Chapter 1

Dimer coverings

The dimer problem occurs experimentally when a diatomic gas is adsorbed onto a crystalline substrate. Given some lattice, we ask for the number of ways that its vertices can be completely covered by non-overlapping *dimers* that each occupy two neighbouring vertices. More generally, we can assign a fugacity to each type of dimers, for instance depending on its orientation on the lattice (or maybe even on its exact position), and the goal is to compute the corresponding partition function.

Dimer are also known in mathematics as dominos. When the lattice is bipartite,[1] dimer coverings are known in graph theory as perfect matchings.

The dimer problem was solved in 1961, almost simultaneously and independently by Kasteleyn [Ka61], Fisher [Fi61] and Temperley [TF61]. In all cases it was crucial to realise that the partition function can be conveniently expressed as a Pfaffian (whose square is an ordinary determinant). The crux of the problem was to get all configurations counted with the same sign.

The detailed solution for the square lattice was presented in [Ka61, Fi61]. The approach of [Ka61] to deal with the sign problem appears to generalise most easily to other lattices, and we shall follow this method with subsequent simplifications [Ka63]. Several fine points were further discussed in chapter 4 of the book by McCoy and Wu [MW73]. Certain correlation functions were obtained by Fisher and Stephenson [FS63] as Toeplitz determinants.

Results on other lattices, scattered throughout the literature, have been reviewed in [Wu06]. Recent generalisations of the problem include quantum dimer models [RK88] with applications to superconductivity, aligning interactions [Al05], and the inclusion of a single monomer on the boundary. The solution with an single monomer in an arbitrary position is a difficult open problem. Including several monomers appears to be intractable, although the essential physics in known in the continuum limit [Al06].

1.1 Solution by determinants

1.1.1 The partition function as a determinant

Let us first define the problem. We consider a graph $G = (V, E)$ where V is the set of vertices and E is the set of edges. A dimer configuration $\mathcal{C}$ on G is a subset of edges which covers all the vertices, and where no overlap occurs between two edges. An example is shown in Fig. 1.1. Each edge e of the graph carries a positive weight $\pi(e)$, and to each

[1] By definition a bipartite lattice is one for which the set of vertices V can be written as a disjoint union $V = A \cup B$, so that any edge connects an A-vertex with a B-vertex.

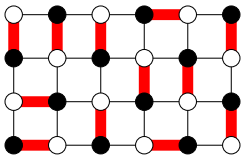

Figure 1.1: An example dimer covering of a portion of the square lattice.

configuration $\mathcal{C}$, we assign a Boltzmann weight

$$\pi(\mathcal{C}) := \prod_{e \in \mathcal{C}} \pi(e)\,.$$

The partition function is defined as

$$Z_G := \sum_{\mathcal{C}} \pi(\mathcal{C}) = \sum_{\mathcal{C}} \prod_{e \in \mathcal{C}} \pi(e)\,, \tag{1.1.1}$$

where the sum runs over all possible dimer configurations on G. In particular, if we set $\pi(e) = 1$ for every edge, Z_G is the number of dimer configurations on G.

We now restrict the discussion to the case when G is **bipartite**. This means that the vertices of G can be coloured in black and white, in such a way that every white vertex is only adjacent to black vertices, and *vice-versa*. Examples of bipartite graphs are the square lattice and the honeycomb lattice. The triangular lattice is *not* bipartite. We denote by $(w_1, \dots, w_N)$ the white vertices, and $(b_1, \dots, b_N)$ the black ones. Also, we choose an orientation of the edges of G, and we introduce the $N \times N$ weighted adjacency matrix K (called the **Kasteleyn matrix**), defined by:

$$K_{ij} = \begin{cases} +\pi(w_i, b_j) & \text{if } w_i \to b_j, \\ -\pi(w_i, b_j) & \text{if } w_i \leftarrow b_j, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1.2}$$

Consider the determinant of this matrix:

$$\det K = \sum_{\sigma \in \mathcal{S}_N} \operatorname{sgn}(\sigma) K_{1,\sigma(1)} \dots K_{N,\sigma(N)}\,,$$

where $\mathcal{S}_N$ is the group of permutations of N elements. The permutations σ which contribute to $\det K$ are those which satisfy the condition

$$\forall i \in \{1, \dots, N\}\,, \qquad w_i \text{ is adjacent to } b_{\sigma(i)}\,.$$

This is equivalent to saying that the set of edges

$$\mathcal{C}(\sigma) := \{(w_1, b_{\sigma(1)}), \dots, (w_N, b_{\sigma(N)})\}$$

is a dimer configuration! Also, the contribution of σ to $\det K$ equals the weight $\pi[\mathcal{C}(\sigma)]$, **up to a sign**. We will now show that, if the orientation of G is well chosen, every contribution to $\det K$ picks the same sign, and we have

$$\boxed{Z_G = |\det K|\,.} \tag{1.1.3}$$

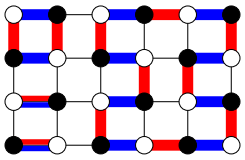

Figure 1.2: Superposition of two dimer configurations.

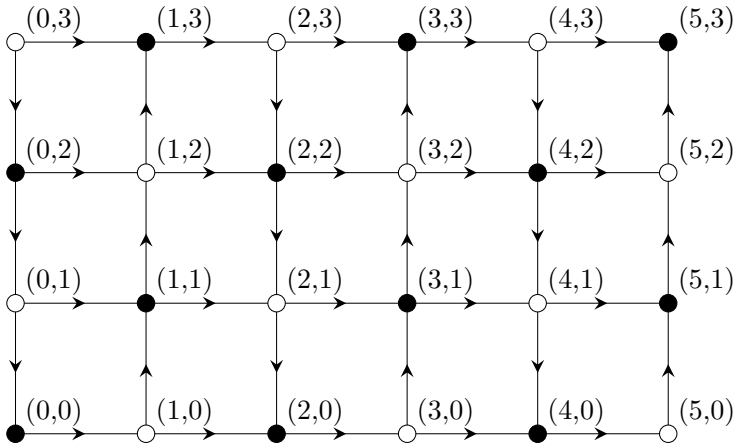

Figure 1.3: A Kasteleyn orientation of the square lattice. The coordinates of black and white vertices are indicated.

To show this, we take two arbitrary dimer configurations $\mathcal{C}$ and $\mathcal{C}'$, corresponding to the permutations σ and σ', and draw their superposition, as in Fig. 1.2. The resulting graph is made of doubly-covered edges, together with closed cycles of even length, and is called the transition graph. For simplicity, let us assume first that there is only one closed cycle of length 2ℓ. If we number the vertices around this cycle $(w_1, b_1, w_2, b_2, \ldots, w_\ell, b_\ell)$, it is easy to see that the permutations σ and σ' differ by a cyclic permutation:

$$\sigma' = \begin{pmatrix} 1 & 2 & 3 & \ldots & \ell \\ \ell & 1 & 2 & \ldots & \ell - 1 \end{pmatrix} \circ \sigma \,,$$

and hence $\operatorname{sgn} \sigma' = (-1)^{\ell+1} \operatorname{sgn} \sigma$. In order to compensate the sign $(-1)^{\ell+1}$ with those coming from (1.1.2), we impose the condition:

$$\textit{In any cycle of length } 2\ell \textit{ on } G, \begin{cases} \#\text{ edges oriented } b \to w \\ (\ell + 1) \end{cases} \textit{ have the same parity.}$$

An orientation satisfying this condition is called a **Kasteleyn orientation** of the graph G. Actually, it is sufficient to impose the above condition on the elementary cycles, *i.e.* cycles which enclose a single face of G. In the determinant of the corresponding matrix K, every contribution has the same sign, and (1.1.3) follows.

1.1.2 Explicit computation on the square lattice

We consider a portion of the square lattice G_{mn} of size $m \times n$. For simplicity, we restrict to even m. Once a Kasteleyn orientation is found, we can use (1.1.3) to compute the

partition function of dimers on the square lattice. We shall use the orientation shown in Fig. 1.3, and give a weight $\pi(e) = z_1$ to every horizontal edge e and $\pi(e) = z_2$ to every vertical edge e. The resulting adjacency matrix is

$$K_{(x,y),(x',y')} = z_1(\delta_{x+1,x'} - \delta_{x-1,x'})\delta_{yy'} + z_2\delta_{xx'}(-1)^x(\delta_{y+1,y'} - \delta_{y-1,y'})\,,$$

where (x, y) are the coordinates of a white vertex ($x + y$ is odd), and (x', y') are the coordinates of a black vertex ($x' + y'$ is even). Let us rewrite K as

$$K_{(x,y),(x',y')} = (-1)^x(z_1 Q_{xx'}\delta_{yy'} + z_2\delta_{xx'}R_{yy'})\,, \tag{1.1.4}$$

with

$$Q_{xx'} := (-1)^x(\delta_{x+1,x'} - \delta_{x-1,x'})\,, \qquad R_{yy'} := (\delta_{y+1,y'} - \delta_{y-1,y'})\,. \tag{1.1.5}$$

Since it is more convenient to work with operators acting on the full space of vertices V, we introduce the $(mn) \times (mn)$ matrix $\widetilde{K}$ defined by (1.1.4–1.1.5) for any $0 \le x, x' < m$ and $0 \le y, y' < n$. We have then $|\det \widetilde{K}| = (\det K)^2$.

