Chapter 2

The two-dimensional Ising model 1/2

2.1 An exactly solvable model of phase transition

2.1.1 Introduction

One of the main concerns in Statistical Mechanics is the study of phase transitions, when the state of a system changes dramatically. In this Course, we will restrict to the study of classical statistical systems at equilibrium, in two dimensions. We will cover two distinct approaches to the *exact* description of a phase transition.

First, we can start from a lattice model, and study the mathematical structure arising from its symmetries. This structure is studied through simple combinatorics, or more elaborate methods, like integrability. It is then used to find the exact location of the critical point(s), to compute exactly the physical quantities (free energy, correlation functions ...), and to determine their behaviour in the scaling limit. This will give the critical exponents, correlation functions, operator algebra, etc. and will allow us to build an effective Quantum Field Theory (QFT) describing the phase transition.

A second approach is to consider directly the scaling limit, and to use the symmetries of the system to determine the scaling properties in a given universality class. Instead of writing a Lagrangian and computing exponents and correlation functions, the approach of Conformal Field Theory (CFT) consists in postulating some strong symmetry constraints, and classifying their possible solutions according to symmetries, to finally describe the scaling properties of a phase transition.

Of course, the two approaches are complementary, and it is a very fruitful exercise to identify the operators in the lattice model which correspond to a set of operators in the CFT. Let us start by recalling some important elements in the theory of phase transitions, and the general framework in which they are conceptually described, namely the renormalisation group.

2.1.2 Critical phenomena at equilibrium

In the canonical ensemble, each possible configuration C occurs with a probability proportional to $\exp[-E(C)/(k_B T)]$, where T is the temperature, k_B is the Boltzmann constant and $E(C)$ is the energy of the configuration. The Boltzmann weights $W(C)$ and the

partition function are defined as

$$W(C) = \exp\left[-\frac{E(C)}{k_B T}\right], \qquad Z = \sum_C W(C), \tag{2.1.1}$$

where the sum is over all possible configurations of the system. The thermal average of any quantity $A(C)$ which depends on the configuration is then

$$\langle A\rangle := \frac{1}{Z}\sum_C W(C)\ A(C). \tag{2.1.2}$$

In particular, we will focus on correlation functions, where the function $A(C)$ only depends on the degrees of freedom at N points $\vec{r}_1, \ldots, \vec{r}_N$:

$$G(\vec{r}_1, \ldots, \vec{r}_N) = \langle \mathcal{O}_1(\vec{r}_1) \ldots \mathcal{O}_N(\vec{r}_N)\rangle\ . \tag{2.1.3}$$

The "operators" $\mathcal{O}_j$ in the above expression are simply some functions which depend on the configuration at a single point $\vec{r}_j$.

To describe the important concepts of critical phenomena, let us consider specifically the Ising model at zero field on some regular lattice $\mathcal{L}$. In this Section we will state some well-known results on the Ising model without deriving them, in order to expose the concepts on a simple example. The configurations are spins s_j living on the vertices of $\mathcal{L}$ which can take the values $s_j \in \{1, -1\}$, and the Boltzmann weight of a configuration $\{s_j\}$ is

$$W[s] = \prod_{\langle ij\rangle} \exp\left(\frac{s_i s_j}{k_B T}\right), \tag{2.1.4}$$

where the product is on nearest neighbours. The system is ordered for small T (ferromagnetic phase), and becomes disordered for large T (paramagnetic phase). The two phases are separated by a phase transition at a finite value $T = T_c$.

To characterise more precisely the phase transition, we consider the Ising model on a square lattice $\mathcal{L}_N$ of $N \times N$ sites with periodic boundary conditions (BC). The **order parameter** is defined by the mean magnetisation in the scaling limit:

$$M = \lim_{N\to\infty} \left| \frac{1}{N^2} \sum_{j\in\mathcal{L}_N} \langle s_j\rangle \right|. \tag{2.1.5}$$

The order parameter is positive for $T < T_c$, and zero for $T \geq T_c$. At $T = T_c$, it has an essential singularity in the low-temperature phase:

$$M \propto |T - T_c|^\beta, \qquad \text{with} \quad \beta = 1/8\,. \tag{2.1.6}$$

These features are summarised in Fig. 2.1. The Ising model has an obvious $\mathbb{Z}_2$ symmetry under reversal of all spins ($s_j \to -s_j$). However, in the low-temperature phase, the typical configurations are those where there is a majority of $s_j = +1$, or a majority of $s_j = -1$: this is called **spontaneous symmetry breaking**. In this context, M is often called the spontaneous magnetisation.

In terms of typical configurations in the scaling limit, there are three distinct cases:

1. In the low-temperature phase $T < T_c$, most spins are in the same state (say $s = +1$), and some domains of spin $s = -1$ appear, with typical extension ξ.

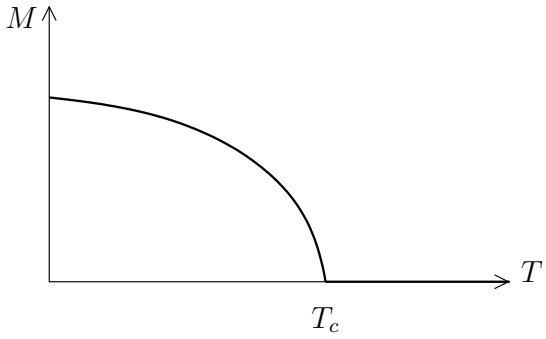

Figure 2.1: Mean magnetisation of the Ising model as a function of temperature.

2. At $T = T_c$, the typical extension $\xi \to \infty$, and all domain sizes coexist. The system is scale-invariant: one cannot distinguish a typical configuration from its image under a scale transformation $\vec{r} \to \lambda\vec{r}$.

3. In the high-temperature phase $T > T_c$, the system is made of a collection of small random $s = \pm 1$ domains, each with typical extension ξ.

Moreover, in the vicinity of T_c, the typical extension ξ of a spin domain, also called the **correlation length**, follows the scaling law:

$$\xi \propto |T - T_c|^{-\nu}\,, \qquad \text{with} \quad \nu = 1\,. \tag{2.1.7}$$

We have characterised the phase transition in terms of the order parameter and the typical configuration. A third characterisation is given by the two-point correlation functions. Before we explain this, let us recall the precise definition of the scaling limit for correlation functions of the form (2.1.3). We consider a domain Ω, and discretise it by a square lattice with lattice spacing a. The operators $\mathcal{O}_j$ are placed at the points $\vec{r}_j = (m_j a, n_j a)$, where m_j and n_j are some integers. We let $a \to 0$ and $m_j, n_j \to \infty$, while keeping the domain Ω and all the $\vec{r}_j$ fixed. For instance, if Ω is square of side L, it can be discretised by a square lattice of $N \times N$ sites, where $L = Na, N \to \infty, a \to 0$ and L is fixed.

In the scaling limit, the spin correlation function has the behaviour:

$$\langle s(\vec{r}_1)s(\vec{r}_2)\rangle - \langle s\rangle^2 \propto \begin{cases} \exp(-|\vec{r}_1 - \vec{r}_2|/\xi) & \text{for } T \neq T_c \\ |\vec{r}_1 - \vec{r}_2|^{-2x_{\rm spin}} & \text{for } T = T_c\,, \quad x_{\rm spin} = 1/8\,. \end{cases} \tag{2.1.8}$$

The critical exponent $x_{\rm spin}$ is called the scaling dimension of the spin. One may consider the two-point function of other local operators. For example, we can define the local energy density at site j as

$$\epsilon_j := \sum_{i|\langle ij\rangle} s_i s_j\,, \tag{2.1.9}$$

where the sum is over all the sites i adjacent to j. The two-point correlation function of ϵ follows the behaviour of (2.1.8), with $x_{\rm spin}$ replaced by $x_\epsilon = 1$, the scaling dimension of the energy operator.

