8 Potts model

When writing the integrable *R*-matrix of the six-vertex model, in (6.18), we have briefly come across a new type of algebraic structure: the Temperley-Lieb algebra. This is an example of a lattice algebra [Ma91], or more generally a partition algebra [HR05]. Other examples include the dilute Temperley-Lieb algebra, the Brauer algebra, and various types of multi-colour braid-monoid algebras [GP93].

The R-matrix based on each of these algebras generates the transfer matrix of a corresponding statistical mechanics model. Obviously one can gather important information about the statistical mechanics model by studying the underlying algebra and its representation theory.

In what follows we shall focus on the (open, or non-periodic) Temperley-Lieb (TL) algebra. This algebra has many different representations, each of which is related to a particular stat-mech model: Potts, Ising, six-vertex, restricted solid-on solid (RSOS) model,.... Historically, each model was introduced independently, but with hindsight the unifying algebraic framework can be used to understand better the relations among them.

Most of the corresponding representations of the TL algebra are not faithful, i.e., they obey additional relations than those defining the TL algebra. The Potts model—to be precise: in its formulation as a loop model—furnishes a faithful representation. Since it is also an extremely interesting and wellstudied model in statistical mechanics, it is natural to study it in some detail. The selfdual Potts model on a square lattice will turn out to be closely related to the six-vertex model, so that the results of preceding chapters imply the exact solution of the Potts model. Also on the triangular lattice can the selfdual model be exactly solved.

For the mathematically inclined, let us briefly mention an important connection to representation theory. A central result, known as Schur-Weyl duality, states that:

- 1. The general linear group $GL_n(\mathbb{C})$ and the symmetric group \mathfrak{S}_k both act on the tensor product $V^{\otimes k}$ with dim V = n. (We interpret $V^{\otimes k}$ as the quantum space.)
- 2. These two actions commute and each action generates the full centraliser of the other.
- 3. As a $(GL_n(\mathbb{C}), \mathfrak{S}_k)$ -bimodule, the tensor space has a multiplicity free

decomposition

$$V^{\otimes k} \simeq \bigoplus_{\lambda} L_{GL_n}(\lambda) \otimes L_{\mathfrak{S}_k}(\lambda) , \qquad (8.1)$$

where $L_{GL_n}(\lambda)$ are irreducible $GL_n(\mathbb{C})$ -modules and $L_{\mathfrak{S}_k}(\lambda)$ are irreducible \mathfrak{S}_k -modules.

Similar results hold when taking subgroups of $GL_n(\mathbb{C})$, in which case the centraliser algebras become bigger. The TL algebra occurs in this hierarchy of dualities, and its centraliser is the quantum algebra $U_q(\mathfrak{sl}_2)$.

In this chapter we define the Potts model in various representations and exhibit its equivalence to the six-vertex model. Even though we are mainly interested in the model defined on a square lattice, it turns out that many of the transformations that we need hold when the model on more general graphs. Since it hardly more complicated—and a lot more instructive—to work in the "correct" generality, we shall choose to do so and specialise only when needed.

8.1 Spin representation

Let G = (V, E) be an arbitrary connected graph with vertex set V and edge set E. The Q-state Potts model is initially defined by assigning a spin variable σ_i to each vertex $i \in V$. Each spin can take Q different values, by convention chosen as $\sigma_i = 1, 2, \ldots, Q$. We denote by σ the collection of all spin variables on the graph. Two spins i and j are called nearest neighbours if they are incident on a common edge $e = (ij) \in E$. In any given configuration σ , a pair of nearest neighbour spins is assigned an energy -J if they take identical values, $\sigma_i = \sigma_j$. The Hamiltonian (dimensionless energy functional) of the Potts model is thus

$$\mathcal{H} = -K \sum_{(i,j)\in E} \delta(\sigma_i, \sigma_j) \,. \tag{8.2}$$

where the Kronecker delta function is defined as

$$\delta(\sigma_i, \sigma_j) = \begin{cases} 1 & \text{if } \sigma_i = \sigma_j \\ 0 & \text{otherwise} \end{cases}$$
(8.3)

and $K = J/k_{\rm B}T$ is a dimensionless coupling constant (interaction energy).

