9 Temperley-Lieb algebra

The Temperley-Lieb algebra $TL_N(n)$ is a unital associative algebra over $\mathbb{C}$. Its $N-1$ generators are denoted E_m for $m = 1, 2, \ldots, N-1$. They satisfy the relations

$$\begin{aligned} (E_m)^2 &= n\, E_m \,, \\ E_m E_{m\pm 1} E_m &= E_m \,, \\ E_m E_{m'} &= E_{m'} E_m \text{ for } |m-m'| > 1 \,. \end{aligned} \tag{9.1}$$

It will turn out useful to define the q-deformed numbers

$$[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}} \tag{9.2}$$

and parameterise

$$n = q + q^{-1} = [2]_q \,. \tag{9.3}$$

Notice that in the previous chapter we had $n = 2\cos\gamma$, so that $q = \mathrm{e}^{i\gamma}$. For $k \in \mathbb{N}$ the q-number is actually a polynomial in n:

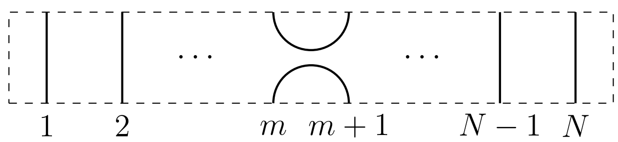
$$[k+1]_q = U_k(n/2)\,, \tag{9.4}$$

where $U_k(x)$ is the kth order Chebyshev polynomial of the second kind

$$U_k(\cos\theta) = \frac{\sin\big((k+1)\theta\big)}{\sin(\theta)}\,. \tag{9.5}$$

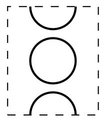
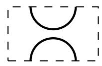
The algebra $TL_N(n)$ can be represented in many ways. We shall be particularly interested in its *loop-model representation*, since this permits us to make contact with the previous chapter. In this representation, $TL_N(n)$ is viewed as an algebra of diagrams acting on N numbered vertical strands (for convenience depicted inside a dashed box) as

$E_m =$

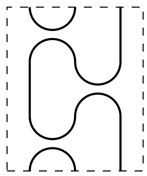
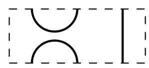


Multiplication in $TL_N(n)$ is defined by stacking diagrams vertically. More precisely, the product of two generators $g_2 g_1$ is defined by placing the diagram for g_2 above the diagram for g_1, identifying the bottom points of g_2 with the top points of g_1. The resulting diagram is considered up to smooth isotopies that keep fixed the surrounding box, and any closed loop is replaced by the factor n.

In this way we have for instance (omitting strands on which the action is trivial)

$(E_m)^2 =$  $= n$  $= n\, E_m$

and

$E_m E_{m+1} E_m =$  $=$  $= E_m \,.$

It is thus readily seen that all the defining relations (9.1) are satisfied. Moreover, for generic values of n no further relations hold: the loop-model representation is faithful.

9.1 Integrable $\check{R}$-matrix

Starting from first principles, we now construct an integrable model based on the TL algebra. Let us suppose that the $\check{R}$-matrix has the form

$$\check{R}_{m,m+1}(u) = f(u)I + g(u)E_m \,, \tag{9.6}$$

where $f(u)$ and $g(u)$ are some functions of the spectral parameter u to be determined. Inserting this into the Yang-Baxter equation (6.8) yields

$$\begin{aligned}&\big(f(u)I + g(u)E_2\big)\big(f(u+v)I + g(u+v)E_1\big)\big(f(v)I + g(v)E_2\big) = \\ &\big(f(v)I + g(v)E_1\big)\big(f(u+v)I + g(u+v)E_2\big)\big(f(u)I + g(u)E_1\big)\,. \end{aligned} \tag{9.7}$$