2.1.3 Critical interfaces

Let us now give a fourth description of the phase transition, through the geometry of interfaces. We now consider the Ising model on the domain Ω, and we fix two points u

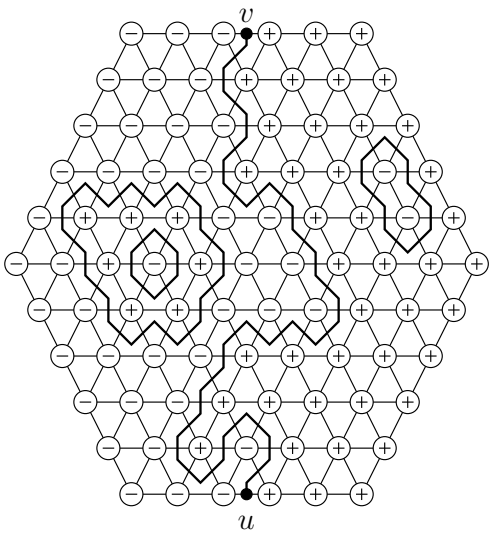

Figure 2.2: The interface γ for the Ising model on the triangular lattice.

and v on the boundary: this divides the boundary into two arcs. We fix the spins to $s = +1$ on one arc, and $s = -1$ on the other. This forces the existence of an interface γ joining u and v, and separating the spin domains adjacent to the $s = +1$ and $s = -1$ boundaries: see Fig. 2.2.

The typical geometry of the interface γ in the scaling limit is the following.

1. In the low-temperature phase $T < T_c$, the interface γ is a smooth curve with small fluctuations around the configuration of minimal length.

2. At $T = T_c$, the interface γ is **scale-invariant**: if we take a portion of γ of spatial extension ℓ in the range $a \ll \ell \ll |\Omega|$, it is statistically equivalent to a portion of spatial extension $\lambda\ell$ (for finite λ). The curve γ is a fractal object, with fractal dimension (or Hausdorff dimension) $d_f = 11/8$ (see the definition below).

3. In the high-temperature phase $T > T_c$, neighbouring spins become decorrelated, and the interface γ behaves like the boundary of a site-percolation cluster. It remains a fractal, with dimension $d_f = 7/4$.

We close this paragraph by recalling the definition of the **fractal dimension** of a curve γ. We call L the spatial extension of γ (*i.e.* L is the diameter of the smallest disc containing γ), and we try to cover γ by small discs of diameter a: see Fig. 2.3. Then the minimum number $\mathcal{N}$ of small discs that we need scales like

$$\mathcal{N} \propto (L/a)^{d_f} . \tag{2.1.10}$$

Of course, if γ is an ordinary smooth curve, $d_f = 1$, but in general the fractal dimension of a curve can have any value in the range $1 \leq d_f \leq 2$, as it is the case for Ising interfaces.

2.1.4 Critical exponents and scaling dimensions

Let us summarise the critical exponents that we have defined so far, and add, for completeness, a few more to the list:

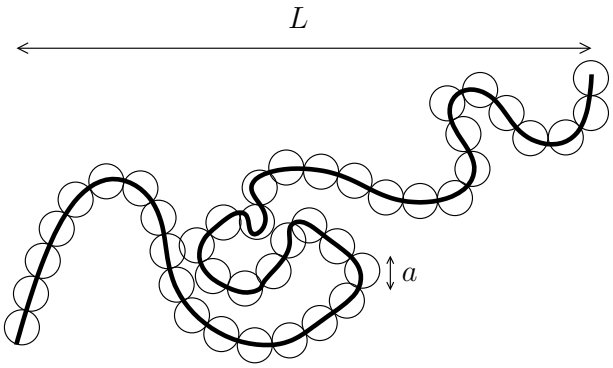

Figure 2.3: Covering of a curve of extension L by $\mathcal{N}$ discs of diameter a.

- *Exponents defined by the order parameter M.* We have seen that the spontaneous magnetisation is $M \propto |T - T_c|^\beta$. Also, if we add a term $-H \sum_i s_i$ to the energy, *i.e.* a magnetic field H, the susceptibility at $H = 0$ scales like:

$$\chi := \left.\frac{\partial M}{\partial H}\right|_{H=0} \propto |T - T_c|^{-\gamma} . \tag{2.1.11}$$

  Also, at the critical temperature, we have $M \propto |H|^{1/\delta}$.

- *Exponents defined by correlations.* The correlation length diverges as $\xi \propto |T - T_c|^{-\nu}$. At the critical temperature, two-point correlation functions follow power laws:

$$\langle \mathcal{O}_j(\vec{r}_1)\mathcal{O}_j(\vec{r}_2)\rangle \propto |\vec{r}_1 - \vec{r}_2|^{-2x_j} , \tag{2.1.12}$$

  where x_j is the scaling dimension of the operator x_j.

- *Exponents defined by interfaces.* At the critical temperature, the spin interfaces become fractal curves, with fractal dimension d_f. For these scale-invariant curves, one can define a larger set of dimensions which describe their geometry, namely the "multifractal spectrum".

2.1.5 The renormalisation group

TO DO: topo sur les op relevants/irrelevants, points (multi-)critiques

2.2 Graphical expansions, Kramers-Wannier duality, and disorder operators

2.2.1 High-temperature expansion

In this section, we will consider the Ising model on the square lattice $\mathcal{L}$, with coupling constant J_1 on the horizontal edges and J_2 on the vertical edges. The Boltzmann weights then read:

$$W[s] = \prod_{\substack{\langle ij\rangle \\ \text{horizontal}}} \exp(J_1\ s_i s_j) \prod_{\substack{\langle ij\rangle \\ \text{vertical}}} \exp(J_2\ s_i s_j) . \tag{2.2.1}$$

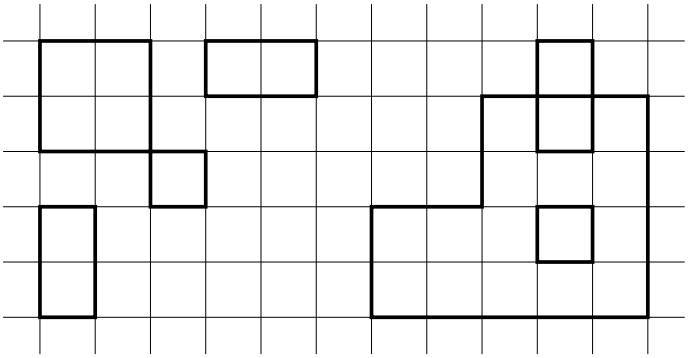

Figure 2.4: A polygon configuration contributing to the high-temperature expansion of the partition function Z.

For two Ising spins (s, s'), since the product $ss' \in \{1, -1\}$, we have the identity

$$\exp(Jss') = \cosh J + \sinh J \; ss' = \cosh J \times \left(1 + w \; ss'\right) , \tag{2.2.2}$$

where $w := \tanh J$. If we drop the overall multiplicative factor, we get:

$$W[s] \propto \prod_{\substack{\langle ij \rangle \\ \text{horizontal}}} (1 + w_1 \; s_i s_j) \prod_{\substack{\langle ij \rangle \\ \text{vertical}}} (1 + w_2 \; s_i s_j) , \tag{2.2.3}$$

with

$$w_1 = \tanh J_1 , \qquad w_2 = \tanh J_2 . \tag{2.2.4}$$

When the product over edges is expanded, each edge $\langle ij \rangle$ can contribute by two terms: the term 1 is then represented by an empty edge, and the term $w s_i s_j$ by an occupied edge. We get the partition function:

$$Z = (\cosh J_1)^{N_{e1}(\mathcal{L})} (\cosh J_2)^{N_{e2}(\mathcal{L})} \times \sum_{\{s\}} \sum_{G \subseteq \mathcal{L}} \left[w_1^{N_{e1}(G)} w_2^{N_{e2}(G)} \prod_{\langle ij \rangle \in G} s_i s_j \right] ,$$