The case Q = 2 corresponds to the Ising model. Indeed, if $S_i = \pm 1$ we have

$$2\delta(S_i, S_j) = S_i S_j + 1.$$
(8.4)

The second term amounts to an unimportant shift of the interaction energy, and so the models are equivalent if we set $K_{\text{Potts}} = 2K_{\text{Ising}}$.

The thermodynamic information about the Potts model is encoded in the partition function

$$Z = \sum_{\sigma} e^{-\mathcal{H}} = \sum_{\sigma} \prod_{(ij)\in E} e^{K\delta(\sigma_i,\sigma_j)}$$
(8.5)

and in various correlation functions. By a correlation function we understand the probability that a given set of vertices are assigned fixed values of the spins.

In the ferromagnetic case K > 0 the spins tend to align at low temperatures $(K \gg 1)$, defining a phase of ferromagnetic order. Conversely, at high temperatures $(K \ll 1)$ the spins are almost independent, leading to a paramagnetic phase where entropic effects prevail. On physical grounds, one expects the two phases to be separated by a critical point K_c where the effective interactions between spins becomes long ranged.

For certain regular planar lattices K_c can be determined exactly by duality considerations. Moreover, K_c will turn out to be the locus of a second order phase transition if $0 \le Q \le 4$. In that case the Potts model enjoys conformal invariance in the limit of an infinite lattice, allowing its critical properties to be determined exactly by a variety of techniques. These properties turn out to be *universal*, i.e., independent of the lattice used for defining the model microscopically.

8.2 Fortuin-Kasteleyn cluster representation

The initial definition (8.2) of the Potts model requires the number of spins Q to be a positive integer. It is possible to rewrite the partition function and correlation functions so that Q appears only as a parameter. This makes its possible to assign to Q arbitrary real (or even complex) values.

Notice first that by (8.3) we have the identity

$$e^{K\delta(\sigma_i,\sigma_j)} = 1 + v\delta(\sigma_i,\sigma_j), \qquad (8.6)$$

where we have defined $v = e^{K} - 1$. Now, it is obvious that for any edgedependent factors h_e one has

$$\prod_{e \in E} (1+h_e) = \sum_{E' \subseteq E} \prod_{e \in E'} h_e \,. \tag{8.7}$$

where the subset E' is defined as the set of edges for which we have taken the term h_e in the development of the product $\prod_{e \in E}$. In particular, taking $h_e = v\delta(\sigma_i, \sigma_j)$ we obtain for the partition function (8.5)

$$Z = \sum_{E' \subseteq E} v^{|E'|} \sum_{\sigma} \prod_{(ij) \in E'} \delta(\sigma_i, \sigma_j) = \sum_{E' \subseteq E} v^{|E'|} Q^{k(E')}, \qquad (8.8)$$

where k(E') is the number of connected components in the graph G' = (V, E'), i.e., the graph obtained from G by removing the edges in $E \setminus E'$. Those connected components are called *clusters*, and (8.8) is the Fortuin-Kasteleyn *cluster representation* of the Potts model partition function. The sum over spins σ in (8.5) has now been replaced by a sum over edge subsets, and Q appears as a parameter in (8.8) and no longer as a summation limit.

8.3 Duality of the partition function

Consider now the case where G = (V, E) is a connected *planar* graph. Any planar graph possesses a *dual graph* $G^* = (V^*, E^*)$ which is constructed by placing a dual vertex $i^* \in V^*$ in each face of G, and connecting a pair of dual vertices by a dual edge $e^* \in E^*$ if and only if the corresponding faces are adjacent in G. In other words, there is a bijection between edges and dual edges, since each edge $e \in E$ intersects precisely one dual edge $e^* \in E^*$. Note that by the Euler relation

$$|V| + |V^*| = |E| + 2. (8.9)$$

By construction, the dual graph is also connected and planar. Note also that duality is an involution, i.e., $(G^*)^* = G$.