Using the algebraic relations (9.1) we can expand both sides of (9.7). The left-hand side produces

$$\begin{aligned}&f(u)f(u+v)f(v)I + f(u)g(u+v)f(v)E_1 + \\ &g(u)g(u+v)f(v)E_2E_1 + f(u)g(u+v)g(v)E_1E_2 + \\ &\big[g(u)g(v)\big(g(u+v) + nf(u+v)\big) + f(u+v)\big(f(u)g(v) + f(v)g(u)\big)\big]\, E_2 \,,\end{aligned}$$

and the right-hand side becomes

$$\begin{aligned}
&f(v)f(u+v)f(u)I + f(v)g(u+v)f(u)E_2 + \\
&f(v)g(u+v)g(u)E_2E_1 + g(v)g(u+v)f(u)E_1E_2 + \\
&\big[g(u)g(v)\big(g(u+v)+nf(u+v)\big) + f(u+v)\big(f(u)g(v)+f(v)g(u)\big)\big]\, E_1 \,.
\end{aligned}$$

These expressions must be identical in $TL_3(n)$, and so we can identify the coefficients for each of the five possible words in the algebra. The relations resulting from the words I, E_1E_2 and E_2E_1 are trivial. The relations coming from E_1 and E_2 are identical—related via an exchange of the left- and right-hand sides—and read

$$\begin{aligned}
&g(u)g(v)\big(g(u+v)+nf(u+v)\big) + f(u+v)\big(f(u)g(v)+f(v)g(u)\big) = \\
&f(u)f(v)g(u+v)\,. \qquad (9.8)
\end{aligned}$$

The functional relation (9.8) is a typical outcome of this way of solving the Yang-Baxter equations. It is in general not easy to solve this type of relation, and even if one finds solutions it is often difficult to make sure that one has found *all* the solutions. Worse, in more complicated cases than the one considered here the Ansatz for the $\check{R}$-matrix will involve more terms and the functions $f(u)$, $g(u)$,... must satisfy several coupled functional equations.

It is useful to rewrite (9.8) in terms of the parameters $z = \mathrm{e}^{iu}$, $w = \mathrm{e}^{iv}$ and $q = \mathrm{e}^{i\gamma}$. That is, instead of the *additive* spectral parameters u, v we have now *multiplicative* spectral parameters z, w. Thus

$$\begin{aligned}
&g(z)g(w)\big(g(zw)+(q+q^{-1})f(zw)\big) + f(zw)\big(f(z)g(w)+f(w)g(z)\big) = \\
&f(z)f(w)g(zw)\,. \qquad (9.9)
\end{aligned}$$

It is tempting to set $f(z) = 1$, since the overall normalisation of the $\check{R}$-matrix is unimportant, but in general this is *not* a good idea. A time proven strategy is to suppose that $f(z)$ and $g(z)$ are polynomials of some small degree in the variables z, z^{-1}, q and q^{-1}. (In some cases one needs to try fractional powers of q as well.) In this case we are lucky: there is a solution of degree one

$$f(z) = \frac{q}{z} - \frac{z}{q}\,, \qquad (9.10)$$

$$g(z) = z - z^{-1}\,. \qquad (9.11)$$

Verifying that this *is* a solution is of course easy. Finding it from scratch already calls for the use of symbolic algebra software such as MATHEMATICA.

Going back to additive spectral parameters, we thus have a trigonometric solution of (9.7):

$$f(u) = \sin(\gamma - u)\,, \qquad g(u) = \sin(u)\,. \tag{9.12}$$

Note that this agrees with (6.18).

In general, solutions to the Yang-Baxter equation turn out to be polynomial, trigonometric or elliptic (in order of increasing difficulty).

9.2 Transfer matrix decomposition

In the remainder of this chapter we shall be interested in the Q-state Potts model defined on an $L \times M$ annulus of width L spins and of circumference M spins. The boundary conditions are free in the space (L, horizontal) direction and periodic in the time (M, vertical) direction.

We work in the loop representation in order to make contact with the Temperley-Lieb algebra $TL_N(n)$ defined on $N = 2L$ strands and with loop weight $n = \sqrt{Q}$. The transfer matrix can be read off from (8.25):

$$T = Q^{L/2} \left(\prod_{m=1}^{L-1} (I + x_1 E_{2m}) \right) \left(\prod_{m=1}^{L} (x_2 I + E_{2m-1}) \right)\,, \tag{9.13}$$

where x_1 (resp. x_2) defines the horizontal (resp. vertical) coupling constant through (8.29).