where $N_{e1}(G)$ [resp. $N_{e2}(G)$] is the number of horizontal (resp. vertical) edges of the graph G, and the second sum is over all subgraphs of $\mathcal{L}$. To get a non-zero contribution, every factor s_j should occur with an even power, and the sum over s_j then gives a factor 2: this selects the graphs G which are collections of closed polygons (see Fig. 2.4). Performing the sum over spins gives:

$$Z = (\cosh J_1)^{N_{e1}(\mathcal{L})} (\cosh J_2)^{N_{e2}(\mathcal{L})} \; 2^{N_v(\mathcal{L})} \times \sum_{G \in \mathcal{C}} w_1^{N_{e1}(G)} w_2^{N_{e2}(G)} , \tag{2.2.5}$$

where the sum is over closed polygon configurations, and $N_v(\mathcal{L})$ is the number of vertices of the lattice $\mathcal{L}$. This is often referred to as a **high-temperature expansion**, since $w = \tanh J \ll 1$ when $J \ll 1$, but we stress that this is an exact rewriting of Z that holds for any J. Hence, we have shown that the Ising partition function can be rewritten as the **generating function of closed polygon configurations** weighted by the number of occupied edges.

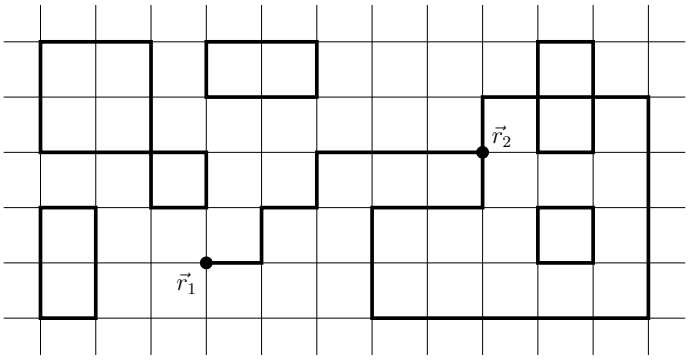

Figure 2.5: A polygon configuration contributing to the high-temperature expansion of $\langle s(\vec{r}_1)s(\vec{r}_2)\rangle$.

Suppose now that we apply the same expansion to the correlation function

$$\langle s(\vec{r}_1)s(\vec{r}_2)\rangle = \frac{1}{Z}\sum_{\{s\}} W[s]\ s(\vec{r}_1)s(\vec{r}_2)\,. \tag{2.2.6}$$

All the above steps can be reproduced, except for the summation over the spins $s(\vec{r}_1)$ and $s(\vec{r}_2)$. We get:

$$\langle s(\vec{r}_1)s(\vec{r}_2)\rangle = \frac{\sum_{G\in\mathcal{C}(\vec{r}_1,\vec{r}_2)} w_1^{N_{e1}(G)}\ w_2^{N_{e2}(G)}}{\sum_{G\in\mathcal{C}} w_1^{N_{e1}(G)}\ w_2^{N_{e2}(G)}}\,, \tag{2.2.7}$$

where the sum in the numerator is over polygon configurations in which all the polygons are closed, except at the points $\vec{r}_1$ and $\vec{r}_2$, which must be adjacent to an *odd* number of edges.

2.2.2 Low-temperature expansion

On the other hand, one can expand Z around one of the totally ordered states, say $s \equiv 1$. The excitations are then domain walls surrounding domains of the opposite spin value ($s = -1$): see Fig. 2.6.

The domain walls live on the **dual lattice** $\mathcal{L}^*$, whose sites are the centers of the faces of $\mathcal{L}$. They separate regions of opposite spin values. Suppose we fix the value of the spin s_0 at the origin: then the spin configurations are in one-to-one correspondence with the domain wall configurations. For the partition function, we have

$$Z = 2e^{J_1 N_{e1}(\mathcal{L})+J_2 N_{e2}(\mathcal{L})} \times {\sum_{G^*\subseteq\mathcal{L}^*}}' (w_1^*)^{N_{e1}(G^*)}(w_2^*)^{N_{e2}(G^*)}\,, \tag{2.2.8}$$

where the symbol $\sum'$ has the same meaning as in (2.2.5), and

$$w_1^* = e^{-2J_1}\,, \qquad w_2^* = e^{-2J_2}\,. \tag{2.2.9}$$

This formulation in terms of domain walls corresponds to the expansion of the local Boltzmann weights:

$$W[s] \propto \prod_{\substack{\langle ij\rangle \\ \text{horizontal}}} \left[\delta_{s_i s_j} + w_1^*\ (1-\delta_{s_i s_j})\right] \prod_{\substack{\langle ij\rangle \\ \text{vertical}}} \left[\delta_{s_i s_j} + w_2^*\ (1-\delta_{s_i s_j})\right]. \tag{2.2.10}$$

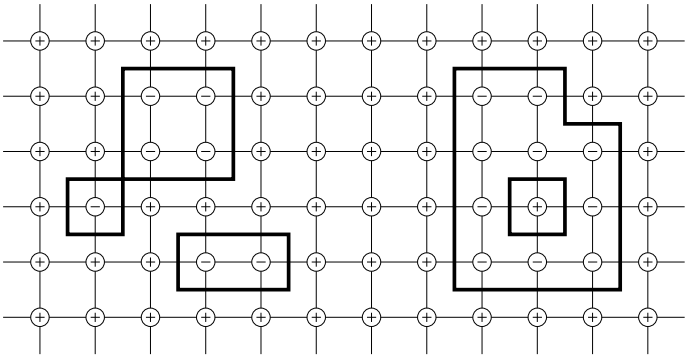

Figure 2.6: A spin configuration and the corresponding domain-wall configuration contributing to the low-temperature expansion.

2.2.3 Kramers-Wannier duality

TO DO:

2.2.4 Disorder operators

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2.3 Appendix: History

The so-called Ising model was suggested by Wilhelm Lenz in 1920 [Le20] as a simple model of ferromagnetism. The one-dimensional case was studied in detail by Lenz' Ph.D. student Ernst Ising in 1925 [Is25], who found that it exhibits no phase transition[1] at $T > 0$.

The situation is two dimensions is however much richer. The exact transition temperature on the square lattice was found in 1941 through a duality argument by Hendrik Kramers and Gregory Wannier [KW41]. This was followed by the exact solution for the free energy in 1944 by Lars Onsanger [On44]. The expression for the spontaneous magnetisation

$$M = \left(1 - (\sinh(2\beta J_1)\sinh(2\beta J_2))^{-2}\right)^{\frac{1}{8}} \tag{2.3.1}$$

and hence the exact value of the critical exponent $\beta = \frac{1}{8}$ was announced by Onsager in sibylline form in the discussion section at a conference in 1949 [On49], but a proof by Chen Ning Yang only appeared in written form in 1952 [Ya52].

Onsager's solution in terms of quaternion algebras is not easy reading, and it took researchers many years to extract from it the simplest and most convenient formulation of the algebraic facts that make an exact solution possible.[2] Also, it was not easy to see if there was any hope of generalising the solution to the experimentally most relevant case of three dimensions, or to solve more general classes of models (such as the Potts model).

For these reasons, many alternative—and simpler—solutions appeared subsequently. Among the most influential and useful we can mention the derivation of correlation functions in terms of Pfaffians by Montroll, Potts and Ward [MPW63], and the formulation of

[1] Perhaps as a consequence, he then decided to quit physics!

