e}_w = -2$, and going through the same steps as above we see that one can simply maintain (11.11), but take the coupling in the interval $1 \leq g \leq 2$ for the tricritical Potts model.

The electric charge e_w whose vertex operator is required to be exactly marginal is known as the *screening charge* in standard CG terminology.

The central charge (11.12) can now be formally identified with that of the Kac table (10.87), with $m' = m + 1$. The result is a formal relation between the minimal model index m and the CG coupling g, valid for integer m. We have

$$m = \begin{cases} \frac{g}{1-g} & \text{for the critical Potts model} \\ \frac{1}{g-1} & \text{for the tricritical Potts model} \end{cases} \tag{11.13}$$

11.4 Critical exponents

We shall now see how to use the Coulomb gas technology to compute a variety of critical exponents in loop models.

The watermelon exponents were derived by Nienhuis [Nh87] and by Duplantier and Saleur (see [DS87] and references therein). The issues of their relation to the standard exponents of polymer physics [Ge79], and to the Kac table (10.81), were discussed in [DS87].

Although the watermelon exponents are essentially magnetic-type exponents in the CG, they do not produce the standard magnetic exponent of the Potts model. The latter was derived by den Nijs [Ni83], but we present here a somewhat different argument.

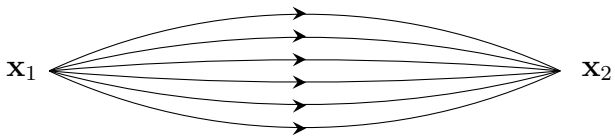

Figure 26: Watermelon configuration with $\ell = 6$ legs.

11.4.1 Watermelon exponents

An important object in loop models is the operator $\mathcal{O}_\ell(\mathbf{x}_1)$ that inserts ℓ oriented lines at a given point $\mathbf{x}_1$. Microscopically, this can be achieved by violating the arrow conservation constraint at $\mathbf{x}_1$. In the Potts model, or rather the equivalent six-vertex model, a vertex with four outgoing and zero ingoing arrows furnishes a microscopic realisation of the case $\ell = 2$. Higher ℓ can be obtained by inserting several defects in a small region around $\mathbf{x}_1$.[33]

If one had strict arrow conservation at all other vertices, the insertion of $\mathcal{O}_\ell(\mathbf{x}_1)$ would not lead to a consistent configuration. However, also inserting $\mathcal{O}_{-\ell}(\mathbf{x}_2)$, the operator that absorbs ℓ oriented lines in a small region around $\mathbf{y}$, will lead to consistent configurations (see Fig. 26) in which ℓ defect lines propagate from $\mathbf{x}_1$ to $\mathbf{x}_2$. Let $Z_\ell(\mathbf{x}_1, \mathbf{x}_2)$ be the corresponding constrained partition function. One then expects

$$\langle \mathcal{O}_\ell(\mathbf{x}_1)\mathcal{O}_{-\ell}(\mathbf{x}_2)\rangle \equiv \frac{Z_\ell(\mathbf{x}_1,\mathbf{x}_2)}{Z} \sim \frac{1}{|\mathbf{x}_1-\mathbf{x}_2|^{2\Delta_\ell}} \text{ for } |\mathbf{x}_1-\mathbf{x}_2| \gg 1\,. \quad (11.14)$$

The corresponding critical exponents Δ_ℓ are known as watermelon (or *fuseau*, or ℓ-leg) exponents. To compute them we first notice that the sum of the height differences around a closed contour encircling $\mathbf{x}_1$ but not $\mathbf{x}_2$ will be $a\ell$. Equivalently, one could place the two defects at the extremities of a cylinder [i.e., taking $\mathbf{x}_1 = (x, -\infty)$ and $\mathbf{x}_2 = (x, \infty)$], and the height difference would be picked up by any non-contractible loop separating $\mathbf{x}_1$ and $\mathbf{x}_2$. This latter formulation makes contact with the defect lines (10.97) introduced when studying the compactified boson, the equivalent magnetic charge being $m_\ell = \frac{\ell a}{2\pi} = \frac{\ell}{2}$.

A little care is needed to interpret the configurations of $Z_\ell(\mathbf{x}_1, \mathbf{x}_2)$ in the model of un-oriented loops. The fact that all ℓ lines are oriented away

[33] The Potts model only allows for defects with even ℓ. In the closely relate O(n) model, any parity of ℓ is permitted.