The Euler relation can easily be proved by induction. If $E = \emptyset$, since G was supposed connected we must have $|V| = |V^*| = 1$, so (8.9) indeed holds. Each time a further edge is added to E, there are two possibilities. Either it connects an existing vertex to a new vertex, in which case |V| increases by one and $|V^*|$ is unchanged. Or it connects two existing vertices, meaning that a cycle is closed in G. In this case |V| is unchanged and V^* increases by one. In both cases (8.9) remains valid.

Recalling the cluster representation (8.8)

$$Z_{G}(Q, v) = \sum_{E_{1} \subseteq E} v^{|E_{1}|} Q^{k(E_{1})}$$
$$Z_{G^{*}}(Q, v^{*}) = \sum_{E_{2} \subseteq E^{*}} (v^{*})^{|E_{2}|} Q^{k(E_{2})}$$
(8.10)

we now claim that it is possible to chose v^* so that

$$Z_G(Q, v) = k Z_{G^*}(Q, v^*)$$
(8.11)

where k is an unimportant multiplicative constant.

To prove this claim, we show that the proportionality (8.11) holds term by term in the summations (8.10). To this end, we first define a bijection between the terms by $E_2 = (E \setminus E_1)^*$, i.e., an edge is present in E_1 if its dual edge is absent from E_2 , and vice versa. This implies

$$|E_1| + |E_2| = |E|. (8.12)$$

We have moreover the topological identity for the induced (not necessarily connected) graphs $G_1 = (V, E_1)$ and $G_2 = (V^*, E_2)$

$$k(E_1) = |V| - |E_1| + c(E_1) = |V| - |E_1| + k(E_2) - 1, \qquad (8.13)$$

where we $k(E_1)$ and $c(E_1)$ are respectively the number of connected components and the number of independent cycles²³ in the graph G_1 .

The proof of (8.13) is again by induction. If $E_1 = \emptyset$, we have $k(E_1) = |V|$, $c(E_1) = 0$ and $k(E_2) = 1$. Each time an edge is added to E_1 there are two possibilities. Either $c(E_1)$ stays constant, in which case $k(E_1)$ is reduced by one and $k(E_2)$ is unchanged. Or $c(E_1)$ increases by one, in which case $k(E_1)$ is unchanged and $k(E_2)$ increases by one. In both cases (8.13) remains valid.

²³The number of independent cycles—also known as the circuit rank, or the cyclomatic number—is the smallest number of edges to be removed from a graph in order that no graph cycle remains.

Combining (8.12)–(8.13) gives

$$v^{|E_1|}Q^{k(E_1)} = k (v^*)^{|E_2|}Q^{k(E_2)}, \qquad (8.14)$$

where we have defined

$$k = Q^{1-|V^*|}v^{|E|} = Q^{|V|-|E|-1}v^{|E|}$$
(8.15)

and $v^* = Q/v$. Comparing (8.14) with (8.10) completes the demonstration of (8.11) and furnishes the desired duality relation

$$vv^* = Q. ag{8.16}$$

The duality relation (8.16) is particularly useful when the graph is selfdual, $G^* = G$. This is the case of the regular square lattice. Assuming the uniqueness of the phase transition, the critical point is given by the selfdual coupling:

$$v_{\rm c} = \pm \sqrt{Q}$$
 (square lattice) (8.17)

In the Ising case Q = 2, the solution $v_c = +\sqrt{Q}$ gives $K_c = \log(\sqrt{2} + 1)$ in agreement with (4.12).

8.4 Special cases

One of the strengths of the Q-state Potts model is that it contains a large number of interesting special cases. Many of those make manifest the geometrical content of the partition function (8.8). The equivalence between Q = 2 and the Ising model has already been discussed. We shall concentrate here on a couple of more subtle equivalences, that explicitly exploit the fact that Q can now be used as a continuous variable.

8.4.1 Bond percolation

For Q = 1 the Potts model is seemingly trivial, with partition function $Z = (1+v)^{|E|}$. Instead of setting Q = 1 brutally, one can however consider taking the *limit* $Q \to 1$. This leads to the important special case of *bond* percolation.