[2] With hindsight, one can now see in Onsager's paper [On44] the germs of what was later to be known as the Yang-Baxter equations—the most important ingredient in the study of integrable systems.

the Ising model as a quantum spin chain involving fermion operators

$$aa^\dagger + a^\dagger a = 1 \tag{2.3.2}$$

by Schultz, Mattis and Lieb [SML64]. It is this latter work that most clearly characterises the field-theoretical content of the Ising model: it is a theory of free fermions. The exact way in which the fermion sign problem is solved in the quantum spin chain makes precise the current understanding that the Ising model is not solvable in three or higher dimensions.

A closely related—but somehow simpler—fermionic formulation in terms of Grassmann variables

$$aa^* + a^* a = 0 \tag{2.3.3}$$

was found by Berezin [Be69]. We shall present this approach (following [Pl88]) below. A rather different alternative that makes direct contact with Combinatorics is an elegant reformulation by Kasteleyn [Ka63] of the Ising model as a dimer covering problem.

In some sense, the Ising model is to statistical physics what the hydrogen atom is to atomic physics. Although originally solved on a square lattice with periodic boundary conditions, the Ising model remains solvable when defined on other lattices, or when subjected to various kinds of modifications (such as the inclusion of certain interactions with the boundary or certain multi-spin interactions). For this reason it roles as a testing bed on which new theoretical ideas, approximation schemes or numerical calculations can be tried out.

Note also that despite of all the activity mentioned above (see the book [MW73] for a rather complete account as of 1973), seemingly simple questions about the Ising model remain unanswered to this day. For example, an exact solution in a rectangle with free boundary conditions does not seem to have been uncovered yet.

2.4 Appendix: Relation to dimer coverings

The Ising model, viewed as polygon configurations on the square lattice, can be related to dimer configurations on a decorated square lattice, as shown in Fig. 2.7. In each line, the occupation of external edges by polygons or dimers is identical.

There appears to be two problems about this bijection. First, the correspondence is not bijective, since a vertex with no polygons corresponds to three (not one) dimer configurations on the internal decoration. Second, the decorated lattice is non-planar and so it is not guaranteed to possess a Kasteleyn orientation.

Fortunately it turns out that these two apparent complications compensate one another, resulting in an exact equivalence. To see this, consider the orientation of the decorated lattice shown in Fig. 2.8. One can verify that with this orientation the orientation parity of all even cycles *without self-intersections* is odd. Thus, the orientation parity of the transition cycles connecting non-intersecting dimer configurations is odd as required. On the other hand, in the third line of Fig. 2.7, the orientation parity connecting either of the first two configurations with the third one is *even*, meaning that the third configuration is counted with a minus sign. This implies that the total count of the three configurations is $1 + 1 - 1 = 1$, and the bijection between polygon configurations and dimer coverings is established.

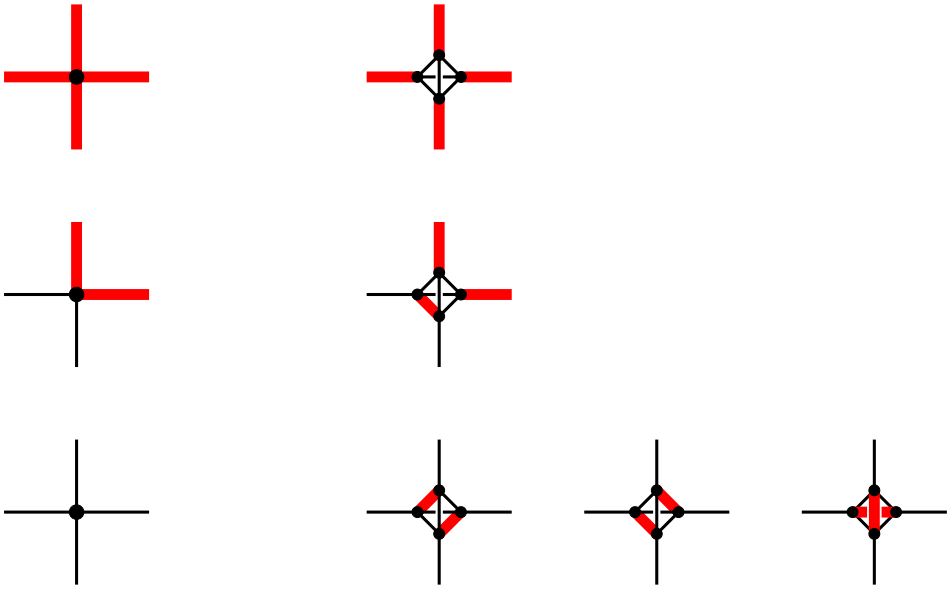

Figure 2.7: Relation between polygon configurations on the square lattice (left) and dimer coverings on a decorated square lattice (right).

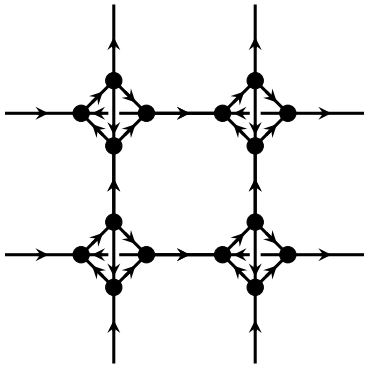

Figure 2.8: An oriented square lattice that permits us to solve the square-lattice Ising model as a dimer covering problem.

One can write down the matrix elements $d(k, k')$ of the matrix D by using the orientation of Fig. 2.8. When doing this, one can readily distinguish horizontal and vertical couplings. It turns out that D is not easily diagonalised for an $M \times N$ lattice with free boundary conditions. However, on the torus this is easily done (as usual one needs then four Pfaffians). Going through the analysis one finds finally the same expression for the free energy in the $M, N \to \infty$ limit as obtained by other methods [On44, SML64, Be69].

The problem of computing the Ising partition function on a rectangle with free boundary conditions appears to be an open problem to this date. It can however be done in the conformal limit [KV92].

The configurations of the zero-temperature antiferromagnetic Ising model on the triangular lattice are bijectively related to dimer coverings of the hexagonal lattice.

Chapter 3

The two-dimensional Ising model 2/2

TO DO: Simplifier argument (reseau carre uniquement), et developper la relation avec l'action de Majorana

3.1 Solution by lattice path integral

We now present a detailed solution of the Ising model using the rather different approach of Grassmann integrations [Be69]. The main motivation for this approach is that it will enable us to make a precise connection between the Ising model and free fermions. More generally, it is always convenient to understand exact solvability as the consequence of some underlying algebraic structure, and we want to make this algebraic link clear.

Rather than restricting to the square-lattice Hamiltonian we might as well consider a more general situation [Pl88] in which a general class of interactions between *pairs* of spins, all situated within the shaded triangle in Fig. 3.1.d, take place within the elementary cells of an underlying square lattice.

Define the normalised trace over a spin $\sigma_{m,n} = \pm 1$ at lattice position (m, n) as

$$\operatorname*{Tr}_{\sigma_{m,n}} (\cdots) = \frac{1}{2} \sum_{\sigma_{m,n}=\pm 1} (\cdots) \tag{3.1.1}$$

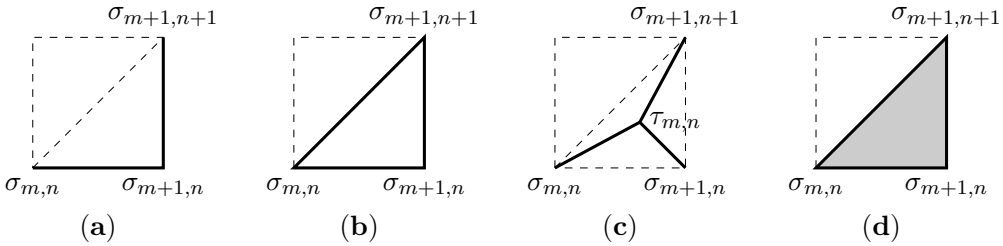

Figure 3.1: Four possible choices for the interactions within an elementary cell. This permits us to treat the a) square, b) triangular and c) hexagonal lattices in one single calculation. In d) the grey region stands for an arbitrary interaction between *pairs* of spins, possibly including one or more internal spins τ.

and the normalised trace over all spins on the $M \times N$ lattice as

$$\operatorname*{Tr}_{\sigma} = \prod_{m=1}^{M} \prod_{n=1}^{N} \operatorname*{Tr}_{\sigma_{m,n}} . \tag{3.1.2}$$

3.1.1 Parameterisation of the chosen lattice

We first show that the partition function of any Ising model with interactions as in Fig. 3.1 can be written, up to an unimportant multiplicative factor, as

$$Z = \operatorname*{Tr}_{\sigma} \left\{ \prod_{mn} (\alpha_0 + \alpha_1 \sigma_1 \sigma_2 + \alpha_2 \sigma_2 \sigma_3 + \alpha_3 \sigma_1 \sigma_3)_{mn} \right\} , \tag{3.1.3}$$

where

$$(\sigma_1, \sigma_2, \sigma_3)_{mn} = (\sigma_{m,n}, \sigma_{m+1,n}, \sigma_{m+1,n+1}) \tag{3.1.4}$$

and α_i is a set of four coefficients which can easily be determined for any given lattice.