from $\mathbf{x}_1$ prevents them from annihilating at any other vertex than $\mathbf{x}_2$. One should therefore like to think about them as ℓ marked lines linking $\mathbf{x}_1$ and $\mathbf{x}_2$, where each line carries the Boltzmann weight 1. This is consistent with not summing over the orientations of the defect lines in the oriented loop model. However, each oriented line can also pick up spurious phase factors $w(\alpha/2\pi)$, due to the local redistribution of loop weights, whenever it turns around the end points $\mathbf{x}_1$ and $\mathbf{x}_2$. These factors are however exactly cancelled if we insert in addition a vertex operator $\mathrm{e}^{ie_0\phi}$ (resp. $\mathrm{e}^{-ie_0\phi}$) at $\mathbf{x}_1$ (resp. $\mathbf{x}_2$) [Nh84]. Note that these vertex operators do not modify the weighting of closed loops: these must encircle either none of both of $\mathbf{x}_1$, $\mathbf{x}_2$, since otherwise they would intersect the ℓ watermelon legs. We conclude that $\Delta_\ell = \Delta_{e_0,m_\ell}$, and using (11.10) this gives

$$\Delta_\ell = \frac{1}{8} g\ell^2 - \frac{(1-g)^2}{2g} \,. \tag{11.15}$$

Interestingly, these exponents can be attributed to the Kac table under the identification (11.13). One has

$$\Delta_\ell = \begin{cases} 2h_{0,\ell/2} & \text{for the dense O}(n) \text{ model} \\ 2h_{\ell/2,0} & \text{for the dilute O}(n) \text{ model} \end{cases} \tag{11.16}$$

The Kac indices (r,s) appearing in $h_{r,s}$ are integer valued, since $\ell \in 2\mathbb{N}$. When the loop model coincides with a minimal model—i.e., when g is such that $m \in \mathbb{N}$ in (11.13)—some of these exponents are located outside the fundamental domain of the Kac table. This reflects the fact that the corresponding watermelon operators are of intrinsic non-local nature.

11.4.2 Application to percolation clusters

The watermelon exponents can be used to elucidate the fractal properties of the Fortuin-Kasteleyn (FK) clusters defined in section 8.2. Here we limit the discussion to the special case of percolation clusters.

We have seen in section 8.4.1 that bond percolation is the $Q \to 1$ limit of FK clusters. We have therefore $g = \frac{2}{3}$ from (11.11). The watermelon exponents (11.15) are therefore

$$\Delta_\ell = \frac{\ell^2 - 1}{12} \,. \tag{11.17}$$

Marking a point $\mathbf{x}$ on the hull of a percolation cluster corresponds to the insertion of the operator $\mathcal{O}_2(\mathbf{x})$. The fractal dimension of the hull is therefore

$$d_{\rm h} = 2 - \Delta_2 = \frac{7}{4}\,. \tag{11.18}$$

A *pivotal edge* is defined as an edge belonging to a percolation cluster which is such that the removal of the edge makes the cluster break into two connected components. In the literature on percolation pivotal edges are also known as *red bonds*. Cutting the loop strands on either side of any edge belonging to the cluster looks like an $\ell = 4$ leg insertion. Note however that only is the edge is pivotal will the four legs propagate to "infinity" without contracting among themselves. Therefore the fractal dimension of red bonds is

$$d_{\rm rb} = 2 - \Delta_4 = \frac{3}{4}\,. \tag{11.19}$$

11.4.3 Magnetic exponent

The watermelon exponents can be said to be of the "magnetic" type, since they induce a magnetic type defect charge m_ℓ in the CG. The standard magnetic exponent, describing the decay of the spin-spin correlation function in the Potts model, is however not of the watermelon type. It can nevertheless be inferred from (11.10) as follows:

The probability that two spins situated at $\mathbf{x}_1$ and $\mathbf{x}_2$ are in the same Potts state is proportional, in the random cluster picture, to the probability that they belong to the same cluster. In the cylinder geometry this means that no winding loop separates $\mathbf{x}_1$ from $\mathbf{x}_2$. This can be attained in the CG by giving a weight $n_1 = 0$ to such loops. We have seen that inserting a pair of vertex operators with charge $\pm e$ at $\mathbf{x}_1$ and $\mathbf{x}_2$ leads exactly to this situation with $n_1 = 2\cos(\pi e)$, and so we need $e = \frac{1}{2}$. The scaling dimension of this excitation, with respect to the ground state which has $e = e_0$, is then

$$\Delta_{\rm m} = \Delta_{\frac{1}{2},0} - \Delta_{e_0,0} = \frac{1 - 4(1-g)^2}{8g}\,. \tag{11.20}$$

In particular for the Ising model, with $g = \frac{3}{4}$, this yields the magnetic exponent $\Delta_{\rm m} = \frac{1}{8}$, or in standard notation

$$\beta = \frac{1}{8}\,. \qquad \text{(Ising model)} \tag{11.21}$$

For bond percolation, with $g = \frac{2}{3}$, we find $\Delta_{\rm m} = \frac{5}{48}$. The fractal dimension of a percolation cluster is thus

$$d_{\rm c} = 2 - \Delta_m = \frac{91}{48}\,. \tag{11.22}$$

The location in the Kac table (10.81) of the magnetic exponent (11.20) can be found using (11.13):

$$\Delta_{\rm m} = 2h_{1/2,0}\,. \tag{11.23}$$

Note that this differs from the lowest possible watermelon excitation $\Delta_2 = 2h_{0,1}$. Indeed, the two-leg excitation corresponds to a cluster that propagates along the length direction of the cylinder *without* wrapping around the transverse periodic direction. The dominant configurations participating in the magnetic correlation function have *no* propagating legs, since the cluster containing $\mathbf{x}_1$ and $\mathbf{x}_2$ will typically wrap around the cylinder.