Let $p \in [0,1]$ and set v = p/(1-p). We then consider the rescaled partition function

$$\widetilde{Z}(Q) \equiv (1-p)^{|E|} Z = \sum_{E' \subseteq E} p^{|E'|} (1-p)^{|E|-|E'|} Q^{k(E')}.$$
(8.18)

We have of course $\widetilde{Z}(1) = 1$. But formally, what is written here is that each edge is present in E' (i.e., percolating) with probability p and absent (i.e., non percolating) with probability 1 - p. Appropriate correlation functions and derivatives of $\widetilde{Z}(Q)$ in the limit $Q \to 1$ furnish valuable information about the geometry of the percolation clusters. For instance

$$\lim_{Q \to 1} Q \frac{\mathrm{d}\bar{Z}(Q)}{\mathrm{d}Q} = \langle k(E') \rangle \tag{8.19}$$

gives the average number of clusters.

8.4.2 Trees and forests

Using (8.13), and defining $w = \frac{Q}{v}$, one can rewrite (8.8) as

$$Z = \sum_{E_1 \subseteq E} \left(\frac{Q}{w}\right)^{|V| + c(E_1) - k(E_1)} Q^{|V| - |E_1| + c(E_1)}$$
$$= v^{|V|} \sum_{E_1 \subseteq E} w^{k(E_1) - c(E_1)} Q^{c(E_1)}.$$
(8.20)

Take now the limit $Q \to 0$ and $v \to 0$ in such a way that the ratio w = Q/v is fixed and finite, and consider the rescaled partition function $\widetilde{Z} = Zv^{-|V|}$. The limit $Q \to 0$ will suppress any term with $c(E_1) > 0$, and we are left with

$$\widetilde{Z} = \sum_{E_1 \subseteq E}' w^{k(E_1)},$$
(8.21)

where the prime indicates that the summation is over edge sets such that the graphs $G_1 = (V, E_1)$ have no cycles, $c(E_1) = 0$. Such graphs are known as forests, or more precisely (since the vertex set V is that of G), spanning forests of G. Each connected component carries a weight w.

For $w \to 0$, the surviving terms are spanning trees, i.e., forests with a single connected component. Note that the critical curve on the square lattice (8.17) goes through the point (Q, v) = (0, 0) with a vertical tangent (i.e., $w \to 0$) and thus describes spanning trees.

8.5 Loop representation

We now transform the Potts model defined on a planar graph G into a model of self-avoiding loops on a related graph $\mathcal{M}(G)$, known in graph theory as



Figure 19: (a) A planar graph G (black circles and solid lines) and its dual graph G^* (white circles and dashed lines). (b) The medial graph $\mathcal{M}(G) = \mathcal{M}(G^*)$. (c) The plane quadrangulation $\widehat{G} = \mathcal{M}(G)^*$.

the medial graph. Each term E' in the cluster representation (8.8) is in bijection with a term in the loop representation. The correspondence is, roughly speaking, that the loops turn around the connected components in G' = (V, E') as well as their elementary internal cycles. More precisely, the loops separate the clusters from their duals.

To make this transformation precise, we first need to define the medial graph $\mathcal{M}(G) = (\tilde{V}, \tilde{E})$ carefully. Let G = (V, E) be a connected planar graph with dual $G^* = (V^*, E^*)$. The pair (G, G^*) can be drawn in the plane such that each edge $e \in E$ intersects its corresponding dual edge $e^* \in E^*$ exactly once, see Figure 19a. To each of these intersections corresponds a vertex $\tilde{i} \in \tilde{V}$ of $\mathcal{M}(G)$.

Consider now the union $G \cup G^*$. This is in fact a quadrangulation of the plane. Each quadrangle consists of a pair of half-edges and one vertex from G, and a pair of half-edges and one vertex from G^* . These two pairs of half-edges meet in a pair of vertices from \widetilde{V} . An edge of $\mathcal{M}(G)$ is drawn diagonally inside each quadrangle, joining the pair of vertices from \widetilde{V} . This defines the edge set \widetilde{E} and completes the definition of the medial graph. An example is shown in Figure 19b.