In the original paper of Kasteleyn, the dimer partition function on an arbitrary graph (not necessarily bipartite) is related to the *Pfaffian* of an antisymmetric matrix D (see Appendix). In the case of a bipartite graph, one has $D = \widetilde{K}$, and since $\mathrm{Pf}\ \widetilde{K} = \sqrt{\det \widetilde{K}}$, the two approaches are consistent!

We decompose the space of vertices as a tensor product $V = \mathbb{R}^m \otimes \mathbb{R}^n$, where the two factors represent the x and y coordinates. Using (1.1.4), we can write

$$\widetilde{K} = (S \otimes \mathbf{1}_n)(z_1\ Q \otimes \mathbf{1}_n + z_2\ \mathbf{1}_m \otimes R)\,,$$

where Q and R are defined by their matrix elements (1.1.5), and $S_{xx'} := (-1)^x$. Let us now compute the determinant of $\widetilde{K}$.

We first find the eigenvalues and eigenvectors of R. The matrix R is invariant under translation, up to boundary terms. Thus it can be diagonalised using "plane waves". We define the vector

$$|u_\alpha\rangle := \begin{pmatrix} \alpha \\ \alpha^2 \\ \vdots \\ \alpha^n \end{pmatrix}.$$

We then have

$$R\,|u_\alpha\rangle = (\alpha^{-1} - \alpha)\,|u_\alpha\rangle + \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ \alpha^{n+1} \end{pmatrix}.$$

Hence, the combination $|\widehat{u}_\alpha\rangle := |u_\alpha\rangle - |u_{-\alpha^{-1}}\rangle$ is an eigenvector of R, under the condition that $\alpha^{n+1} - (\alpha^{-1})^{n+1} = 0$. Writing $\alpha = -ie^{iq}$, this becomes $\sin(n+1)q = 0$, and the eigenvalues of R are

$$\mu_\ell = 2i\cos\frac{\pi\ell}{n+1}\,, \qquad \ell \in \{1, \dots, n\}\,.$$

Similarly, the eigenvectors of Q are of the form $|\widehat{v}_\beta\rangle := |v_\beta\rangle - |v_{-\beta^{-1}}\rangle$, where

$$|v_\beta\rangle = \begin{pmatrix} \beta \\ i\beta^2 \\ \beta^3 \\ \vdots \\ i\beta^m \end{pmatrix},$$

and the eigenvalues of Q are

$$\lambda_k = 2\cos\frac{\pi k}{m+1}, \qquad k \in \{1, \ldots, m\}.$$

The eigenvectors of $(z_1 Q \otimes \mathbf{1}_n + z_2 \mathbf{1}_m \otimes R)$ are given by $|\widehat{u}_\alpha\rangle \otimes |\widehat{v}_\beta\rangle$, and we obtain

$$|\det \widetilde{K}| = \prod_{k=1}^{m}\prod_{\ell=1}^{n} |z_1\lambda_k + z_2\mu_\ell| = \prod_{k=1}^{m/2}\prod_{\ell=1}^{n} |(z_1\lambda_k)^2 - (z_2\mu_\ell)^2|,$$

where we have used $\lambda_{m+1-k} = -\lambda_k$. The final step is to substitute the values of λ_k and μ_ℓ into this expression.

We obtain

$$\begin{aligned} Z_{mn}(z_1, z_2) &= \sqrt{|\det \widehat{K}|} \\ &= 2^{mn/2} \prod_{k=1}^{m/2}\prod_{\ell=1}^{n} \left[z_1^2 \cos^2\left(\frac{\pi k}{m+1}\right) + z_2^2 \cos^2\left(\frac{\pi \ell}{n+1}\right)\right]^{1/2}. \end{aligned} \tag{1.1.6}$$

In particular we get the number of ways to tile a chessboard by 32 dominos [TF61]:

$$Z_{8,8}(1,1) = 12\,988\,816 = 2^4 \times 17^2 \times 53^2. \tag{1.1.7}$$

1.1.3 Thermodynamical limit

The above combinatorial derivation has produced an expression (1.1.6) for the partition function which is explicit and exact in finite size. This is a rather *unusual* situation in statistical physics, where more often that not one can obtain exact results only in the thermodynamical limit $m, n \to \infty$.

In that limit one is typically interested in the free energy per site

$$f(z_1, z_2) = -\lim_{m,n\to\infty} \frac{1}{mn} \log Z_{mn}(z_1, z_2) \tag{1.1.8}$$

for which one expects a finite limit. The result (1.1.6) needs some manipulation in order to extract an analytical expression for f. Replacing first

$$\frac{1}{m}\sum_{k=1}^{m/2} \longrightarrow \frac{1}{\pi}\int_0^{\pi/2} \mathrm{d}\omega,$$

we obtain

$$f(z_1, z_2) = -\frac{1}{\pi^2} \int_0^{\pi/2} \mathrm{d}\omega \int_0^{\pi/2} \mathrm{d}\omega' \log \left[4(z_1^2 \cos^2 \omega + z_2^2 \cos^2 \omega') \right] . \tag{1.1.9}$$

This expression can be simplified by using a few tricks of analysis.

To this end we first perform explicitly the integral over ω. Defining the ratio $\zeta = \frac{z_2}{z_1}$ we have

$$\begin{aligned} & \frac{1}{\pi} \int_0^{\pi/2} \mathrm{d}\omega \, \log \left[4(z_1^2 \cos^2 \omega + z_2^2 \cos^2 \omega') \right] \\ = \; & \frac{1}{2} \log (2 z_2 \cos \omega')^2 + \frac{1}{\pi} \int_0^{\pi/2} \mathrm{d}\omega \, \log \left(1 + \frac{\cos^2 \omega}{\zeta^2 \cos^2 \omega'} \right) \\ = \; & \log z_1 + \log (2\zeta \cos \omega') + \log \left(\frac{1 + \sqrt{1 + \frac{1}{\zeta^2 \cos^2 \omega'}}}{2} \right) \\ = \; & \log z_1 + \log \left(\zeta \cos \omega' + \sqrt{1 + \zeta^2 \cos^2 \omega'} \right) . \end{aligned}$$

Renaming $\omega' \mapsto \omega$, (1.1.9) then becomes

$$f(z_1, z_2) = -\frac{1}{2} \log z_1 - \frac{1}{\pi} \int_0^{\pi/2} \mathrm{d}\omega \, g(\zeta \cos \omega) \tag{1.1.10}$$

where we have defined

$$g(x) = \log (x + \sqrt{1 + x^2}) .$$

We can suppose that $|z_2| \leq |z_1|$, since if this were not the case we could simply exchange z_1 and z_2. Setting $x = \zeta \cos \omega$ we have then $|x| < 1$ (except for $\omega = 0$, which does not matter under the integral). Therefore the integrand $g(x)$ can be replaced by its expansion as an entire series and we can integrate term by term. After a little work one finds

$$g(x) = \sum_{j=0}^{\infty} \left(\begin{array}{c} 2j \\ j \end{array} \right) \frac{(-1)^j}{(2j+1)\, 2^{2j}} \, x^{2j+1} .$$

Using now

$$\int_0^{\pi/2} \mathrm{d}\omega \, (\zeta \cos \omega)^{2j+1} = \zeta^{2j+1} \frac{\sqrt{\pi}\, \Gamma(j+1)}{2\Gamma(j+\frac{3}{2})} = \zeta^{2j+1} \frac{j!\, 2^j}{(2j+1)!!}$$

we arrive at

$$\begin{aligned} \int_0^{\pi/2} \mathrm{d}\omega \, g(\zeta \cos \omega) \; &= \; \sum_{j=0}^{\infty} \left(\begin{array}{c} 2j \\ j \end{array} \right) \frac{(-1)^j}{(2j+1) 2^{2j}} \, \frac{j!\, 2^j}{(2j+1)!!} \zeta^{2j+1} \\ &= \; \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^2} \, \zeta^{2j+1} . \end{aligned}$$

This expression is reminiscent of $\arctan \zeta$. Indeed we have

$$\arctan x = \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} \, x^{2j+1}$$

and so

$$\int_0^\zeta \mathrm{d}x\, \frac{\arctan x}{x} = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^2} \zeta^{2j+1} .$$

Putting the pieces together we obtain

$$f(z_1, z_2) = -\frac{1}{2} \log z_1 - \frac{1}{\pi} \int_0^\zeta \mathrm{d}x\, \frac{\arctan x}{x} . \tag{1.1.11}$$

To simplify further we can use that

$$\arctan x = \frac{1}{2i} \log \left(\frac{1+ix}{1-ix} \right) ,$$

as is easily seen by taking tan on both sides. The integral in (1.1.11) can therefore be expressed in terms of the Euler dilogarithm[2]

$$\mathrm{Li}_2(u) = -\int_0^u \mathrm{d}x\, \frac{\log(1-x)}{x} = \sum_{k=1}^{\infty} \frac{u^k}{k^2}$$

as follows

$$\int_0^\zeta \mathrm{d}x\, \frac{\arctan x}{x} = \frac{1}{2i} \left(\mathrm{Li}_2(i\zeta) - \mathrm{Li}_2(-i\zeta) \right) \equiv \mathrm{Ti}_2(\zeta) , \tag{1.1.12}$$

where we have finally introduced the inverse tangent integral $\mathrm{Ti}_2(\zeta)$.