Consider as an example the triangular lattice (see Fig. 3.1.b) with Hamiltonian

$$\mathcal{H} = -\sum_{m,n} (J_1 \sigma_{m,n} \sigma_{m+1,n} + J_2 \sigma_{m+1,n} \sigma_{m+1,n+1} + J_3 \sigma_{m,n} \sigma_{m+1,n+1}) . \tag{3.1.5}$$

Using the identity (2.2.2) three times, we find that the Boltzmann weight of an elementary cell is

$$\left(\mathrm{e}^{\beta(J_1 \sigma_1 \sigma_2 + J_2 \sigma_2 \sigma_3 + J_3 \sigma_1 \sigma_3)} \right)_{mn} = R \left[(1 + t_1 \sigma_1 \sigma_2)(1 + t_2 \sigma_2 \sigma_3)(1 + t_3 \sigma_1 \sigma_3) \right]_{mn} , \tag{3.1.6}$$

where we have set

$$\begin{aligned} R &= \cosh(\beta J_1) \cosh(\beta J_2) \cosh(\beta J_3) , \\ t_i &= \tanh(\beta J_i) , \text{ for } i = 1, 2, 3 . \end{aligned}$$

Expanding (3.1.6), and using $(\sigma_i)^2 = 1$, we find that indeed the partition function on the triangular lattice reads

$$Z_{\text{tri}} = (2R)^{MN} Z , \tag{3.1.7}$$

where Z is the general form (3.1.3) with coefficients

$$\begin{aligned} \alpha_0 &= 1 + t_1 t_2 t_3 , \\ \alpha_i &= t_i + t_{i+1} t_{i+2} , \text{ for } i = 1, 2, 3 \pmod 3 . \end{aligned} \tag{3.1.8}$$

In the most general setting of Fig. 3.1.d one must first obtain the form (3.1.3) by tracing over the spins in the interior of the shaded region. The simplest example of this is the hexagonal lattice, shown in Fig. 3.1.c. It is easily shown that

$$Z_{\text{hex}} = (4R)^{MN} Z , \tag{3.1.9}$$

with R as before and coefficients

$$\begin{aligned} \alpha_0 &= 1 , \\ \alpha_i &= t_i t_{i+1} , \text{ for } i = 1, 2, 3 \pmod 3 . \end{aligned} \tag{3.1.10}$$

3.1.2 Grassmann variables

We now introduce two Grassmann variables per site, $c_{m,n}$ and $c^*_{m,n}$. By definition, any two Grassmann variables anticommute

$$c_i c_j + c_j c_i = 0\,, \tag{3.1.11}$$

and in particular these variables are nilpotent

$$c_i^2 = c_j^2 = 0\,. \tag{3.1.12}$$

Any function of Grassmann variables is defined by its Taylor expansion, and because of the nilpotency such expansions are automatically truncated to the first order. The most general function of k Grassmann variables is defined by the coefficients of the 2^k possible monomials, e.g.,

$$f(c, c^*) = f_0 + f_1 c + f_2 c^* + f_3 c c^*\,. \tag{3.1.13}$$

Finally we introduce an integration measure, such that

$$\int \mathrm{d}c \cdot 1 = 0\,, \qquad \int \mathrm{d}c \cdot c = 1 \tag{3.1.14}$$

and extended by linearity. The differentials $\mathrm{d}c$ are themselves Grassmann variables and thus anticommute. Note that Grassmann integration works like ordinary differentiation.

In particular one finds

$$\int \mathrm{d}c^*\, \mathrm{d}c\, \mathrm{e}^{\lambda c c^*} f(c, c^*) = \lambda f_0 + f_3\,. \tag{3.1.15}$$

3.1.3 Density matrix

In parallel with the trace over spin variables (3.1.1)–(3.1.2) we introduce the trace over a pair of Grassmann variables

$$\operatorname*{Tr}_{c_{m,n}} (\cdots) = \int \mathrm{d}c^*_{m,n}\, \mathrm{d}c_{m,n}\, \mathrm{e}^{\lambda c_{m,n} c^*_{m,n}} (\cdots) \tag{3.1.16}$$

as a Gaussian integral with some weight factor λ. We shall also need the trace over all $2MN$ Grassmann variables, and it turns out that the relevant weight factor is $\lambda = \alpha_0$. We therefore define

$$\operatorname*{Tr}_{c} (\cdots) = \int \prod_{m=1}^{M} \prod_{n=1}^{N} \mathrm{d}c^*_{m,n}\, \mathrm{d}c_{m,n}\, \mathrm{e}^{\alpha_0 c_{m,n} c^*_{m,n}} (\cdots)\,. \tag{3.1.17}$$

Because of the anticommuting nature of the differentials, it is quite important to understand what order of integrations is implied by our notation. In (3.1.16) the first integral is over $\mathrm{d}c_{m,n}$, followed by a second integration over $\mathrm{d}c^*_{m,n}$. More generally, the first integration is over the rightmost differential. On the other hand, any fixed *pair* of Grassmann variables commutes with the whole algebra, so the order in which the product in (3.1.17) is written out with respect to the indexing variable has no importance. We shall soon encounter situations where this order is of the highest importance, so let us define that $\prod_{m=1}^{M}$ means writing the first term (with $m = 1$) to the left. Sometimes we shall need the opposite order, and in that case we would write $\prod_{m=M}^{1}$.

Let us first concentrate on the Boltzmann weight of a single elementary cell in (3.1.3), viz.

$$(P_{123})_{mn} = (\alpha_0 + \alpha_1\sigma_1\sigma_2 + \alpha_2\sigma_2\sigma_3 + \alpha_3\sigma_1\sigma_3)_{mn}\,. \tag{3.1.18}$$

This can be rewritten in factorised form

$$(P_{123})_{mn} = \int \mathrm{d}c^*_{mn}\,\mathrm{d}c_{mn}\mathrm{e}^{\alpha_0 c_{mn}c^*_{mn}}\,B^{(1)}_{m,n}B^{(2)}_{m+1,n}B^{(3)}_{m+1,n+1}\,, \tag{3.1.19}$$

where we have defined

$$\begin{aligned} B^{(1)}_{m,n} &= 1 + \frac{\alpha_1}{\sqrt{\eta}}c_{m,n}\sigma_{m,n}\,, \\ B^{(2)}_{m+1,n} &= 1 + \sqrt{\eta}(c_{m,n} + c^*_{m,n})\sigma_{m+1,n}\,, \\ B^{(3)}_{m+1,n+1} &= 1 + \frac{\alpha_2}{\sqrt{\eta}}c^*_{m,n}\sigma_{m+1,n+1}\,. \end{aligned} \tag{3.1.20}$$

We have here defined $\eta = \frac{\alpha_1\alpha_2}{\alpha_3}$. Note that in these expressions for B, the subscript refers to the spin variable σ, whereas the subscript of the fermion variables c is always m, n.