We have seen in (8.35) that the Potts model is solvable if $x_2 = (x_1)^{-1}$. In that case we have

$$T = \left(\frac{\sqrt{Q}}{x_1} \right)^L \left(\prod_{m=1}^{L-1} (I + x_1 E_{2m}) \right) \left(\prod_{m=1}^{L} (I + x_1 E_{2m-1}) \right)\,. \tag{9.14}$$

We recognise here the integrable $\check{R}$-matrix (9.6) and identify

$$x_1 = \frac{g(u)}{f(u)} = \frac{\sin(u)}{\sin(\gamma - u)}\,. \tag{9.15}$$

According to (8.31) we have also $x_1 = \frac{\omega_3}{\omega_1}$, and we note that this agrees precisely with the parameterisation (6.13) used when studying the six-vertex model.

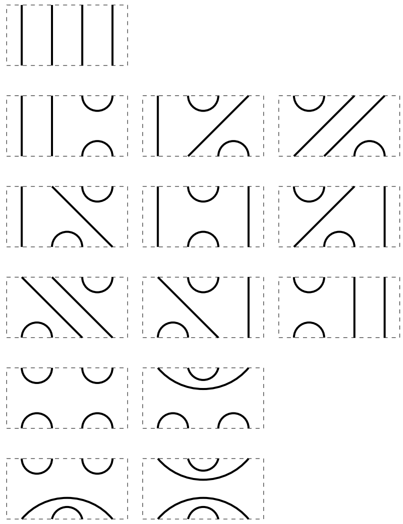


Figure 23: List of all TL states on $N = 4$ strands. Each row corresponds to a definite sector of the transfer matrix.

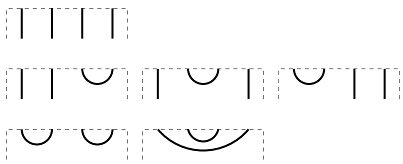


Figure 24: List of all TL reduced states on $N = 4$ strands. Each row corresponds to a definite sector of the transfer matrix.

The transfer matrix T acts on states which can be depicted diagramatically as non-crossing link patterns within a box bordered by two horizontal rows, each of N points. The complete list of states for $N = 4$ is shown in Fig. 23. The bottom (resp. top) side of the box corresponds to time $t = 0$ (resp. $t = t_0$); the transfer matrix propagates the states from t_0 to $t_0 + 1$ and thus acts on the top of the box only.

A link joining the top and the bottom of the box is called a *string*, and any other link is called an *arc*. We denote by s the number of strings in a given state. Any state can be turned into a pair of *reduced states* by cutting all its strings and pulling apart the upper and lower parts. For convenience, a cut string will still be called a string with respect to the reduced state. The complete list of reduced states for $N = 4$ is shown in Fig. 24.

Conversely, a state can be obtained by adjoining two reduced states, gluing together their strings in a unique fashion. Thus, if we define d_{2j} as the number of reduced states with $s = 2j$ strings, the number of states with $s = 2j$ strings is simply $(d_{2j})^2$.

The partition function $Z_{N,M}$ on an annulus of width N strands and height M units of time cannot be immediately expressed in terms of reduced states only, since these do not contain the information about how many loops (contractible or non-contractible) are formed when the periodic boundary condition is imposed. We can however write it in terms of states as

$$Z_{N,M}(n, \ell) = \langle u | T^M | v \rangle \, . \tag{9.16}$$

It is useful to slightly generalise the problem by giving the weight n to contractible loops and a different weight ℓ to non-contractible loops. Recalling (9.3) we shall parameterise the latter as[27]

$$\ell = t + t^{-1} = [2]_t \, . \tag{9.17}$$

At time $t_0 = 0$ the top and the bottom of the box must be identified. Therefore, the right vector $|v\rangle$ is just the unit vector corresponding to the unique state that contains no arcs and N strings (i.e., each link connects a point on the bottom to the point immediately above it on the top).