The final result thus reads

$$\boxed{ f(z_1, z_2) = -\frac{1}{2} \log z_1 - \frac{1}{\pi} \mathrm{Ti}_2(\zeta) , \qquad \zeta = \frac{z_2}{z_1} . } \tag{1.1.13}$$

For the combinatorial counting problem $\zeta = 1$. Recalling the series expansion (1.1.12) of $\mathrm{Li}_2(u)$ we obtain

$$f(1,1) = -\frac{1}{\pi} \mathrm{Ti}_2(1) = -\frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \equiv -\frac{G}{\pi} , \tag{1.1.14}$$

where we have introduced the Catalan constant G. The number of arrangements per dimer (sometimes known in the theoretical chemistry literature as the *molecular freedom*) is then

$$\exp[-2f(1,1)] = \exp(2G/\pi) \simeq 1.791\,622\cdots \tag{1.1.15}$$

This is of course smaller than 2, since the effective number of arrangements of a given dimer is constrained by the other dimers.

TO DO: calculer un exposant?

1.1.4 Toroidal boundary conditions

Consider now a square lattice $\widetilde{G}_{mn}$ embedded in the torus. The results of Sec. 1.1 do not apply directly, since the lattice is no longer planar. In particular, one cannot find a

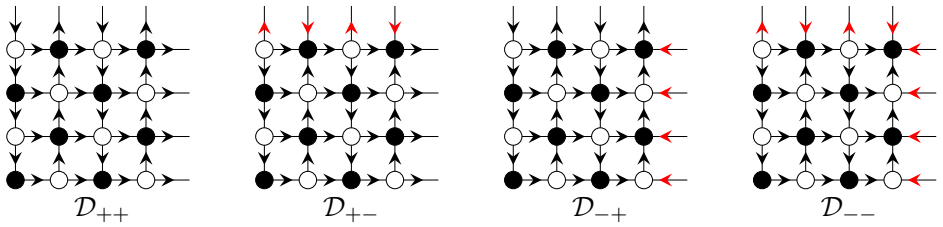

Figure 1.4: Four different orientations of the square lattice with toroidal boundary conditions.

Kasteleyn orientation for this lattice. It is however still possible to express the partition function (1.1.1) as a sum of *four* different determinants.

Consider the four different orientations of $\widetilde{G}_{mn}$ shown in Fig. 1.4. The first orientation $\mathcal{D}_{++}$ is just the Kasteleyn orientation of G_{mn} endowed with periodic boundary conditions (The other three orientations are obtained by applying antiperiodic boundary conditions in one or both directions).

Pick an arbitrary dimer configuration $\mathcal{C}_0$, say the blue dimers in Fig. 1.2, and construct the transition graph between $\mathcal{C}_0$ and the other dimer configurations. Orientation $\mathcal{D}_{++}$ gives the correct parity to all transition cycles that are homotopic to a point. However, an incorrect sign is given to some of the dimer configurations for which the transition cycles have non-trivial homotopy. Note that since different transition cycles cannot intersect, all non-trivial transition cycles have in fact the same homotopy, *i.e.*, they all wrap the horizontal and vertical directions the same number of times.

Let us divide the possible dimer configurations on $\widetilde{G}_{mn}$ into four disjoint classes. Class (e,e) comprises dimer configuration for which the set of transition cycles wrap both the horizontal and vertical directions an even number of times. Similarly we define classes (o,e), (e,o) and (o,o), where e = even and o = odd, and the first (resp. second) symbol refers to the horizontal (resp. vertical) direction. We denote by $Z_{\rm ee}, Z_{\rm oe}, Z_{\rm eo}, Z_{\rm oo}$ the corresponding contributions to the partition function. By examining one example per case and arguing that local deformations of the transition cycles do not alter the results, one establishes that the signs with which the four different classes of dimer configurations are counted in the four different determinants are as follows:

$$\begin{aligned}
\det \mathcal{D}_{++} &= +Z_{\rm ee} - Z_{\rm oe} - Z_{\rm eo} - Z_{\rm oo}\,,\\
\det \mathcal{D}_{+-} &= +Z_{\rm ee} - Z_{\rm oe} + Z_{\rm eo} + Z_{\rm oo}\,,\\
\det \mathcal{D}_{-+} &= +Z_{\rm ee} + Z_{\rm oe} - Z_{\rm eo} + Z_{\rm oo}\,,\\
\det \mathcal{D}_{--} &= +Z_{\rm ee} + Z_{\rm oe} + Z_{\rm eo} - Z_{\rm oo}\,.
\end{aligned}$$

It follows that

$$\boxed{Z_{mn}(z_1,z_2) = \frac{1}{2}\left(-\det \mathcal{D}_{++} + \det \mathcal{D}_{+-} + \det \mathcal{D}_{-+} + \det \mathcal{D}_{--}\right)\,.} \tag{1.1.16}$$

The four determinants in (1.1.16) can be computed explicitly as before, except the set of eigenvalues of U and V are slightly different: they are typically of the form $\lambda_k = 2\cos(2\pi k/m)$. Explicit results are given in [Ka61].

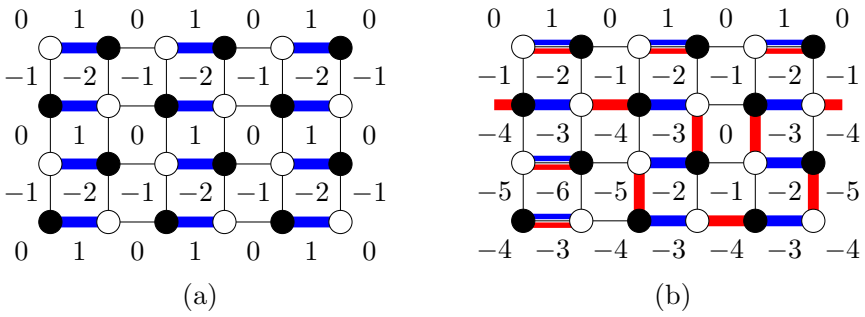

Figure 1.5: (a) The reference dimer configuration $\mathcal{C}_0$ and the corresponding height $4h_0(\vec{r})$; (b) An example dimer configuration $\mathcal{C}$ (in red), superposed with $\mathcal{C}_0$ (in blue), and the values of $4h(\vec{r})$.

1.2 Effective Gaussian scaling theory

1.2.1 Lattice height function

To any dimer covering we can associate a height function $h(\vec{r})$ on the dual lattice, which is defined as follows. First note that the square lattice is bipartite. When encircling a black vertex in the positive (counterclockwise) direction, the height h changes by $+1/4$ upon crossing an empty edge and by $-3/4$ upon crossing an edge covered by a dimer. The same rule holds when encircling a white vertex in the negative direction. By fixing the height at the origin, e.g., $h(\vec{0}) = 0$, these rules define the entire height function $h(\vec{r})$ uniquely. An example is shown in Fig. 1.5.

Let us consider the model on a rectangular lattice of $m \times n$ sites, with m and n even and periodic boundary conditions. In the reference configuration $\mathcal{C}_0$, the height function alternates between two values at each lattice step in both directions. Hence, since m and n are even, the height function satisfies the periodic BCs. Consider now a dimer configuration $\mathcal{C}$, such that the transition graph between $\mathcal{C}_0$ and $\mathcal{C}$ has one cycle winding in the horizontal direction. Then the height function for $\mathcal{C}$ acquires a defect $\delta h = \pm 1$ across this cycle, where the sign depends on the exact position of the cycle. More generally, sereval cycles with non-trivial topology are allowed, producing integer defects δh and $\delta' h$ in the horizontal and vertical directions.

1.2.2 Effective action

The dimer model is a purely entropic problem, and so one shall construct the effective scaling theory from entropy arguments. The height function h_0 for the reference dimer configuration $\mathcal{C}_0$ is almost flat, and it is the configuration where a maximum of elementary square transition cycles can be inserted. Let us now consider the coarse-grained version of $h(\vec{r})$ for generic configurations, obtained by averaging the discrete h over small domains. Microscopic height configurations obtained from a given $h(\vec{r})$ by adding a finite number of elementary cycles all contribute to the same coarse-grained configuration, and thus the flat configuration will be dominant, since it maximises the entropy. We expect the scaling theory to describe the small fluctuations, and it is natural to assume a Gaussian action. Moreover, as we have seen above, the height variable on the lattice can have defects of integer amplitude, so the coarse-grained variable h must be considered as periodic. These

arguments suggest the effective action:

$$\boxed{S_{\text{eff}} = \pi g \int d^2 r \ \left[(\partial_x h)^2 + (\partial_y h)^2\right] , \qquad h \equiv h + 1 \, .} \tag{1.2.1}$$

Here g is the coupling constant, which controls the stiffness of the interface model. It is *a priori* unknown.

1.2.3 Local operators

Let us describe the two types of elementary local operators in the scaling theory (1.2.1). First, the correct way of measuring slow variations of the height is to insert "spin wave" or "electric" operators of the form

$$V_\alpha(\vec{r}) = : \exp[i\alpha h(\vec{r})] : \tag{1.2.2}$$

Because of the identification $h \equiv h + 1$, this operator is well-defined only if $\alpha = 2i\pi e$, where $e \in \mathbb{Z}$ is the electric charge.

Additionally, a "magnetic" or "vortex" operator $W_m(\vec{r})$ inserts a height defect $\delta h = m$ as one follows the function h along a small circle centered at $\vec{r}$. Again, the magnetic charge $m \in \mathbb{Z}$ is quantised due to the identification $h \equiv h + 1$.

More generally, a generic operator $\mathcal{O}_{em}(\vec{r})$ in the theory is a combination of a height defect $\delta h = m$ and a spin-wave factor $\exp(2i\pi e h)$, both located at $\vec{r}$.