To prove this, we first rewrite (3.1.18) as

$$P_{123} = (\alpha_0 - \eta) + \eta\left(\frac{\alpha_1}{\eta}\sigma_1 + \sigma_2\right)\left(\sigma_2 + \frac{\alpha_2}{\eta}\sigma_3\right)\,,$$

where the subscript m, n has be omitted. By (3.1.15) we have

$$\begin{aligned} P_{123} &= \int \mathrm{d}c^*\,\mathrm{d}c\,\mathrm{e}^{(\alpha_0-\eta)cc^*}\left[1 + c\sqrt{\eta}\left(\frac{\alpha_1}{\eta}\sigma_1 + \sigma_2\right)\right]\times \\ &\qquad \left[1 + c^*\sqrt{\eta}\left(\sigma_2 + \frac{\alpha_2}{\eta}\sigma_3\right)\right]\,. \end{aligned}$$

Using now the nilpotency, each $[\cdots]$ can be factorised:

$$\begin{aligned} \left[1 + c\sqrt{\eta}\left(\frac{\alpha_1}{\eta}\sigma_1 + \sigma_2\right)\right] &= \left(1 + c\frac{\alpha_1}{\sqrt{\eta}}\sigma_1\right)(1 + c\sqrt{\eta}\sigma_2) \\ \left[1 + c^*\sqrt{\eta}\left(\sigma_2 + \frac{\alpha_2}{\eta}\sigma_3\right)\right] &= (1 + c^*\sqrt{\eta}\sigma_2)\left(1 + c^*\frac{\alpha_2}{\sqrt{\eta}}\sigma_3\right)\,. \end{aligned}$$

Multiplying now the 2nd and 3rd factors

$$(1 + c\sqrt{\eta}\sigma_2)\,(1 + c^*\sqrt{\eta}\sigma_2) = 1 + \sqrt{\eta}(c + c^*)\sigma_2 + \eta cc^*$$

one gets a commuting term ηcc^* which can be absorbed into the integration measure:

$$\mathrm{e}^{(\alpha_0-\eta)cc^*} + \eta cc^* = \mathrm{e}^{\alpha cc^*}\,.$$

Assembling the pieces, this implies

$$\begin{aligned} P_{123} &= \int \mathrm{d}c^*\,\mathrm{d}c\,\mathrm{e}^{\alpha_0 cc^*}\left(1 + c\frac{\alpha_1}{\sqrt{\eta}}\sigma_1\right)\times \\ &\qquad (1 + \sqrt{\eta}(c + c^*)\sigma_2)\left(1 + c^*\frac{\alpha_2}{\sqrt{\eta}}\sigma_3\right)\,. \end{aligned}$$

This establishes the factorisation (3.1.19)–(3.1.20).

In terms of the trace (3.1.17) we have therefore

$$Z = \operatorname*{Tr}_{\sigma} \hat{Q}\,, \tag{3.1.21}$$

$$\hat{Q} = \operatorname*{Tr}_{c} \left\{ \prod_{m,n} \left(B^{(1)}_{m,n} B^{(2)}_{m+1,n} B^{(3)}_{m+1,n+1} \right) \right\}, \tag{3.1.22}$$

where $\hat{Q}$ will be referred to as the density matrix.

3.1.4 Mirror factorisation

The strategy will now be to perform $\operatorname*{Tr}_{\sigma}$ while keeping $\operatorname*{Tr}_{c}$, so as to obtain a Grassmann representation of Z. This cannot be done directly with the form (3.1.22), since factors referring to the same $\sigma_{m,n}$ do not occur in adjacent positions in the product (cf. the different subscripts on the B factors). We therefore first aim at rearranging the product (3.1.22) in order to obtain the required adjacency.

It is convenient in this subsection to omit writing the integration over Grassmann variables. We thus write instead of (3.1.19)

$$(P_{123})_{mn} = B^{(1)}_{m,n} B^{(2)}_{m+1,n} B^{(3)}_{m+1,n+1}\,, \tag{3.1.23}$$

keeping in mind that the result will eventually be integrated. Note that the Grassmann pair $(c_{m,n}, c^*_{m,n})$ occurs *only* in this factor, and since terms in $c_{m,n}$ and $c^*_{m,n}$ will vanish under the integration—as in (3.1.15)—eventually only the commuting terms 1 and $c_{m,n}c^*_{m,n}$ will matter anyway. In this sense, the factor (3.1.23) can be considered to commute with the whole algebra.

Suppose that the products $\mathcal{O}_i\mathcal{O}_i^*$ are commuting terms, whereas individual factors are not. Then we can write

$$(\mathcal{O}_1\mathcal{O}_1^*)(\mathcal{O}_2\mathcal{O}_2^*)(\mathcal{O}_3\mathcal{O}_3^*) = (\mathcal{O}_1\mathcal{O}_1^*)(\mathcal{O}_2(\mathcal{O}_3\mathcal{O}_3^*)\mathcal{O}_2^*) = (\mathcal{O}_1(\mathcal{O}_2(\mathcal{O}_3\mathcal{O}_3^*)\mathcal{O}_2^*)\mathcal{O}_1^*)$$

and more generally

$$\prod_{i=1}^{L} \mathcal{O}_i\mathcal{O}_i^* = \prod_{i=1}^{L} \mathcal{O}_i \cdot \prod_{i=L}^{1} \mathcal{O}_i^*\,. \tag{3.1.24}$$

Using this property repeatedly suffices to bring (3.1.22) into the desired form.

Let us see in details how this is done. It is convenient sometimes to include factors $B^{(i)}_{m,n}$ in the product for which the indices $m = 0$ or $m = M+1$ (and $n = 0$ or $n = N+1$). Imposing formally that spins "beyond the boundary" vanish ($\sigma_{m,n} = 0$) we have $B^{(i)}_{m,n} = 1$, and the inclusion of such factors does not alter the result.

Consider first the product of a row of $(P_{123})_{mn}$ for fixed n. Using (3.1.24) we have

$$\prod_{m=0}^{M} (P_{123})_{mn} = \prod_{m=0}^{M} B^{(1)}_{m,n} B^{(2)}_{m+1,n} \cdot \prod_{m=M}^{0} B^{(3)}_{m+1,n+1}\,.$$

In the both factors on the right-hand side we can move the parenthesis and eliminate boundary terms:

$$\begin{aligned}\prod_{m=0}^{M} B^{(1)}_{m,n}B^{(2)}_{m+1,n} &= B^{(1)}_{0,n}\cdot\prod_{m=1}^{M} B^{(2)}_{m,n}B^{(1)}_{m,n}\cdot B^{(2)}_{M+1,n} = \prod_{m=1}^{M} B^{(2)}_{m,n}B^{(1)}_{m,n}\,,\\ \prod_{m=M}^{0} B^{(3)}_{m+1,n+1} &= B^{(3)}_{M+1,n+1}\cdot\prod_{m=M}^{1} B^{(3)}_{m,n+1} = \prod_{m=M}^{1} B^{(3)}_{m,n+1}\,. \end{aligned} \tag{3.1.25}$$

Thus we have for one row (neglecting boundary effects)

$$\prod_{m=1}^{M}(P_{123})_{mn} = \prod_{m=1}^{M} B^{(2)}_{m,n}B^{(1)}_{m,n}\cdot\prod_{m=M}^{1} B^{(3)}_{m,n+1}\,,$$

where now the m indices are nicely organised.