At time $t_0 = M$ the top and the bottom of the box must be reglued. Therefore, the left vector $\langle u|$ is obtained by identifying the top and bottom sides for each state. Counting the number of loops of each type gives the corresponding weight as a monomial in the loop weights n and ℓ.

The reduced states can be ordered according to a decreasing number of strings. The states can be ordered first according to a decreasing number of strings, and next, for a fixed number of strings, according to its bottom half reduced state. These orderings are brought out by the rows in Figs. 23–24.

With this ordering, T has a blockwise lower triangular structure in the basis of reduced states, since the generator e_i can annihilate two strings (if their position on the top of the box are i and $i + 1$) but cannot create any strings.

In the basis of states, T is blockwise lower triangular with respect to the number of strings, for the same reason. Each block on the diagonal in

[27]This t has of course nothing to do with the "time" discussed above.

this decomposition corresponds to a definite number of strings. The block corresponding to $s = 2j$ strings is denoted $\widetilde{T}_{2j}$. But since T acts only on the top of the box, each $\widetilde{T}_{2j} = T_{2j} \oplus \ldots \oplus T_{2j}$ is in turn a direct sum of d_{2j} identical blocks T_{2j} which correspond simply to the action of T on the reduced states with $2j$ strings.

In particular, the eigenvalues of T are the union of the eigenvalues of T_{2j}, where the T_{2j} now act in the much smaller basis of reduced states. This observation is particularly useful in numerical studies.

9.3 The dimensions d_k and D_k

In spite of the periodic boundary conditions, $Z_{N,M}(n,\ell)$ is obviously not a usual matrix trace. It can however be decomposed on standard traces by constructing the transfer matrix blocks T_k algebraically within $TL_N(n)$. We shall come back to this issue in the following sections.

For each block T_k we define the corresponding character as

$$K_k = \text{tr } (T_k)^M \ , \tag{9.18}$$

where we stress that the trace is over *reduced* states. Obviously we have

$$K_k = \sum_{i=1}^{d_k} \left(\lambda_i^{(k)}\right)^M \ , \tag{9.19}$$

where $\lambda_i^{(k)}$ are the eigenvalues of T_k. We recall that $d_k = \dim T_k$.

The expression of the partition function in terms of transfer matrix eigenvalues is more involved, due essentially to the non-local nature of the loops, and reads

$$Z_{N,M} = \sum_{j=0}^{L} D_{2j} K_{2j} \, , \tag{9.20}$$

where D_k are some eigenvalue amplitudes to be determined. We shall provide the answer in the next sections, using algebraic means.

In view of the Schur-Weyl duality (mentioned briefly in the introduction to chapter 8) the D_k can also be interpreted as the (quantum) dimensions of the commutant of $TL_N(n)$, which is the quantum algebra $U_q(\mathfrak{sl}_2)$. In the corresponding bimodule, the partition function (9.20) therefore has a *multiplicity free* decomposition.

Determining the d_k is an exercise of elementary combinatorics that we deal with now. Let $E(j,k)$ denote the number of reduced states on $2j$ strands, and using $2k$ strings, so that $d_{2j} = E(L,j)$. The corresponding generating function reads

$$E^{(k)}(z) = \sum_{j=0}^{\infty} E(j,k) z^j \,, \tag{9.21}$$

where z is a formal parameter representing the weight of an arc, or of a pair of strings. When $k=0$, a reduced state with no strings is either empty, or has a leftmost arc which divides the space into two parts (inside the arc and to its right) each of which can accomodate an independent arc state. The generating function $f(z) \equiv E^{(0)}(z)$ therefore satisfies $f(z) = 1 + z f(z)^2$ with regular solution

$$f(z) = \frac{1-\sqrt{1-4z}}{2z} = \sum_{j=0}^{\infty} \frac{(2j)!}{j!\,(j+1)!} z^j \,. \tag{9.22}$$

When $k \neq 0$, the strings simply divide the space into $2k+1$ parts each of which contains an independent arc state. Therefore,

$$E^{(k)}(z) = z^k f(z)^{2k+1} = \sum_{j=k}^{\infty} \left[\binom{2j}{j+k} - \binom{2j}{j+1+k} \right] z^j \tag{9.23}$$

and in particular we have

$$d_{2j} = E(L,j) = \binom{2L}{L+j} - \binom{2L}{1+L+j} \,. \tag{9.24}$$

Note that d_{2j} depends on the number of strands $N=2L$, but we usually will not mention this explicitly.