It is manifest in these definitions that G and G^* are used in a completely



Figure 20: Square and triangular lattices (solid lines) with their corresponding medial lattices (dashed lines).

symmetric way. Thus, a graph and its dual has the same medial graph, $\mathcal{M}(G) = \mathcal{M}(G^*)$. Moreover, it is easy to see that every vertex of $\mathcal{M}(G)$ has degree four.²⁴

The medial of the square lattice is another (tilted) square lattice. The medial of the triangular lattice (or of its dual hexagonal lattice) is known as the kagome lattice.²⁵ These two medial lattices, shown in Figure 20, are particularly important for subsequent applications.

To each term E' in appearing in the sum (8.8) we now define a system of self-avoiding loops that completely cover the edges of $\mathcal{M}(G)$. Let $\tilde{i} \in \tilde{V}$ be a vertex of $\mathcal{M}(G)$ and write its adjacent (half)edges from E, E^* and \tilde{E} in cyclic order as $\tilde{e}_1 e \tilde{e}_2 e^* \tilde{e}_3 e \tilde{e}_4 e^*$. Now if $e \in E'$, link up the half-edges of \tilde{E} in two pairs as $(\tilde{e}_4 \tilde{e}_1)(\tilde{e}_2 \tilde{e}_3)$. Conversely, if $e \in E \setminus E'$, we link $(\tilde{e}_1 \tilde{e}_2)(\tilde{e}_3 \tilde{e}_4)$. Note that we do not allow the non-planar (crossing) linking $(\tilde{e}_1 \tilde{e}_3)(\tilde{e}_2 \tilde{e}_4)$. The set of linkings at all vertices \tilde{V} defines the desired system of loops.

In concrete terms, this definition means that the loops bounce off all edges E' and cut through the corresponding dual edges. The complete correspondence is illustrated in Figure 21.

To complete the transformation, note that the number of loops l(E') is the sum of the number of connected components k(E') and the number of

²⁴This implies that the dual of $\mathcal{M}(G)$ is a quadrangulation \widehat{G} , which is however different from the quadrangulation $G \cup G^*$. See Figure 19c. The Potts model admits yet another representation, namely as a height model—or RSOS model—on \widehat{G} .

²⁵Literally "eye basket". This refers to a type of traditional Japanese wicker basket weave.



Figure 21: (a) A subset $E' \subseteq E$ (thick black solid lines) and its complementary subset $(E')^* \subseteq E^*$ (thick black dashed lines). (b) The corresponding system of self-avoiding loops on the medial graph (blue curves).

independent cycles c(E'):

$$l(E') = k(E') + c(E').$$
(8.22)

Inserting this and the topological identity (8.13), which reads in the present notation

$$k(E') = |V| - |E'| + c(E'), \qquad (8.23)$$

into (8.8) we arrive at

$$Z = Q^{|V|/2} \sum_{E' \subseteq E} x^{|E'|} Q^{l(E')/2}, \qquad (8.24)$$

where we have defined $x = vQ^{-1/2}$.

This is the *loop representation* of the Potts partition function. It importance stems from the fact that the loops, their local connectivities (called linkings in the above argument), and the non-local quantity l(E') all admit an algebraic interpretation within the Temperley-Lieb algebra.

The duality transformation in the loop representation consists in shifting the linking at each vertex cyclically by one step. In terms of the x variables the duality relation (8.16) reads simply

$$xx^* = 1.$$
 (8.25)

In the case of the square lattice, the self-dual points are $x_c = \pm 1$, and the usual critical point is $x_{c+} = 1$. The loop model (8.24) then becomes extremely simple: there is just a weight $n = \sqrt{Q}$ for each loop.

8.6 Vertex model representation

In the definition of the Q-state Potts model, Q was originally a positive integers. However, in the corresponding loop model (8.24) it appears as formal parameters and may thus take arbitrary complex values. The price to pay for this generalisation is the appearance of a non-locally defined quantity, the number of loops l. The locality of the model may be recovered by transforming it to a vertex model with complex Boltzmann weights as we now show.

The following argument supposes that G = (V, E) is a (connected) planar graph. Most applications however suppose a regular lattice, a situation to which we shall return shortly.