The n indices are still not the same, but this is settled by taking the product over n and rearranging the expression in the same way as we just saw. The result is

$$\prod_{n=1}^{N}\prod_{m=1}^{M}(P_{123})_{mn} = \prod_{n=1}^{N}\left[\prod_{m=M}^{1} B^{(3)}_{m,n}\cdot\prod_{m=1}^{M} B^{(2)}_{m,n}B^{(1)}_{m,n}\right]. \tag{3.1.26}$$

Putting back the trace over Grassmann variables, the density matrix therefore has the mirror factorised form

$$\hat{Q} = \operatorname*{Tr}_{c}\left\{\prod_{n=1}^{N}\left[\prod_{m=M}^{1} B^{(3)}_{m,n}\cdot\prod_{m=1}^{M} B^{(2)}_{m,n}B^{(1)}_{m,n}\right]\right\}. \tag{3.1.27}$$

3.1.5 Partition function

At the junction of the two products in (3.1.27) we have three B with the same subscripts $(m,n) = (1,n)$, i.e., referring to the same spin $\sigma_{m,n}$. We can now trace over that spin:

$$\begin{aligned}\operatorname*{Tr}_{\sigma_{m,n}}\left\{B^{(3)}_{m,n}B^{(2)}_{m,n}B^{(1)}_{m,n}\right\} &= \tfrac{1}{2}\textstyle\sum_{\sigma_{m,n}=\pm1}\left(1+\frac{\alpha_2}{\sqrt{\eta}}c^*_{m-1,n-1}\sigma_{m,n}\right)\times\\ &\quad\left(1+\sqrt{\eta}(c_{m-1,n}+c^*_{m-1,n})\sigma_{m,n})\right)\left(1+\frac{\alpha_1}{\sqrt{\eta}}c_{m,n}\sigma_{m,n}\right).\end{aligned}$$

Only terms in 1 and $\sigma^2_{m,n}$ survive the trace:

$$\begin{aligned}\ldots &= 1+(c_{m-1,n}+c^*_{m-1,n})(\alpha_1 c_{m,n}-\alpha_2 c^*_{m-1,n-1})+\alpha_3 c^*_{m-1,n-1}c_{m,n}\\ &= \exp\left[\alpha_3 c^*_{m-1,n-1}c_{m,n}+(c_{m-1,n}+c^*_{m-1,n})(\alpha_1 c_{m,n}-\alpha_2 c^*_{m-1,n-1})\right].\end{aligned}$$

For that same reason, this expression is quadratic in the fermion operators, hence commutes with the algebra. It can therefore be taken out in front of the expression.

In the remainder of the product the three B factors with $(m,n) = (2,n)$ are now adjacent, and we can trace next over that $\sigma_{m,n}$. Repeating the operation until nothing remains, we arrive at a purely fermionic expression for the partition function

$$\begin{aligned}Z &= \int\prod_{m=1}^{M}\prod_{n=1}^{N}\mathrm{d}c^*_{m,n}\,\mathrm{d}c_{m,n}\,\exp\left\{\sum_{m=1}^{M}\sum_{n=1}^{N}\left[\alpha_0 c_{m,n}c^*_{m,n}\right.\right.\\ &+ \left.\left.(c_{m-1,n}+c^*_{m-1,n})(\alpha_1 c_{m,n}-\alpha_2 c^*_{m-1,n-1})+\alpha_3 c^*_{m-1,n-1}c_{m,n}\right]\right\}.\end{aligned} \tag{3.1.28}$$

3.1.6 Diagonalisation

One can now diagonalise by performing a discrete Fourier transformation of the Grassmann variables. Let us suppose that $M = N \equiv L$:

$$c_{mn} = \frac{1}{L}\sum_{p=0}^{L-1}\sum_{q=0}^{L-1}\widetilde{c}_{pq}\exp\left(\frac{2\pi i}{L}(mp+nq)\right), \tag{3.1.29}$$

$$c^*_{mn} = \frac{1}{L}\sum_{p=0}^{L-1}\sum_{q=0}^{L-1}\widetilde{c}^*_{pq}\exp\left(-\frac{2\pi i}{L}(mp+nq)\right). \tag{3.1.30}$$

This is a rather standard exercise. Neglecting a few delicate effects having to do with the boundary, the result is

$$\begin{aligned} Z &= \prod_{p=0}^{L-1}\prod_{q=0}^{L-1}\Big[\left(\alpha_0^2+\alpha_1^2+\alpha_2^2+\alpha_3^2\right) - 2(\alpha_0\alpha_1-\alpha_2\alpha_3)\cos\left(\frac{2\pi p}{L}\right) \\ &- 2(\alpha_0\alpha_2-\alpha_1\alpha_3)\cos\left(\frac{2\pi q}{L}\right) - 2(\alpha_0\alpha_3-\alpha_1\alpha_2)\cos\left(\frac{2\pi(p+q)}{L}\right)\Big]^{1/2}. \end{aligned} \tag{3.1.31}$$

3.2 Thermodynamical limit

3.2.1 Partition function

In the thermodynamical limit we then have for the free energy

$$\begin{aligned} \frac{\log Z}{L^2} \overset{L\to\infty}{=} & \frac{1}{2}\int_0^{2\pi}\frac{\mathrm{d}p}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}q}{2\pi} \\ & \log\left[\left(\alpha_0^2+\alpha_1^2+\alpha_2^2+\alpha_3^2\right) - 2(\alpha_0\alpha_1-\alpha_2\alpha_3)\cos p\right. \\ & \left. -2(\alpha_0\alpha_2-\alpha_1\alpha_3)\cos q - 2(\alpha_0\alpha_3-\alpha_1\alpha_2)\cos(p+q)\right]. \end{aligned} \tag{3.2.1}$$

This expression can be specialised to yield the result for the square, triangular, hexagonal and other lattices by inserting the coefficients α_i.

One could think that this exact result could be converted into an enumeration of the lattice polygons of the low/high temperature expansions. This does not appear to be feasible in practice.

3.2.2 Critical point

There are several obvious symmetries in the expressions (3.1.31)–(3.2.1) allowing arbitrary permutations of the α_i and sign changes of any two of them, provided that one simultaneously shifts the integration angles p and q. A less obvious symmetry is to change α_i to the conjugated parameters α_i^* defined by

$$\begin{bmatrix}\alpha_0^*\\ \alpha_1^*\\ \alpha_2^*\\ \alpha_3^*\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix}. \tag{3.2.2}$$

Specialising to the square and triangular lattices, this transformation turns out to coincide with the Kramers-Wannier duality transformation.

At the critical point, (3.2.1) must exhibit a singularity. A moment's reflection shows that this can happen if and only if the argument $Q(p,q)$ of the logarithm vanishes for certain exceptional modes $(p,q) = (0,0), (0,\pi), (\pi,0), (\pi,\pi)$.[1] One can check the following rewriting:

$$Q(p,q) = \begin{bmatrix} \bar{\alpha}_0^2 & \bar{\alpha}_1^2 & \bar{\alpha}_2^2 & \bar{\alpha}_3^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \cos p \\ \cos q \\ \cos(p+q) \end{bmatrix}, \tag{3.2.3}$$

where we have defined

$$\bar{\alpha}_i = \alpha_0 - \alpha_i^* \,, \text{ for } i = 0,1,2,3\,. \tag{3.2.4}$$

In the product of the matrix and the right vector, three out of four trigonometric polynomials (i.e., the terms with coefficient $\bar{\alpha}_i^2$ for $i = 0,1,2,3$) vanish at any one of the exceptional (p,q) values. Therefore $Q(p,q) = 0$ if and only if the prefactor $(\alpha_0 - \alpha_i^*)^2$ of the last trigonometric polynomial vanishes. The criticality criterion can thus be written in compact form as

$$\bar{\alpha}_0\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3 = 0 \tag{3.2.5}$$

or equivalently

$$\alpha_0\alpha_1\alpha_2\alpha_3 = \alpha_0^*\alpha_1^*\alpha_2^*\alpha_3^*\,. \tag{3.2.6}$$