The total number of reduced states is

$$\sum_{j=0}^{L} d_{2j} = \binom{2L}{L} \,, \tag{9.25}$$

while the total number of (non-reduced) states is

$$\sum_{j=0}^{L} (d_{2j})^2 = E(2L,0) = \frac{1}{2L+1} \binom{4L}{2L} \,. \tag{9.26}$$

In particular for $N = 2L = 4$ we have

$$d_4 = 1\,, \qquad d_2 = 3\,, \qquad d_0 = 2\,,$$

in agreement with the number of reduced states shown in each row of Fig. 24. The total number of states is $1^2 + 3^2 + 2^2 = 14$ in agreement with Fig. 23.

9.4 Jones-Wenzl projectors

We have decomposed the full transfer matrix T to elementary blocks T_k that have the property that the number of strings is precisely k and cannot be lowered by the action of $TL_N(n)$. In more algebraic terms, T_k is the restriction of T to a representation with precisely k strings. This representation is known as the standard module $\mathcal{V}_k$. It can be shown that $\mathcal{V}_k$ is irreducible when q is not a root of unity.

Within $\mathcal{V}_k$, the generator E_m annihilates any (reduced) state for which the strands at positions m and $m+1$ are both strings, and acts in the usual way on any other state. There exists an algebraic object that imposes this restriction: the Jones-Wenzl (JW) projector.

The JW projector $P_k \in TL_k(n)$ is defined by the recursion relation

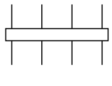
$$P_{k+1} = P_k - \frac{[k]_q}{[k+1]_q} P_k E_k P_k\,, \qquad \text{for } k \geq 1\,, \tag{9.27}$$

and the initial condition $P_1 = I$. Both sides of this equation act in $TL_{k+1}(n)$. To keep the notation simple it is implicitly understood that the projector P_k acts only on the k leftmost strands.

The first few JW projectors read explicitly

$$\begin{aligned} P_1 &= I\,, \\ P_2 &= I - \frac{1}{n} E_1\,, \\ P_3 &= I - \frac{n}{n^2-1}(E_1 + E_2) + \frac{1}{n^2-1}(E_1 E_2 + E_2 E_1)\,. \end{aligned} \tag{9.28}$$

In the sequel it will turn out useful to have a diagrammatic representation of P_k. We shall represent it as a bar across the strands being projected (here and in the following all pictures are for $k = 4$):

$P_k =$  (9.29)

The JW has two crucial properties. First, it is idempotent and bigger projectors swallow smaller ones:

$$P_m P_k = P_k P_m = P_k \,, \qquad \text{for } 1 \leq m \leq k \,. \tag{9.30}$$

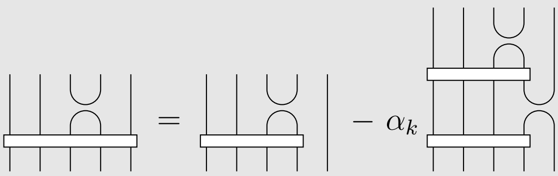
Second, no contractions are allowed among the strands having been projected:

$$E_m P_k = P_k E_m = 0 \,, \qquad \text{for } 1 \leq m \leq k-1 \,. \tag{9.31}$$

The proof of the properties (9.30)–(9.31) is by induction in k. The case $k = 1$ is obvious: since $P_1 = I$, the first property (9.30) is trivial, and for the second property (9.31) there is nothing to be shown. Suppose therefore that both properties hold for k and let us show them for $k+1$. For convenience we write $\alpha_k = [k]_q/[k+1]_q$.