1.2.4 Critical exponents

Since the action (1.2.1) is massless, we expect the two-point correlation functions to decay asymptotically – i.e., for distances $|\vec{r}_1 - \vec{r}_2|$ satisfying $1 \ll r \ll N$ on an $N \times N$ lattice, and with operator positions far from the boundaries – as

$$\langle \mathcal{O}_{em}(\vec{r}_1) \mathcal{O}_{-e,-m}(\vec{r}_2) \rangle \propto \frac{1}{|\vec{r}_1 - \vec{r}_2|^{2x_{em}}} \, , \tag{1.2.3}$$

where x_{em} is the scaling dimension of $\mathcal{O}_{em}$.

Purely electric operator V_α. The correlation function $\langle V_\alpha(\vec{r}_1) V_{-\alpha}(\vec{r}_2) \rangle$ measures slow variations of the height function, and thus it is not sensitive to the compactification condition, and can be computed like an ordinary Gaussian integral. After integration by parts, the action takes the form

$$S_{\text{eff}} = -\pi g \int d^2 r \ h \Delta h \, , \tag{1.2.4}$$

and the Green's function of the Laplacian in 2d is

$$G(\vec{r}) = \langle h(\vec{r}) h(0) \rangle = -\frac{1}{4\pi^2 g} \log |\vec{r}| \, . \tag{1.2.5}$$

Then, since the theory is Gaussian:

$$\langle V_\alpha(\vec{r}_1) V_{-\alpha}(\vec{r}_2) \rangle = \left\langle e^{i\alpha(h(\vec{r}_1) - h(\vec{r}_2))} \right\rangle = e^{\alpha^2 \langle h(\vec{r}_1) h(\vec{r}_2) \rangle} = |\vec{r}_1 - \vec{r}_2|^{-e^2/g} \, , \tag{1.2.6}$$

where we have set $\alpha = 2\pi e$. Hence, the scaling dimension of $V_{\alpha = 2\pi e}$ is $x_e = e^2/(2g)$.

Purely magnetic operator W_m. For this, we consider a "vortex" configuration of charge m around the origin, corresponding to the classical solution $h(r,\theta) = m\theta/(2\pi)$, in polar coordinates. Neglecting the fluctuations around this solution, the contribution of the annulus $\epsilon < r < R$ to the action is

$$\pi g \int_\epsilon^R r dr \int_0^{2\pi} d\theta \left[(\partial_r h)^2 + \left(\frac{1}{r} \partial_\theta h \right)^2 \right] = \frac{g m^2}{2} \log \frac{R}{\epsilon} , \tag{1.2.7}$$

and hence the corresponding free energy is $(R/\epsilon)^{-gm^2/2}$. This shows that the scaling dimension of W_m is $x_m = gm^2/2$.

General case. In general, the scaling dimension of an operator $\mathcal{O}_{em}$ is given by

$$\boxed{x_{em} = \frac{e^2}{2g} + \frac{gm^2}{2} .} \tag{1.2.8}$$

For example, two monomer defects on opposite sublattices correspond to $m = \pm 1$. It is known from exact results [FS63] that $x_{01} = \frac{1}{4}$, and this fixes $g = \frac{1}{2}$. The exponents for correlation function of all possible charges then follow from (1.2.8). In particular, the dimer-dimer correlation function—i.e., the probability that two widely separated dimers have the same orientation, after subtraction of the trivial $r \to \infty$ limit of $\frac{1}{2}$—then decays with exponent $x_{10} = 1$, and this is confirmed by the exact solution [FS63].

Let us also note that in CFT there is a link between (1.1.16) and modular invariant partition functions for the free boson on the torus.

1.3 Appendix: Pfaffian formulation

We consider the dimer problem on an $m \times n$ square lattice Q_{mn}. Obviously a dimer covering exists only if mn is even, so we shall suppose m even. An example on Q_{64} is shown in Fig. 1.1.

The number of dimer coverings will of course depend on the boundary conditions. In this section we concentrate on free boundary conditions (i.e., free both along the horizontal and vertical directions). In this case the result can be written as the Pfaffian of an appropriate matrix. An expression in terms of a single Pfaffian also exists for cylindrical boundary conditions (i.e., free along one lattice direction and periodic along the other). In Sec. 1.1.4 we shall show that toroidal boundary conditions (i.e., periodic in both directions) leads to a linear combination of *four* different Pfaffians. More generally, the number of Pfaffians needed will be 4^g for a lattice embedded into a surface of genus g.

Let the fugacity of horizontal and vertical dimers be respectively z_1 and z_2. The weight of the configuration shown in Fig. 1.1 is then $(z_1)^4 (z_2)^8$. Let $g(N_1, N_2)$ be the number of dimer coverings of Q_{mn} using N_1 horizontal and N_2 vertical dimers. We have necessarily $g(N_1, N_2) = 0$ unless $2(N_1 + N_2) = mn$. The goal is then to compute the partition function

$$Z_{mn}(z_1, z_2) = \sum_{N_1, N_2} g(N_1, N_2) z_1^{N_1} z_2^{N_2} . \tag{1.3.1}$$

The Pfaffian of an $2N \times 2N$ skew-symmetric matrix A with elements $a(k,k') = -a(k',k)$

is defined by

$$\begin{aligned} \text{Pf } A &= \sum_P{}' \varepsilon(P) a(k_1,k_2)a(k_3,k_4)\cdots a(k_{2N-1,2N}) \\ &= \frac{1}{N!\,2^N} \sum_P \varepsilon(P) a(k_1,k_2)a(k_3,k_4)\cdots a(k_{2N-1,2N})\,. \end{aligned} \tag{1.3.2}$$

Here $\sum_P$ runs over all permutations $P : (1,2,\ldots,2N) \to (k_1,k_2,\ldots,k_{2N})$, whereas $\sum'_P$ is constrained to those permutations satisfying the constraint

$$\begin{aligned} &k_1 < k_2\,, \quad k_3 < k_4\,, \quad \cdots\,, \quad k_{2N-1} < k_{2N} \\ &k_1 < k_3 < k_5 < \cdots < k_{2N-1} \end{aligned} \tag{1.3.3}$$

and $\varepsilon(P) = \pm 1$ is the sign of P. It is easy to see that there are $\frac{(2N)!}{N!\,2^N} = (2N-1)!!$ terms in the sum $\sum'_P$.

The constraint (1.3.3) is very natural from the point of view of dimers. Suppose we assing to the vertices of the lattice (i,j), where $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$, some numbering, for instance

$$(i,j) \mapsto k = (j-1)m + i\,. \tag{1.3.4}$$

Let a configuration of dimers be denoted

$$C = [k_1,k_2]\,[k_3,k_4]\,\cdots\,[k_{2N-1},k_{2N}]\,, \tag{1.3.5}$$

where $[k,k']$ means that there is a dimer covering vertices k and k'. Then the constraint (1.3.3) expresses simply that C is considered modulo exchanges of dimers, and modulo exchanges of the two end points within each dimer.

This suggests an obvious strategy for computing $Z_{mn}(z_1,z_2)$ as a Pfaffian. Indeed we will have

$$|\text{Pf } D| = Z_{mn}(z_1,z_2) \tag{1.3.6}$$

provided that we can define a $2N \times 2N$ skew-symmetric matrix D, with $2N = mn$, that fulfills three requirements:

1. There should be a bijection between the non-vanishing contributions to Pf D and the dimer configurations on Q_{mn}.

2. The weight of each non-vanishing contribution to Pf D should be equal, *up to a sign*, to the corresponding statistical weight in $Z_{mn}(z_1,z_2)$.

3. All contributions to Pf D should have *the same sign*.

Requirements 1–2 are easy to fulfill. To satisfy requirement 1, we simply set $d(k,k') = 0$ if the vertices k and k' are not adjacent in Q_{mn}. To satisfy requirement 2, we set $d(k,k') = -d(k',k) = \pm z_{kk'}$ if k and k' are adjacent, where $z_{kk'}$ is the desired fugacity of a dimer that covers k and k'. Note that the liberty in choosing $z_{kk'}$ makes it possible to tackle the most general situation of edge-dependent fugacities; in the sequel we shall however only need z_1 and z_2 as in (1.3.1).

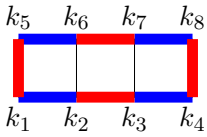

Figure 1.6: Example of a transition cycle on Q_{42} between the standard configuration C (in blue) and another configuration C' (in red). The vertices are labelled according to the canonical order (1.3.4).

Solving the sign problem. The tricky part is requirement 3: how to choose the correct sign of $d(k,k')$ when k and k' are adjacent? It is convenient to represent the signs of the matrix elements $d(k,k')$ by an orientation of the edges of Q_{mn}. If the edge (kk') is oriented from k to k' (resp. from k' to k), we take $d(k,k') = +z_{kk'}$ (resp. $d(k,k') = -z_{kk'}$).

The question is then whether a lattice orientation exists that will fulfill requirement 3. The answer is positive, not only for Q_{mn} but in fact for any planar graph. The corresponding orientation is known as a *Kasteleyn orientation*. The goal of this section is to characterise precisely this orientation.

Consider superposing two different dimer configurations C and C' of Q_{mn}. The resulting *transition graph* is made up of doubly occupied edges, where the dimers of C and C' coincide, and of *transition cycles*, which are cycles of even length along which the dimers from C and C' alternate. This is shown in Fig. 1.2. Since the length of a transition cycle is even, the number of clockwise and anticlockwise oriented edges in the cycle is either both even, or both odd: we call this the *orientation parity*.