3.2.3 Critical exponents

The singularity in the free energy at the transition can be inferred by integrating (3.2.1) around one of the exceptional (p,q) values. Consider for instance (p,q) near $(0,0)$ and small $(\alpha_0 - \alpha_0^*)^2$. Taylor expanding the cosines to second order[2] we find for the free energy

$$\begin{aligned} -\beta f &= \frac{1}{8\pi^2} \iint\limits_{0\le p^2+q^2\le r^2} \mathrm{d}p\,\mathrm{d}q \\ &\qquad \log\left[4(\alpha_0-\alpha_0^*)^2 + A_1\frac{p^2}{2} + A_2\frac{q^2}{2} + A_3\frac{(p+q)^2}{2}\right] + \cdots \end{aligned} \tag{3.2.7}$$

with $A_1 = 2(\alpha_0\alpha_1 - \alpha_2\alpha_3)$, $A_2 = 2(\alpha_0\alpha_2 - \alpha_1\alpha_3)$, and $A_3 = 2(\alpha_0\alpha_3 - \alpha_1\alpha_2)$. Integrating this yields

$$-\beta f = \frac{(2\bar{\alpha}_0)^2}{4\pi\sqrt{A_1A_2 + A_2A_3 + A_1A_3}} \log\left(\frac{1}{(2\bar{\alpha}_0)^2}\right) + \cdots, \tag{3.2.8}$$

where the argument of the square root can be evaluated at criticality.

More generally, for the critical point $\bar{\alpha}_i = 0$ the dominant singular behaviour of the free energy reads

$$-\beta f_{\text{sing}} = \frac{(2\bar{\alpha}_i)^2}{16\pi\sqrt{(\alpha_0\alpha_1\alpha_2\alpha_3)_{\text{c}}}} \log\left(\frac{1}{(2\bar{\alpha}_i)^2}\right). \tag{3.2.9}$$

We infer that the singularity of the specific heat $C = -\beta^2\frac{\partial^2}{\partial\beta^2}(\beta f)$ is

$$C_{\text{sing}} \overset{T\to T_{\text{c}}}{\sim} A_{\text{c}} \log\left|\frac{T_{\text{c}}}{T - T_{\text{c}}}\right|, \tag{3.2.10}$$

[1] This will become clear when we compute the singular part of the free energy f_{sing} below.

[2] It is indeed the absense of first order terms that leads to the singular behaviour.

where the specific-heat critical amplitude is

$$A_{\rm c} = \frac{\beta_{\rm c}^2}{\pi\sqrt{(\alpha_0\alpha_1\alpha_2\alpha_3)_{\rm c}}} \left(\frac{{\rm d}\bar{\alpha}_j(\beta)}{{\rm d}\beta}\right)_{\rm c}^2 . \tag{3.2.11}$$

Thus, the standard critical exponent $\alpha = 0$, but with a logarithmic divergence (i.e., weaker than the usual power-law one).

The Grassmann approach can also be used to compute correlation functions. For example, the one-point function gives the spontaneous magnetisation, $M = \langle\sigma_{m,n}\rangle$. One finds [Pl88] a surprisingly simple expression for the eight power:

$$M^8 = \begin{cases} 1 - \frac{\alpha_0^*\alpha_1^*\alpha_2^*\alpha_3^*}{\alpha_0\alpha_1\alpha_2\alpha_3} & \text{for } T \le T_{\rm c} \\ 0 & \text{for } T \ge T_{\rm c} \end{cases} \tag{3.2.12}$$

This implies that

$$M \stackrel{T\to T_{\rm c}}{\sim} \left|\frac{T - T_{\rm c}}{T_{\rm c}}\right|^{1/8} , \tag{3.2.13}$$

so that the critical exponent $\beta = \frac{1}{8}$.

3.2.4 Fermionic action

To exhibit precisely the fermionic nature of the Ising model, we wish to find the continuum limit of the action S appearing under the exponential in (3.1.28).

We first rewrite the expression for S so that finite differences appear whereever possible:

$$\begin{aligned} S &= \sum_{m,n} \big\{ \widetilde{m} c_{m,n} c^*_{m,n} + \alpha_1 c_{m,n}(c^*_{m,n} - c^*_{m-1,n}) \\ &+ \alpha_2 c_{m,n}(c^*_{m,n} - c^*_{m,n-1}) + \alpha_3 c_{m,n}(c^*_{m,n} - c^*_{m-1,n-1}) \qquad (3.2.14) \\ &+ \alpha_1 c_{m,n}(c_{m,n} - c_{m-1,n}) + \alpha_2 c^*_{m,n}(c^*_{m,n} - c^*_{m,n-1}) \big\} , \qquad (3.2.15) \end{aligned}$$

where $\widetilde{m} = \alpha_0 - \alpha_1 - \alpha_2 - \alpha_3 = 2\bar{\alpha}_0$. We have here used nilpotency and anticommutativity, and some of the summation indices have been shifted by one unit. Introduce now finite lattice derivatives

$$\partial_m x_{m,n} = x_{m,n} - x_{m-1,n} , \qquad \partial_n x_{m,n} = x_{m,n} - x_{m,n-1} , \tag{3.2.16}$$

so that $x_{m,n} - x_{m-1,n-1} = (\partial_m + \partial_n - \partial_m\partial_n)x_{m,n}$. This gives

$$\begin{aligned} S &= \sum_{m,n} \{ \widetilde{m} c_{mn} c^*_{mn} + \lambda_1 c_{mn}\partial_m c^*_{mn} + \lambda_2 c_{mn}\partial_n c^*_{mn} \\ &- \alpha_3 c_{mn}\partial_m\partial_n c^*_{mn} + \alpha_1 c_{mn}\partial_m c_{mn} + \alpha_2 c^*_{mn}\partial_n c^*_{mn} \} , \end{aligned} \tag{3.2.17}$$

where $\lambda_1 = \alpha_1 + \alpha_3$ and $\lambda_2 = \alpha_2 + \alpha_3$.

We now take the continuum limit $(m,n) \to (x_1, x_2) \equiv x$, so that $\partial_m \to \frac{\partial}{\partial x_1} \equiv \partial_1$ and $\partial_n \to \frac{\partial}{\partial x_2} \equiv \partial_2$. The continuum limit of the Grassmann variables defines a two-component field: $c_{mn} \to \psi(x)$ and $c^*_{mn} \to \bar{\psi}(x)$. This leads to a slightly non-standard form of the fermionic action. However, if we rotate the derivatives

$$\partial = \frac{1}{2}(\partial_1 - i\partial_2) , \qquad \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2) \tag{3.2.18}$$

and similarly rotate the field components, we arrive at

$$S = \int \mathrm{d}^2x \, \left\{ i\mathcal{M}\psi(x)\bar{\psi}(x) + \psi(x)\partial\psi(x) + \bar{\psi}(x)\bar{\partial}\bar{\psi}(x) \right\} , \tag{3.2.19}$$

with the rescaled mass

$$\mathcal{M} = \frac{\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3}{\left(2\sqrt{(\alpha_0\alpha_1\alpha_2\alpha_3)_{\mathrm{c}}}\right)^{1/2}} . \tag{3.2.20}$$

Note that (3.2.19) has the standard form for the action of a Majorana fermion.

According to (3.2.5) the mass $\mathcal{M}$ vanishes at the critical point. We can therefore conclude that the critical Ising model corresponds, in the continuum limit, to a massless Majorana fermion. This furnishes a direct link between the lattice model and (conformal) field theory.

We see moreover that all different lattices that we have treated on the same footing are in the same universality class, since the coefficients α_i only enter into the mass $\mathcal{M}$.

Needless to say, one needs to be a little more careful when all the coupling constants are not positive. For instance, it is an interesting exercise to track down where the preceding argument should be changed when dealing with antiferromagnetic cases, such as the $T = 0$ Ising model on the triangular lattice (which is known to belong to a different universality class).