We consider first (9.31). For $m < k$ we have

$E_m P_{k+1} =$

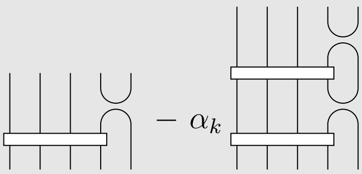


(9.32)

where we have used (9.27). Both diagrams on the right-hand side are zero by the induction hypothesis (9.31), whence $E_m P_{k+1} = 0$ as required.

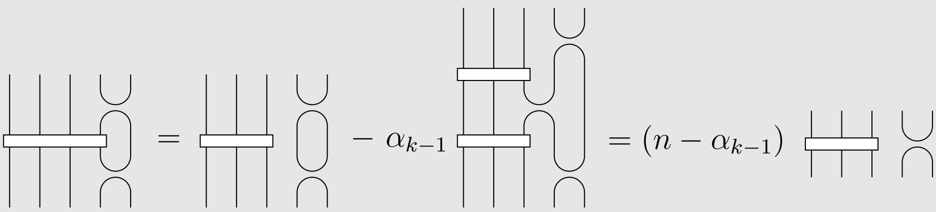
The argument for $m = k$ is slightly more involved. We first use (9.27) to write

$E_k P_{k+1} =$



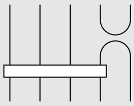
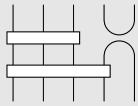
(9.33)

In the second term on the right-hand side, the small loop cannot yet be replace by n since it is "trapped" by the projector. We therefore use (9.27) once more:



(9.34)

where in the last step use was made of the induction hypothesis (9.30). Inserting (9.34) into (9.33) we obtain

$E_k P_{k+1} =$  $- \alpha_k (n - \alpha_{k-1})$  (9.35)

but by (9.30) the two diagrams on the right-hand side are identical. To have $E_k P_{k+1} = 0$ as required, we therefore need the coefficient to vanish

$$1 - \alpha_k (n - \alpha_{k-1}) = 0 \, ,$$

which is easily shown to be equivalent to

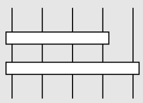
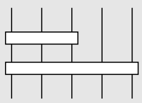
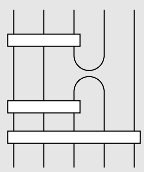
$$[2]_q [k]_q = [k-1]_q + [k+1]_q \, . \tag{9.36}$$

But (9.36) is precisely the recursion relation satisfied by $[k]_q$, so (9.31) is proved.

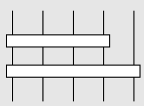
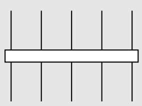
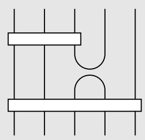
Let us note that (9.34) is actually a quite useful identity. In algebraic terms, and using (9.36), it can be written

$$E_k P_k E_k = \frac{[k+1]_q}{[k]_q} E_k P_{k-1} \, . \tag{9.37}$$

We still need to prove (9.30) for $k+1$ and $m \le k+1$. This is done by induction in m. Since $P_1 = I$ the statement is trivial for $m = 1$. Suppose now $m \le k$ and that the statement has been proved for $m-1$. We have

$P_m P_{k+1} =$  $=$  $- \alpha_{m-1}$  (9.38)

After using the induction hypothesis on both diagrams on the right-hand side we obtain

 $=$  $- \alpha_{m-1}$  (9.39)

But the second diagram on the right-hand side vanishes by (9.31), so we have $P_m P_{k+1} = P_{k+1}$ as required.

Exactly the same argument applies when $m = k+1$, so the proof is complete.

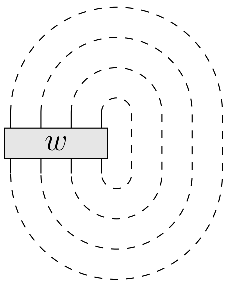
9.5 Markov trace

One of our main objectives is to compute the annulus partition function $Z_{N,M}(n,\ell)$ given by (9.16). This calls for algebraic way of imposing the periodic boundary conditions in the time direction. This motivates the following definition of the Markov trace.