To turn configuration C into C' one needs to shift the dimers one unit along each transition cycle. The factor $\varepsilon(P)$ appearing in (1.3.2) can then be shown (see below) to produce a minus sign for each transition cycle. This sign must be cancelled by another one coming from the signs of the entries $d(k,k') = \pm z_{kk'}$. The terms representing C and C' will therefore have equal signs if *the orientation parity of all transition cycles is odd*.[3] If we can find an orientation of Q_{mn} satisfying this requirement, the sign problem is solved.

For the sake of definiteness, let us consider the case where C is the standard configuration shown in blue in Fig. 1.2, and C' is another arbitrary dimer configuration. We focus on the contribution to $\varepsilon(P)$ of a single transition cycle in $C \cup C'$.

Orient the transition cycle in the counterclockwise direction. It will then pass through the edges of C in any fixed column exactly as many times in the right direction (i.e., in the direction of increasing lablling k) as in the left direction (i.e., in the direction of decreasing labelling k), since otherwise the cycle would not be a closed polygon. So *a fortiori* this is true for the passages through any edge of C. Let r be the number of right (and hence left) passages.

Let us now describe a 5-stage process that permutes the C-term into the C'-term. To follow the argument, it is useful to consider in parallel an example with $r = 2$ on Q_{42}, i.e., with a single transition cycle of length $4r = 8$ (see Fig. 1.6). The initial configuration C is then

$$[k_1, k_2]\,[k_3, k_4]\,[k_5, k_6]\,[k_7, k_8]\,. \tag{1.3.7}$$

[3] Indeed, if the product of all the signs around the cycle is -1, then the products of the subsets of signs corresponding to C and to C' must be opposite.

The stages are as follows:

1. Reverse the r pairs of points within each doublet that correspond to a left passage, so that the order within each pair now corresponds to the cyclical rather than the canonical order (1.3.4):

$$[k_1, k_2]\,[k_3, k_4]\,[k_6, k_5]\,[k_8, k_7]\,. \tag{1.3.8}$$

   This produces a factor $(-1)^r$.

2. Permute the doublets as required to produce the perfect cyclic order:

$$[k_1, k_2]\,[k_3, k_4]\,[k_8, k_7]\,[k_6, k_5]\,. \tag{1.3.9}$$

   Since only doublets are permuted, this results in a factor $+1$.

3. Permute all $4r$ points cyclically:

$$[k_2, k_3]\,[k_4, k_8]\,[k_7, k_6]\,[k_5, k_1]\,. \tag{1.3.10}$$

   This produces a factor $(-1)^{4r-1}$. We now have the C'-term as desired, but the rules (1.3.3) are violated.

4. Permute the doublets so as to respect the second part of rule (1.3.3):

$$[k_5, k_1]\,[k_2, k_3]\,[k_4, k_8]\,[k_7, k_6]\,. \tag{1.3.11}$$

   This is the "opposite of stage 2" and gives a factor $+1$.

5. Reverse r pairs of points within each doublet so as to respect the first part of rule (1.3.2):

$$[k_1, k_5]\,[k_2, k_3]\,[k_4, k_8]\,[k_6, k_7]\,. \tag{1.3.12}$$

   This is the "opposite of stage 1" and gives a factor $(-1)^r$.

The total sign change is then

$$(-1)^r \times (-1)^{4r-1} \times (-1)^r = -1 \tag{1.3.13}$$

as claimed.

This correct choice of orientation parity can indeed be made for any planar graph. This relies on a number of properties that can rather easily be proved by induction in the size of the graph. Let us call a cycle that surrounds a single face of the lattice a *mesh cycle*. The relevant properties are then:

1. A planar graph can be oriented such that the orientation parity off all even mesh cycles is odd.

2. For a planar graph with such an orientation, the orientation parity of any even cycle whose interior contains an even (resp. odd) number of vertices is odd (resp. even).

3. In a planar graph the interior of any transition cycle contains an even number of

vertices.

Leaving this generality and returning to the square lattice Q_{mn}, a possible Kasteleyn orientation is shown in Fig. 1.3.

Let us finally remark, that it is relatively easy to find a Kasteleyn orientation for any regular (Archimedian) lattice. However, despite of the above existence result, there does not seem to be a simple constructive approach for an arbitrary planar graph.

Evaluation of the Pfaffian. We have now established (1.3.6) when the matrix D is chosen according to requirements 1–2 and the Kasteleyn orientation of Fig. 1.3. This reads explicitly

$$d(i,j;i',j') = z_1 \left(\delta_{i+1,i'} - \delta_{i-1,i'}\right) \delta_{j,j'} + (-1)^i z_2 \left(\delta_{j+1,j'} - \delta_{j-1,j'}\right) \delta_{i,i'} \,, \tag{1.3.14}$$

where the subtractions guarantee the proper antisymmetrisation. All of this would be of little avail if the Pfaffian were difficult to compute. Fortunately its square is just a standard determinant. Thus

$$[Z_{mn}(z_1,z_2)]^2 = [\mathrm{Pf}\ D]^2 = \det D \,. \tag{1.3.15}$$

Proving this relation is a little lengthy, and we only give a short outline (full details are provided in [MW73]). Introducing the cofactors D_{jk}, one first applies the Jacobi theorem $D_{jj}D_{kk} - D_{jk}D_{kj} = D_{jk,jk} \det D$ to the skew-symmetric matrix D. An induction argument then shows that $(\det D)^{1/2}$ is a rational function—and actually even a polynomial—of the matrix elements. Exploiting this finally leads to the desired relation with Pf D.

Let us illustrate the main points on a trivial example on Q_{22}. We first choose to orient the edges anticlockwise ($1 \to 2 \to 4 \to 3 \to 1$). Note that this is *not* a Kasteleyn orientation. We choose the most general position-dependent edge weights:

$$\det \begin{pmatrix} 0 & z_{12} & -z_{13} & 0 \\ -z_{12} & 0 & 0 & z_{24} \\ z_{13} & 0 & 0 & -z_{34} \\ 0 & -z_{24} & z_{34} & 0 \end{pmatrix} = (z_{13}z_{24} - z_{12}z_{34})^2 \,. \tag{1.3.16}$$

Changing the orientation of any one edge turns this into a Kasteleyn orientation and makes the two terms have the same sign.

The goal is therefore to compute $\det D$. If D were a cyclic matrix (i.e., with entries that depended on the indices i and j in a periodic fashion) this could be rather easily accomplished by bringing it into diagonal form via a Fourier transformation (see Sec. 1.1.4 for such a computation). In the present case there exists a slightly more complicated transformation that will turn D into a direct sum of 2×2 matrices.

Let us write D as a direct product of $m \times m$ and $n \times n$ matrices that describe the dependence of the weight on the horizontal and vertical coordinates respectively:

$$D = z_1(Q_m \otimes I_n) + z_2(F_m \otimes Q_n) \tag{1.3.17}$$

Here I_n is the $n \times n$ unit matrix, whereas

$$Q_m(i,i') = \delta_{i+1,i'} - \delta_{i-1,i'} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \ddots & 0 & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & -1 & 0 \end{bmatrix},$$

$$F_m(i,i') = (-1)^i \delta_{i,i'} = \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \ddots & 0 & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{1.3.18}$$

Note that the matrix Q is only almost cyclic, since the elements in its upper-right and lower-left corners are zero.

The transformation that we need is

$$\begin{aligned} \widetilde{D} &= U^{-1} D U\,, \\ U &= U_m \otimes U_n\,, \end{aligned} \tag{1.3.19}$$

where

$$U_n(l,l') = \sqrt{\frac{2}{n+1}}\, i^l \sin\left(\frac{ll'\pi}{n+1}\right). \tag{1.3.20}$$

Let us use this transformation to find an explicit formula for $Z_{mn}(z_1, z_2)$. First we note the following orthogonality identity:

$$\frac{2}{n+1} \sum_{l''=1}^{n} \sin\left(\frac{ll''\pi}{n+1}\right) \sin\left(\frac{l''l'\pi}{n+1}\right) = \delta_{l,l'}\,, \tag{1.3.21}$$

which can be easily proved by writing out the sines in terms of complex exponentials, multiplying out, and summing up the resulting geometrical series. This implies that the corresponding inverse matrix is

$$U_n^{-1}(l,l') = \sqrt{\frac{2}{n+1}}\, (-i)^l \sin\left(\frac{ll'\pi}{n+1}\right). \tag{1.3.22}$$

Using this we first diagonalise the matrix Q. We have

$$(QU)(l,l') = \sum_{l''=1}^{n} Q(l,l'')U(l'',l') = U(l+1,l') - U(l-1,l')$$

and further

$$\begin{aligned} (U^{-1}QU)(l,l') &= \sum_{l''=1}^{n} U^{-1}(l,l'')(QU)(l'',l') \\ &= \sum_{l''=1}^{n} \left\{ U^{-1}(l,l'')U(l''+1,l') - U^{-1}(l,l'')U(l''-1,l') \right\}. \end{aligned}$$

Inserting (1.3.20) and (1.3.22) this becomes

$$\begin{aligned}\ldots &= \frac{2i}{n+1}\sum_{l''=1}^{n}\sin\left(\frac{ll''\pi}{n+1}\right)\left\{\sin\left(\frac{l'(l''+1)\pi}{n+1}\right)+\sin\left(\frac{l'(l''-1)\pi}{n+1}\right)\right\}\\ &= \frac{4i}{n+1}\sum_{l''=1}^{n}\sin\left(\frac{ll''\pi}{n+1}\right)\cos\left(\frac{l'\pi}{n+1}\right)\sin\left(\frac{l'l''\pi}{n+1}\right)\\ &= 2i\cos\left(\frac{l\pi}{n+1}\right)\delta_{l,l'}\,, \end{aligned} \tag{1.3.23}$$

where we have first used an addition formula and then the orthogonality relation (1.3.21).