Consider any model of self-avoiding loops defined on G (or some related graph, such as the medial graph $\mathcal{M}(G)$ for the Potts model). The Boltzmann weights are supposed to consist of a local piece—depending on if and how the loops pass through a given vertex—and a non-local piece of the form n^l , where n is the loop weight and l is the number of loops. In the case of the Potts model we have $n = \sqrt{Q}$.

In a first step, each loop is independently decorated by a global orientation $s = \pm 1$, which by planarity and self-avoidance can be described as either counterclockwise (s = 1) or clockwise (s = -1). If each oriented loop is given a weight w(s), we have the requirement

$$n = w(1) + w(-1). (8.26)$$

An obvious possibility, sometimes referred to as the *real loop ensemble*, is w(1) = w(-1) = n/2. This can be interpreted as an O(n/2) model of complex spins.

We are however more interested in the *complex loop ensemble* with $w(s) = e^{is\gamma}$. Note that in the expected critical regime,

$$n = 2\cos\gamma \in [-2, 2],$$
 (8.27)

the parameter $\gamma \in [0, \pi]$ is real. Locality is retrieved by remarking that the weights $w(\pm 1)$ are equivalent to assigning a local weight $w(\alpha/2\pi)$ each time the loop turns an angle α (counted positive for left turns).

Note that a planar graph cannot necessarily be drawn in the plane in such a way that all edges are straight line segments. Therefore, the local weights $w(\alpha/2\pi)$ must in general be assigned both to vertices and to edges. However, it is certainly possible to redraw the graph so that each edge is a succession of several straight line segments. Introducing auxiliary vertices of degree two at the places where two segments join up, the weight for turning can be assigned to those auxiliary vertices. In that sense, any planar graph admits a local redistribution of the loop weight, with local weights $w(\alpha/2\pi)$ assigned only to vertices.

The loop model is now transformed into a *local vertex model* by assigning to each edge traversed by a loop the orientation of that loop. An edge not traversed by any loop is assigned no orientation. The total vertex weight is determined from the configuration of its incident oriented edges: it equals the above local loop weights summed over the possible linkings of oriented loops which are compatible with the given edge orientations. In addition, one must multiply this by any loop-independent local weights, such as x in (8.24).

8.7 Six-vertex model

To see how this is done, we finally specialise to the Potts model defined on the square lattice G. The loop model is defined on the corresponding medial lattice $\mathcal{M}(G)$ which is another (tilted) square lattice. Each edge of the lattice is visited by a loop, and two loop segments (possibly parts of the same loop) meet at each vertex. In the oriented loop representation, each vertex is therefore incident on two outgoing and two ingoing edges.

It is convenient for the subsequent discussion to make the couplings of the Potts model anisotropic. In its original spin formulation (8.5) we therefore let K_1 (resp. K_2) denote respectively the dimensionless coupling in the horizontal (resp. vertical) direction of the square lattice, and we let

$$x_1 = \frac{\mathrm{e}^{K_1} - 1}{\sqrt{Q}}, \qquad x_2 = \frac{\mathrm{e}^{K_2} - 1}{\sqrt{Q}}$$
 (8.28)

be the corresponding parameters appearing in the loop representation (8.24). Note that in all the results obtained this far it is straightforward to generalise to completely inhomogeneous edge dependent couplings, and the only reason that we have chosen not to present the results in this generality is that it tends to make notations slightly cumbersome.



Figure 22: The allowed arrow arrangements (top) around a vertex that define the six-vertex model, with the corresponding particle trajectories (bottom).

The six possible configurations of arrows around a vertex of the medial lattice $\mathcal{M}(G)$ are shown in Fig. 22 (obtained simply by rotating Fig. 15 through $\frac{\pi}{4}$). The corresponding vertex weights are denoted ω_p (resp. ω'_p) on the even (resp. odd) sublattice of $\mathcal{M}(G)$. By definition, a vertex of the even (resp. odd) sublattice of $\mathcal{M}(G)$ is the mid point of an edge with coupling K_1 (resp. K_2) of the original spin lattice G. With respect to Figure 22 we define the even sublattice to be such that an edge $e \in E$ occupies the upper-left and lower-right quadrants, and the corresponding dual edge $e^* \in E^*$ occupies the lower-left and upper-right quadrants. For the odd sublattice, exchange e and e^* .