Let $w \in TL_N(n)$ be a word in the TL algebra. We can represent w as a diagram in a box, cf. Fig. 23. The Markov trace of w is defined as

$$\mathrm{Tr}\, w = n^{N_1} \ell^{N_2} , \tag{9.40}$$

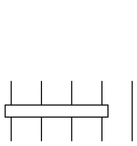
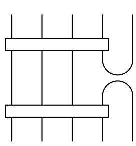
where N_1 (resp. N_2) is the number of contractible (resp. non-contractible) loops formed when identifying the top and the bottom sides of the box. This definition is extended by linearity to the whole algebra $TL_N(n)$. Pictorially we can write

$\mathrm{Tr}\, w =$ 

(9.41)

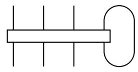
Contractible (resp. non-contractible) loops are those that cover an even (resp. odd) number of dashed lines. The corresponding weights have been defined as $n = [2]_q$ and $\ell = [2]_t$.

In particular wish to know the Markov trace of the JW projectors P_k. This can be found by using (9.27):

$\mathrm{Tr}\, P_{k+1} = \mathrm{Tr}$  $- \alpha_k \ \mathrm{Tr}$  (9.42)

In the second diagram on the right-hand side we can slide the uppermost projector across the periodic boundary condition to the bottom, where it gets swallowed by the other projector. We thus have

$$\mathrm{Tr}\, P_{k+1} = [2]_t \,\mathrm{Tr}\, P_k - \alpha_k \,\mathrm{Tr} \tag{9.43}$$



Note that the last diagram does *not* equal $\mathrm{Tr}\, P_{k-1}$. Indeed, the small loop on the right is contractible. For the moment it is "trapped" by the projector, but we can liberate it by repeating the argument of (9.34). We arrive at

$$\mathrm{Tr}\, P_{k+1} = [2]_t \,\mathrm{Tr}\, P_k - \alpha_k \left([2]_q - \alpha_{k-1}\right) \mathrm{Tr}\, P_{k-1} \,. \tag{9.44}$$

Thanks to (9.36) we have $\alpha_k \left([2]_q - \alpha_{k-1}\right) = 1$, and so we have the recursion relation

$$[2]_t \mathrm{Tr}\, P_k = \mathrm{Tr}\, P_{k-1} + \mathrm{Tr}\, P_{k+1} \tag{9.45}$$

with initial conditions $\mathrm{Tr}\, P_0 = [1]_t = 1$ and $\mathrm{Tr}\, P_1 = [2]_t = \ell$. Invoking again (9.36) the solution is

$$\mathrm{Tr}\, P_k = [k+1]_t = U_k(\ell/2) \,. \tag{9.46}$$

It is rather remarkable that this depends only on ℓ, and not on n.

9.6 Decomposition of the Markov trace

Assume that q is not a root of unity, so that the standard modules $\mathcal{V}_j$ are irreducible. We now define a scalar product in $\mathcal{V}_j$. Given two reduced states $|v_1\rangle, |v_2\rangle \in \mathcal{V}_j$, cf. Fig. 24, each containing j strings. The scalar product $\langle v_1 | v_2 \rangle$ is obtained by reflecting $|v_1\rangle$ in a horizontal mirror, then gluing together the two states. We define $\langle v_1 | v_2 \rangle = 0$ unless each string in v_2 connects onto a string in v_1. Otherwise we attribute a weight $[2]_q = n$ to each closed loop and 1 to each string in the compound diagram.

Let us give some examples in $TL_4(n)$. The Gram matrices of scalar products in $\mathcal{V}_4$, $\mathcal{V}_2$ and $\mathcal{V}_0$, cf. Fig. 24, read

$$M_4 = [1] \,, \qquad M_2 = \begin{bmatrix} n & 1 & 0 \\ 1 & n & 1 \\ 0 & 1 & n \end{bmatrix} , \qquad M_0 = \begin{bmatrix} n^2 & n \\ n & n^2 \end{bmatrix} . \tag{9.47}$$

As a side remark we point out that the determinants of the Gram matrices may vanish if and only if q is a root of unity. We are supposing throughout that this is not the case, so that the representation theory is generic. In more technical terms, the algebra $TL_N(n)$ is supposed to be semi-simple.