Another identity that can be proved in the same way as (1.3.21) is the following:

$$\frac{2}{n+1}\sum_{l''=1}^{n}(-1)^{l''}\sin\left(\frac{ll''\pi}{n+1}\right)\sin\left(\frac{l''l'\pi}{n+1}\right)=\delta_{l+l',n+1}\,, \tag{1.3.24}$$

where the right-hand side is the "mirrored" identity matrix. This can be used to diagonalise the matrix F. We find:

$$(U^{-1}FU)(l,l')=\delta_{l+l',n+1}\,. \tag{1.3.25}$$

The "diagonalised" D-matrix now reads[4], using (1.3.23) and (1.3.25),

$$\begin{aligned}\widetilde{D}(k,l;k',l') &= 2iz_1\delta_{k,k'}\delta_{l,l'}\cos\left(\frac{k\pi}{m+1}\right)\\ &+ 2iz_2\delta_{k+k',m+1}\delta_{l,l'}\cos\left(\frac{l\pi}{n+1}\right)\,. \end{aligned} \tag{1.3.26}$$

This is indeed diagonal in l-space, but not quite in k-space. Rather we have a matrix with the shape

$$\begin{pmatrix} w & & & & w' \\ & w & & w' & \\ & & \ddots & & \\ & w' & & w & \\ w' & & & & w \end{pmatrix}.$$

By changing the labelling $1,2,3,4,5,6,\ldots,m$ of both rows and columns into $1,m,2,m-1,3,m-2,\ldots,m/2,m/2+1$ (recall that m is even) this is turned into the block-diagonal matrix

$$\begin{pmatrix} w & w' & & & \\ w' & \tilde{w} & & & \\ & & \ddots & & \\ & & & w & w' \\ & & & w' & \tilde{w} \end{pmatrix}.$$

Note that in this process some of the entries change sign ($\tilde{w}=-w$), since when $k\mapsto k'\equiv m+1-k$ we get

$$\cos\left(\frac{k'\pi}{m+1}\right)=-\cos\left(\frac{k\pi}{m+1}\right)\,.$$

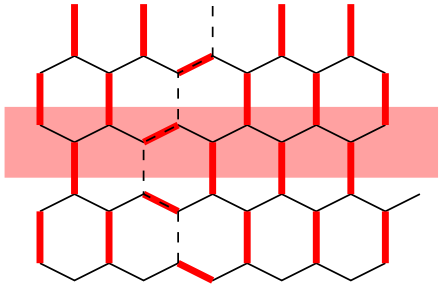

Figure 1.7: Dimers on the honeycomb lattice. The shaded region corresponds to the action of the row-to-row transfer matrix T. The dotted line represents the trajectory of a particle.

Therefore we obtain the result

$$\det D = \det \widetilde{D} = \prod_{k=1}^{m/2} \prod_{l=1}^{n} \begin{vmatrix} 2iz_1 \cos\left(\frac{k\pi}{m+1}\right) & 2iz_2 \cos\left(\frac{l\pi}{n+1}\right) \\ 2iz_2 \cos\left(\frac{l\pi}{n+1}\right) & -2iz_1 \cos\left(\frac{k\pi}{m+1}\right) \end{vmatrix} . \tag{1.3.27}$$

Using finally (1.3.15) we arrive at

$$Z_{mn}(z_1, z_2) = 2^{\frac{mn}{2}} \prod_{k=1}^{m/2} \prod_{l=1}^{n} \sqrt{z_1^2 \cos^2\left(\frac{k\pi}{m+1}\right) + z_2^2 \cos^2\left(\frac{l\pi}{n+1}\right)} . \tag{1.3.28}$$

In particular we can find the number of ways to tile a chessboard by 32 dominos [TF61]:

$$12\,988\,816 = 2^4 \times 17^2 \times 53^2 . \tag{1.3.29}$$

1.4 Appendix: Transfer matrix for the honeycomb lattice

In this section, we consider the dimer problem on the honeycomb lattice, with edge weights z_1, z_2, z_3 on the three different types of edges, as shown in Fig. 1.7. Moreover, we assume periodic boundary conditions in the horizontal direction, and we denote by L the number of vertical edges (type z_3) in any row.

The state of a row of vertical edges is described by the sequence $\alpha = (\alpha_1, \ldots, \alpha_L)$, where $\alpha_j = 1$ (resp. $\alpha_j = 0$) if the edge j is occupied by a dimer (resp. empty). Between two rows of type α, the intermediary state is described by $\beta = (\beta_1, \ldots, \beta_{2L})$, where β_j is defined similarly in terms of occupied/empty edges.

The row-to-row transfer matrix T acts on an α state, and adds a row to the system. More precisely, the matrix element $T_{\alpha,\alpha'}$ is defined as the Boltzmann weight of all possible intermediary states from α' to α:

$$T_{\alpha,\alpha'} = \sum_{\beta | \alpha,\alpha'} \prod_{j=1}^{L} z_1^{\beta_{2j-1}} z_2^{\beta_{2j}} z_3^{(\alpha_j + \alpha'_j)/2} ,$$

where the sum is over the possible intermediary states between configurations α' and α. In our conventions, T acts from bottom to top, so α is placed above α'.

Solution by free fermions. Let us now reformulate the problem in terms of a closed system of particles evolving in the vertical direction. Consider the reference configuration $\mathcal{C}_0$ where all vertical edges are occupied by a dimer, and define the particles associated to any dimer configuration $\mathcal{C}$ as follows: if the dimer occupation of an edge e is the same in $\mathcal{C}$ as in $\mathcal{C}_0$, we say it carries no particle, whereas if the dimer occupations of e are different in $\mathcal{C}$ and $\mathcal{C}_0$, we says it carries a particle. It is easy to see that the matrix element $T_{\alpha,\alpha'}$ vanishes, unless α and α' have the same number of particles: this means that the number of particles is conserved by the action of T. Thus, we can look for the eigenvectors and eigenvalues of T in sectors of fixed number of particles n. A left eigenvector ψ of T is defined by the condition

$$\sum_{\alpha} \psi_\alpha T_{\alpha,\alpha'} = \Lambda \psi_{\alpha'} \,. \tag{1.4.1}$$

For $n = 0$, there is only one vacuum state, with eigenvalue z_3^L for T. We shall use this to normalise our transfer matrix, and set $\tau := z_3^{-L}\, T$.

For $n = 1$, the states are specified by the position x of a single particle. Notice that $x \in \{1, 2, \dots, L\}$ for even rows, and $x \in \{1/2, 3/2, \dots, L - 1/2\}$ for odd rows. We denote by $\tau^{(1)}$ the transfer matrix from even to odd rows, and $\widetilde{\tau}^{(1)}$ the transfer matrix from odd to even rows. The matrix elements of τ read, for $(x, y) \in (\mathbb{N} + 1/2, \mathbb{N})$:

$$\tau^{(1)}_{xy} = (z_1/z_3)\ \delta_{x,y-1/2} + (z_2/z_3)\ \delta_{x,y+1/2} \,,$$

and $\widetilde{\tau}_{xy}$ has the same expression, but with $(x, y) \in (\mathbb{N}, \mathbb{N} + 1/2)$. We wish to find the left eigenvectors of τ and $\widetilde{\tau}$. The eigenvalue equations read for any $y \in \{1, \dots, L\}$

$$\sum_{x=1}^{L} \psi(x - 1/2)\tau^{(1)}_{x-1/2,y} = \Lambda\ \psi(y) \,, \tag{1.4.2}$$

$$\sum_{x=1}^{L} \psi(x)\widetilde{\tau}^{(1)}_{x,y-1/2} = \Lambda\ \psi(y - 1/2) \,, \tag{1.4.3}$$

where we have identified the points $(L+1/2)$ and $1/2$.. Let us focus on (1.4.2) first. Since $\tau^{(1)}$ is invariant by cyclic translations, ψ must have the form of a plane wave:

$$\psi_k(x) := \exp(ikx) \,, \qquad \text{with} \qquad -\pi < k \leq \pi \,.$$

The corresponding eigenvalue is

$$\Lambda_k = (z_1/z_3)e^{-ik/2} + (z_2/z_3)e^{+ik/2} \,.$$

Moreover, periodic boundary conditions impose that $e^{ikL} = 1$. It is easy to see that ψ_k is also an eigenvector of $\widetilde{\tau}^{(1)}$, with the same eigenvalue.