Using (8.24) we then have

$$Z = Q^{|V|/2} \sum_{\text{arrows}} \prod_{p=1}^{6} (\omega_p)^{N_p} (\omega'_p)^{N'_p}, \qquad (8.29)$$

where the sum is over arrow configurations satisfying the constraint "two in, two out" at each vertex, and N_p (resp. N'_p) is the number of vertices on the even (resp. odd) sublattice with arrow configuration p. Thus, the squarelattice Potts model has been represented as a *staggered six-vertex model*.²⁶ The weights read explicitly

$$\omega_1, \dots, \omega_6 = 1, 1, x_1, x_1, e^{i\gamma/2} + x_1 e^{-i\gamma/2}, e^{-i\gamma/2} + x_1 e^{i\gamma/2}$$
(8.30)

$$\omega_1', \dots, \omega_6' = x_2, x_2, 1, 1, e^{-i\gamma/2} + x_2 e^{i\gamma/2}, e^{i\gamma/2} + x_2 e^{-i\gamma/2}$$
(8.31)

To see this, note that configurations i = 1, 2, 3, 4 are compatible with just one linking of the oriented loops:

$$\sum_{\omega_1} = \sum_{1} \left(\begin{array}{c} \\ \\ \\ \\ \\ \end{array} \right)$$
(8.32)

 $^{^{26}}$ The term *staggered* means that the weights alternate between sublattices.

whereas i = 5, 6 are compatible with two different linkings (and the weight is obtained by summing over these two):

$$\sum_{\omega_5} = \sum_{e^{i\gamma/2}} + \sum_{x_1 e^{-i\gamma/2}}$$
(8.33)

Note that the even and odd sublattices are related by a $\pi/2$ rotation of the vertices in Figure 22. This rotation interchanges configurations $(\omega_1, \omega_2) \leftrightarrow (\omega'_3, \omega'_4)$ and $\omega_5 \leftrightarrow \omega'_6$. On the level of the weights it corresponds to $x_1 \leftrightarrow x_2$.

The staggered six-vertex model is not exactly solvable in general. However, if we impose

$$x_2 = (x_1)^{-1} \tag{8.34}$$

we have $\omega'_i = (x_1)^{-1}\omega_i$ for any i = 1, 2, ..., 6. The factors $(x_1)^{-1}$ from each ω'_i can be taken outside the summation in (8.29) and we have effectively $\omega'_i = \omega_i$. The six-vertex model then becomes homogeneous, hence solvable. Note that the solvability condition is nothing else than (8.25): the selfdual square-lattice Potts model is equivalent to a homogeneous six-vertex model. The results for the latter therefore apply. The anisotropy parameter is

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab} = -\cos\gamma, \qquad (8.35)$$

where we have replaced c^2 by $\omega_5\omega_6$ by invoking the usual gauge symmetry.²⁷ This matches perfectly the parameterisation (6.13). Note that the spectral parameter u is precisely what allows us to take different horizontal and vertical couplings.

We stress once more that the square-lattice Potts model is solvable at its selfdual point, but not at arbitrary temperatures. This is in contrast with the Ising model, which is solvable at any temperature. In that sense the Ising model is a rather untypical integrable model.

However the integrable *R*-matrix of the six-vertex model satisfies the Yang-Baxter relations for any choice of the spectral parameters. There is one other choice that also corresponds to a Potts model. If one lets the horizontal spectral parameters alternate like $u, u + \frac{\pi}{2}, u, u + \frac{\pi}{2}, ...$ and the

²⁷In the special case $x_1 = x_2 = 1$ of (8.34) we even have $\omega_5 = \omega_6$.

vertical like $v, v + \frac{\pi}{2}, v, v + \frac{\pi}{2}, \ldots$ one obtains the *antiferromagnetic* transition curve of the Potts model:

$$x_1 = \frac{\sin(u)}{\sin(\gamma - u)}, \qquad x_2 = -\frac{\cos(\gamma - u)}{\cos(u)}.$$
 (8.36)

This has been analysed in [Ba82b, JS06, IJS08].