For $n = 2$, particle states are labelled by two positions (x_1, x_2), with $x_1 < x_2$. The matrix element takes into account the avoiding constraint between particles:

$$\tau^{(2)}_{(x_1,x_2),(y_1,y_2)} = \tau^{(1)}_{x_1,y_1}\tau^{(1)}_{x_2,y_2} - (z_1 z_2/z_3^2)\ \delta_{x_1 x_2}\delta_{x_1,y_1+1/2}\delta_{x_2,y_2-1/2} \,.$$

If we set $\psi_{12}(x_1, x_2) = \psi_{k_1}(x_1)\psi_{k_2}(x_2)$, the left-hand side of (1.4.1) reads

$$\begin{aligned}\sum_{x_1,x_2} \psi_{12}(x_1, x_2)\tau^{(2)}_{(x_1,x_2),(y_1,y_2)} =& \Lambda_{k_1}\Lambda_{k_2}\psi_{12}(y_1, y_2) \\ &- (z_1 z_2/z_3^2)\ \delta_{y_1+1,y_2}\psi_{k_1}(y_1 + 1/2)\psi_{k_2}(y_1 + 1/2) \,.\end{aligned}$$

Similarly, for $\psi_{21}(x_1, x_2) = \psi_{k_2}(x_1)\psi_{k_1}(x_2)$, we get

$$\sum_{x_1,x_2} \psi_{21}(x_1,x_2)\tau^{(2)}_{(x_1,x_2),(y_1,y_2)} = \Lambda_{k_2}\Lambda_{k_1}\psi_{21}(y_1,y_2) - (z_1z_2/z_3^2)\ \delta_{y_1+1,y_2}\psi_{k_2}(y_1+1/2)\psi_{k_1}(y_1+1/2)\,.$$

We can thus simply combine ψ_{12} and ψ_{21} to eliminate the δ_{y_1+1,y_2} terms. We define

$$\psi(x_1,x_2) := \psi_{k_1}(x_1)\psi_{k_2}(x_2) - \psi_{k_2}(x_1)\psi_{k_1}(x_2)\,, \tag{1.4.4}$$

and we have

$$\sum_{x_1,x_2} \psi(x_1,x_2)\tau^{(2)}_{(x_1,x_2),(y_1,y_2)} = \Lambda_{k_1}\Lambda_{k_2}\psi(y_1,y_2)\,, \tag{1.4.5}$$

which is the eigenvalue equation in the two-particle sector. We recognise that (1.4.4) is a fermionic two-body wave function. When a particle goes around the system, the wavefunction picks a factor (-1), and hence the momenta satisfy:

$$e^{iLk_1} = e^{iLk_2} = -1\,,$$

with the additional constraint that $k_1 \neq k_2$. This suggests that the particles behave like free fermions.

For a general value of n, states are specified by the sequence $x_1 < \cdots < x_n$ of particle positions. One can extend the previous discussion, and (1.4.4) is replaced by

$$\psi(x_1,\ldots,x_n) = \sum_{\sigma\in\mathcal{S}_n} \mathrm{sgn}(\sigma)\exp[ik_{\sigma(1)}x_1 + \cdots + ik_{\sigma(n)}x_n]\,, \tag{1.4.6}$$

where $\mathcal{S}_n$ is the set of permutation of n elements, and the momenta $k_1,\ldots,k_n$ take n distinct values, subject to the conditions:

$$\exp(iLk_j) = (-1)^{n-1}\,. \tag{1.4.7}$$

A simple way to interpret this condition is to consider the process of taking one particle around the system: ψ gets a factor (-1) every time two particles are exchanged, and e^{iLk_j} for the translation of particle j around the system. The above equation imposes that these two factors compensate. Again, (1.4.6) is a fermionic wavefunction. The corresponding eigenvalue of τ is

$$\Lambda = \Lambda_{k_1}\ldots\Lambda_{k_n}\,, \qquad \text{where} \qquad \Lambda_k = (z_1/z_3)e^{-ik/2} + (z_2/z_3)e^{+ik/2}\,. \tag{1.4.8}$$

It is important to work we *left* eigenvectors, for the following reason. In the $n = 2$ sector, if we were dealing with the right eigenvector with $\sum_{\alpha'}\tau_{\alpha\alpha'}\psi(\alpha') = \Lambda\psi(\alpha)$, we would get unwanted terms of the form $\delta_{y_1+1,y_2}\psi_{k_1}(y_1)\psi_{k_2}(y_1+1)$. Hence, the right eigenvector is given by $\psi = e^{-ik_2}\psi_{12} - e^{-ik_1}\psi_{21}$, which is not antisymmetric in the positions like (1.4.4). Of course, the eigenvalues are the same, but the relation to a free Fermi system is less obvious.

Dominant eigenvalues. We first restrict to the isotropic case $z_1 = z_2 = z_3$. The one-particle eigenvalues then read $\Lambda_k = 2\cos(k/2)$. The maximal eigenvalue (1.4.8) is obtained when we choose the values of k for which $|\Lambda_k| > 1$. This corresponds to a Fermi sea in the interval $|k| < 2\pi/3$.

Let us show how to compute the dominant eigenvalue $\Lambda_{\max}$ of T as an expansion in $1/L$. For simplicity, we consider the case when L is a multiple of 3. Then $\Lambda_{\max}$ is obtained by taking $n = 2L/3$ particles, with momenta $k_j = 2\pi q_j/L$, and

$$\{q_1, \dots, q_n\} = -\frac{n-1}{2}, -\frac{n-3}{2}, \dots, \frac{n-1}{2}\,. \tag{1.4.9}$$

To evaluate (1.4.8), we can use the Euler-McLaurin formula

$$h \times \left[\frac{f(a)+f(b)}{2} + \sum_{j=1}^{N-1} f(a+jh)\right] = \int_a^b f(k)\mathrm{d}k + \frac{h^2}{12}\left[f'(b) - f'(a)\right] + O(h^4)\,, \tag{1.4.10}$$

where $h := (b-a)/N$, and f is any function such that f''' exists and is continuous on $[a, b]$. This yields the asymptotic expansion:

$$\log \Lambda_{\max} \simeq \frac{L}{2\pi}\int_{-2\pi/3}^{+2\pi/3} \log[2\cos(k/2)]\mathrm{d}k + \frac{\pi\sqrt{3}}{12L}\,. \tag{1.4.11}$$

Consider a system made of M rows. If the hexagons have sides of unit length, then the system has size $(L\sqrt{3}) \times (3M/2)$. For a conformally invariant system, we expect the free energy $F = -\log Z$ to behave like

$$F = LMf_\infty - \frac{M\sqrt{3}}{2L} \times \frac{\pi c}{6}\,,$$

where f_∞ is the free energy density per site, and c is a parameter of the scaling theory, called the central charge in Conformal Field Theory (this will be explained in future lectures). From (1.4.11), and writing $F = -M \log \Lambda_{\max}$, we have

$$f_\infty = -\frac{1}{2\pi}\int_{-2\pi/3}^{+2\pi/3} \log[2\cos(k/2)]\mathrm{d}k\,, \qquad \text{and} \qquad c = 1\,.$$

Proof of (1.4.11)**.**

In (1.4.10), we set $f(k) := \log[2\cos(k/2)]$, $b = -a := (n-1)2\pi/L$ and $N = n-1$. We have immediately $f(\pm 2\pi/3) = 0$. Then, we approximate $f(a) \simeq f'(2\pi/3)h/2$, $f'(a) \simeq f'(2\pi/3)$ and similarly for $f(b)$ and $f'(b)$. Moreover, we write

$$\int_a^b f(k)\mathrm{d}k \simeq \int_{-2\pi/3}^{2\pi/3} f(k)\mathrm{d}k + \frac{h^2}{8}[f'(2\pi/3) - f'(-2\pi/3)]\,.$$

Inserting these approximations into (1.4.10), we get the above result.

We can repeat the calculation for the subdominant eigenvalue Λ_1, corresponding to the momentum distribution (1.4.9), but with one particle removed: $n = 2L/3 - 1$. In terms

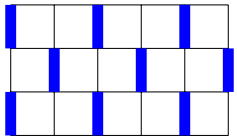

Figure 1.8: A particular dimer configuration C_0 in which all dimers sit on even vertical edges.

of dimers, when superposing configurations with $n = 2L/3$ and $n = 2L/3 - 1$, we get one transition line propagating in the transfer direction, which is equivalent to a shift of the dimers by one site along this line. Therefore, we can identify Λ_1 to the insertion of one monomer at each end of the transition line. The asymptotic expansion for Λ_1 is

$$\log \Lambda_1 \simeq \frac{L}{2\pi} \int_{-2\pi/3}^{+2\pi/3} \log[2\cos(k/2)]\mathrm{d}k - \frac{\pi\sqrt{3}}{6L} \,.$$

The corresponding scaling dimension X_{mon} can be extracted from the CFT prediction

$$\frac{1}{L(\sqrt{3}/2)} \log \frac{\Lambda_1}{\Lambda_{\max}} \simeq -\frac{2\pi X_{\mathrm{mon}}}{L^2} \,,$$

and so we find $X_{\mathrm{mon}} = 1/4$.

Anisotropic system. Finally, suppose we take anisotropic Boltzmann weights, of the form

$$z_1 = z_2 = w^{1/2} \,, \qquad z_3 = w^{-1/2} \,, \qquad \text{where} \quad 2w \geq 1 \,.$$

The eigenvalues for single-particle states are then $\Lambda_k = 2w\cos(k/2)$. The Fermi level k_F is now defined by

$$k_F = 2\mathrm{Arcos}\left(\frac{1}{2w}\right) \,,$$

and the free energy density is given by

$$f_\infty = -\frac{1}{2\pi} \int_{-k_F}^{+k_F} \log[2\cos(k/2)]\mathrm{d}k \,.$$

1.5 Appendix: Transfer matrix for the square lattice

On the square lattice one can assign a definite parity to the vertical edges by alternating even and odd edges throughout a given row, and alternating the convention between even and odd rows. Fig. 1.8 shows a particular dimer configuration C_0 in which all dimers sit on even vertical edges. Note that this corresponds to a maximal height gradient between the left and right rims of the lattice.

Consider now superposing a generic dimer configuration C with C_0 by means of an exclusive or (XOR) operation. For example, when C is the configuration of Fig. 1.1 the resulting superposition is shown in Fig. 1.9.