8.8 Twisted vertex model

Sometimes it is convenient to consider particular correlation functions in which the weight of some of the loops are changed. As an elementary example, consider the Potts loop model defined on a connected planar graph G = (V, E) and let $i_1, i_2 \in V$ be a pair of root vertices. The partition function Z(n) is given by (8.24) with loop weight $n = \sqrt{Q}$ and additional local weights at the vertices.

Define now a modified partition function $Z_1(n, n_1)$ as follows: loops on $\mathcal{M}(G)$ surrounding neither or both of the roots have an unchanged weight n, whereas those surrounding only one of the roots have a modified weight n_1 . This defines the two-point correlation function $Z_1(n, n_1)/Z(n)$. An interesting special case is provided by $n_1 = 0$, which expresses the probability that the two roots belong to the same cluster.

It is possible to produce $Z_1(n, n_1)$ in the vertex model representation, leading to a so-called *twisted vertex model*. To this end, let \mathcal{P}_{12} be a an oriented self-avoiding path on G, going from i_1 to i_2 . Let us parametrise

$$n_1 = 2\cos\gamma_1 \in [-2, 2] \tag{8.37}$$

with real $\gamma_1 \in [0, \pi]$. In the arrow formulation, we then associate a special weight \widetilde{w} to any edge \widetilde{e} of $\mathcal{M}(G)$ that crosses the path \mathcal{P}_{12} . The weight \widetilde{w} depends on the orientation of the arrow on \widetilde{e} : it equals $e^{i\gamma_1}$ (resp. $e^{-i\gamma_1}$) if the arrow points from left to right (resp. from right to left) upon viewing \widetilde{e} along the direction given by \mathcal{P}_{12} .

The path \mathcal{P}_{12} is often called a *seam*, and the edges traversing it are referred to as *seam edges*.

In the oriented loop representation, it is easy to see that a loop surrounding neither or both of the roots will traverse \mathcal{P}_{12} an even number of times, and the phase factors \tilde{w} will cancel out globally. However, a loop surrounding just one of the roots with have one excess factor $e^{\pm i\gamma_1}$ depending on its global orientation (clockwise or counterclockwise), leading to (8.37) once the orientations have been summed over.

Note that the above construction of $Z_1(n, n_1)$ depends on the seam \mathcal{P}_{12} only through its end points i_1 and i_2 . In that sense, the exact shape of the seam is irrelevant and can be deformed at will.

Finally, the weights \tilde{w} can be absorbed in the vertex weights, by incorporating them in the weight of the vertex at the right (with respect to the orientation defined by \mathcal{P}_{12}) end point of \tilde{e} .

These considerations are important when discussing boundary effects. Suppose we wish to define the square-lattice Potts model on a cylinder with periodic boundary conditions. This is still a planar graph, but if we decide to draw it as such, i.e., in cobweb shape



(8.38)

the edges will be curved and additional complex phase factors will be picked up by the loops. These extra phases will cancel out for oriented loops that do not encircle the origin, and the usual weight $n = 2 \cos \gamma$ will result from summing over orientations. However, loops that do encircle the origin will finally a wrong weight $\bar{n} = 2$.

Alternatively, one may draw the cylinder as a standard square lattice with periodic boundary conditions across. In this version, oriented loops that are not homotopic to a point will not turn a total angle $\alpha = \pm 2\pi$, but rather $\bar{\alpha} = 0$. They thus get the weight $\bar{n} = 2$ as above: the two points of view are equivalent.

The introduction of a seam running from the origin to the point at infinity will change the weighting: $\bar{n} = n_1$. In particular, setting $n_1 = n$ we obtain the true Potts model. Such subtleties are important when discussing critical exponents, since these will in fact depend on n_1 .

Note finally that the case of doubly periodic (toroidal) boundary conditions is quite delicate, since most of the equivalences presented in this chapter depend crucially on the planarity of the graph.