This superposition consists in a certain number s of strings (here $s = 3$) along which dimers from C and C_0 alternate. The dynamics under which these strings propagate in the vertical direction has interesting properties:

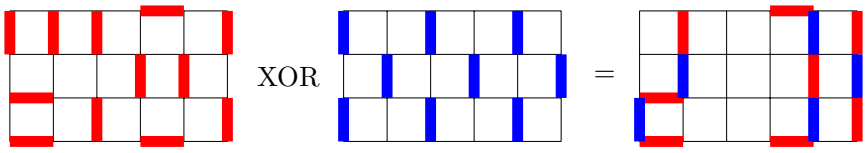

Figure 1.9: Conserved strings in a dimer configuration on $Q_{6,4}$.

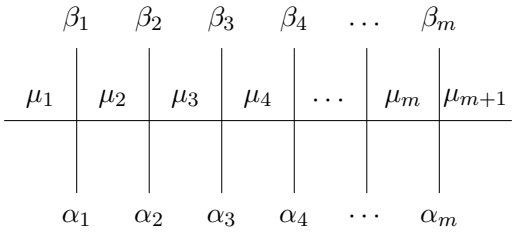

Figure 1.10: Labelling of edges used to define the row-to-row transfer matrix.

1. The number of strings is conserved, and

2. When moving from one horizontal layer to the next, a string can either go straight or move exactly one step to the left or to the right.

These properties follow directly from the definition of the XOR operation and from the definition of a valid dimer covering C.

The properties of strings suggest to view dimer configurations as a discrete time evolution process, where the time increases along the vertical direction. In what follows it is convenient to refer to the horizontal (resp. vertical) direction as *space* (resp. *time*). The time evolution is then accomplished by a linear operator, called the row-to-row transfer matrix $T_{\beta\alpha}$, that we now define.

Let us label the edges of two consecutive rows of vertical edges, as well as the intermediate row of horizontal edges, as shown in Fig. 1.10. The state of a row is specified by the occupation numbers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$, where $\alpha_i = 0$ (resp. $\alpha_i = 1$) means that the ith vertical edge is empty (resp. occupied by a dimer). Given the states α and β of two consecutive rows, the transfer matrix element $T_{\beta\alpha}$ is the part of the Boltzmann weight in (1.1.1) seen locally, summed over all possible intermediate states μ compatible with α and β. Thus

$$T_{\beta\alpha} = \sum_{\mu|(\alpha,\beta)} (z_1)^{\sum_i \mu_i} (z_2)^{\frac{1}{2}\sum_i (\alpha_i+\beta_i)} \,. \tag{1.5.1}$$

The compatibility criterion $\mu|(\alpha,\beta)$ can be expressed formally as

$$\forall i \in \{1,\dots,m\} : \ \mu_i + \alpha_i + \beta_i + \mu_{i+1} = 1 \,, \tag{1.5.2}$$

meaning simply that the sum of occupation numbers around any one vertex is one.

Boundary conditions in the space direction can be specified through an additional constraint on the μ variables. Free boundary conditions mean $\mu_1 = \mu_{m+1} = 0$; periodic boundary conditions are obtained by identifying $\mu_{m+1} \equiv \mu_1$. This implies of course that the transfer matrix is different in the two cases.

Boundary conditions in the time direction are specified by constraints on the first and last row states. Let $|0\rangle$ denote the empty row state, i.e., such that $\forall i \in \{1, \ldots, m\} : \alpha_i = 0$. The partition function (1.1.1) with free boundary conditions in the time direction is then

$$Z_{mn}(z_1, z_2) = \langle 0|T^n|0\rangle \,, \tag{1.5.3}$$

whereas periodic boundary conditions in the time direction lead to

$$Z_{mn}(z_1, z_2) = \mathrm{Tr}\; T^n \,. \tag{1.5.4}$$

In all cases, the free energy per site can be related to the leading eigenvalue of T, and critical exponents can be inferred from various subleading eigenvalues. The eigenvalues can be found either by numerical diagonalisation, or analytically through the Bethe Ansatz technique. For both purposes it is useful to discuss more closely the structure of the transfer matrix.

Sparse matrix factorisation. For all but the smallest m it is inefficient (both in analytical and numerical calculations) to write down the whole transfer matrix in a single go. It is preferable to write $T_{\beta\alpha}$ as a product of matrices $R_{\beta_i,\mu_{i+1};\mu_i,\alpha_i}$ that act locally by adding only the ith vertex in a given row. Organising the pairs of index values in binary order (00, 01, 10, 11) this reads explicitly

$$R = I \otimes \cdots I \otimes \begin{bmatrix} 0 & \sqrt{z_2} & \sqrt{z_1} & 0 \\ \sqrt{z_1} & 0 & 0 & 0 \\ \sqrt{z_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \otimes I \cdots \otimes I \,, \tag{1.5.5}$$

where the identity matrices mean that the action elsewhere in the tensor product of states is trivial.

Before building a row of the lattice, one needs to insert the leftmost horizontal space—sometimes called *auxiliary space*—corresponding to the variable μ_1. Then follows the action of m factors of $R_{\beta_i,\mu_{i+1};\mu_i,\alpha_i}$, each propagating an α_i to a β_i, starting by $i = 1$ and ending by $i = m$. And finally the rightmost horizontal space, corresponding to μ_{m+1}, must be removed. For free boundary conditions in the space direction the insertion and removal of the auxiliary space simply amounts to enforcing $\mu_1 = \mu_{m+1} = 0$. Periodic boundary conditions are slightly more tricky, and require allowing for both possibilities $\mu_1 = 0, 1$ in the insertion, keeping a copy of μ_1 when acting with the factors of R, and finally enforcing $\mu_1 = \mu_{m+1}$ upon removal of the auxiliary space. To say it shortly, one "traces over the auxiliary space".

The advantage of this procedure in numerical calculations is that each application of R generates at most 2 out-states for each in-state, and hence takes time proportional to the dimension of the state space. If the entire T were applied at once, each in-state would produce an exponentially large (in m) number of out-states.

Sector decomposition. Naively it appears that $\dim T = 2^m$. The effective dimension is however greatly reduced by exploiting the conservation of strings. Let the number of strings be $s = m/2 + Q$, with $Q = -m/2, \ldots, m/2$, and denote the corresponding block in the transfer matrix by $T^{(Q)}$. We have then

$$T = \bigoplus_{Q=-\frac{m}{2}}^{\frac{m}{2}} T^{(Q)} \,. \tag{1.5.6}$$

Diagonalising T amounts to diagonalising separately each term $T^{(Q)}$—sometimes called a *sector*—in the direct sum. The states contributing to $T^{(Q)}$ can be specified by giving the position of the strings, whence

$$\dim T^{(Q)} = \begin{pmatrix} m \\ m/2+Q \end{pmatrix} . \tag{1.5.7}$$

An explicit characterisation of the row states follows by noting that that

$$Q = \sum_{i \text{ odd}} \alpha_i - \sum_{i \text{ even}} \alpha_i \, . \tag{1.5.8}$$

In terms of the height mapping, the meaning of the conserved "charge" Q is the height difference Δh between the left and right rims of the lattice. Obviously for free time-like boundary conditions only the $Q = 0$ sector will contribute to $Z_{mn}(z_1, z_2)$, whereas for periodic time-like boundary conditions all sectors participate.

The sectors with $Q \neq 0$ can be used to define correlation functions. For instance, a monomer defect leads to $\Delta h = 1$. The exponential decay of the monomer-monomer correlation function $\mathcal{C}(n)$ in a cylinder geometry (m fixed and $n \to \infty$) is given by the ratio of the largest eigenvalues

$$\mathcal{C}(n) \sim \left(\frac{\Lambda_{\max}^{(Q=1)}}{\Lambda_{\max}^{(Q=0)}} \right)^n \tag{1.5.9}$$

of the transfer matrix sectors $T^{(1)}$ and $T^{(0)}$. Using a standard CFT result (viz., conformally mapping the cylinder to the complex plane) this can be used to infer the corresponding critical exponent $X(0,1)$ as in (1.2.8). This requires obviously finding the $m \to \infty$ limit of the participating eigenvalues, which can be achieved by using Bethe Ansatz techniques.

It is a useful exercise at this point to write down explicitly the row states contributing to the sector $Q = 0$ for a moderately small system, say $m = 4$.

The six possible row states $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ of $T^{(0)}$ read:

$$(0,0,0,0) \quad (1,1,0,0) \quad (0,0,1,1) \quad (1,0,0,1) \quad (0,1,1,0) \quad (1,1,1,1)$$

and the transfer matrix is

$$T^{(0)} = \begin{pmatrix} z_1^2 & z_1 z_2 & z_1 z_2 & z_1 z_2 & 0 & z_2^2 \\ z_1 z_2 & 0 & z_2^2 & 0 & 0 & 0 \\ z_1 z_2 & z_2^2 & 0 & 0 & 0 & 0 \\ z_1 z_2 & 0 & 0 & 0 & z_2^2 & 0 \\ 0 & 0 & 0 & z_2^2 & 0 & 0 \\ z_2^2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

We can use (1.5.3) to compute $Z_{44}(1,1)$. The result is

$$Z_{4,4}(z_1, z_2) = z_1^8 + 9 z_1^6 z_2^2 + 16 z_1^4 z_2^4 + 9 z_1^2 z_2^6 + z_2^8 \, .$$

and this agrees with the exact result (1.1.6). In particular $Z_{44}(1,1) = 36$.