A related observation is that the JW projectors are ill-defined when q is a root of unity.

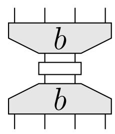
Since $\mathcal{V}_j$ are generic there exists a basis $\mathcal{B}_j$ which is orthonormal with respect to the scalar product:

$$\forall b_k, b_l \in \mathcal{B}_j : \ \langle b_k | b_l \rangle = \delta_{k,l} \,. \tag{9.48}$$

We also extend our definition of the scalar product so that states belonging to different standard modules $\mathcal{V}_j$ and $\mathcal{V}_{j'}$, with $j \neq j'$, are orthogonal.

The next step is to construct an element $P_{N,j} \in TL_N(n)$ that projects on $\mathcal{V}_j$. In other words, $P_{N,j}$ must act as the identity on $\mathcal{V}_j$ and annihilate all states of $\mathcal{V}_{j'}$ with $j' \neq j$. Obviously $P_{N,N} = P_N$ is just the familiar JW projector. For arbitrary j we define (the diagrammatic representation shows the case $N = 4$ and $j = 2$)

$$P_{N,j} = \sum_{b \in \mathcal{B}_j} |b\rangle \circ P_j \circ \langle b| = \sum_{b \in \mathcal{B}_j}$$



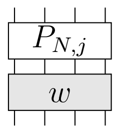
(9.49)

This has the required properties and satisfies the completeness relation

$$\sum_{j=0}^{N} P_{N,j} = I \in TL_N(n) \,. \tag{9.50}$$

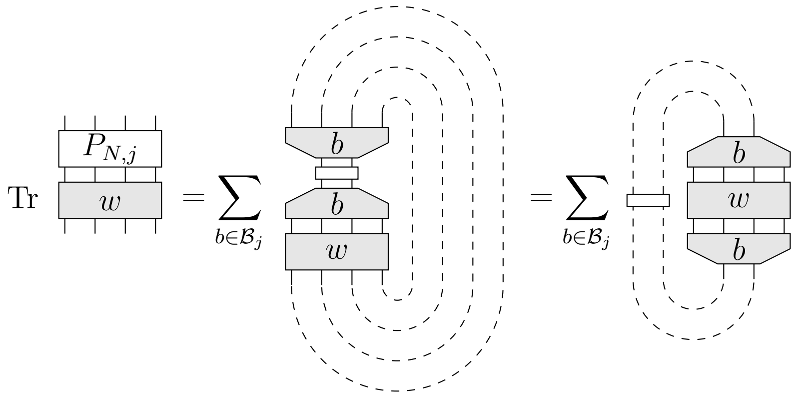
We are now in a position to attain the goal of decomposing the Markov trace of any element $w \in TL_N(n)$ over standard traces—i.e., traces with respect to the basis states $b \in \mathcal{B}_j$. We first decompose w using (9.50):

$$w = \sum_j P_{N,j}\, w = \sum_j$$



(9.51)

Inserting the definition (9.49) and taking the Markov trace yields



(9.52)

Returning to algebraic terms, this means that we have shown

$$\operatorname{Tr} w = \sum_j \operatorname{Tr} P_j \sum_{b\in\mathcal{B}_j} \langle b|w|b\rangle = \sum_j [j+1]_t \operatorname{tr}_{\mathcal{V}_j} w\,. \tag{9.53}$$

The main result (9.53) applies in particular when $w = T^M$, the Mth power of the Potts model partition function. We have thus finished the demonstration that $Z_{N,M}$ indeed has the form (9.20) and identified the eigenvalue amplitudes as $D_j = [j+1]_t$. The characters

$$K_j = \operatorname{tr}_{\mathcal{V}_j} T^M \tag{9.54}$$

defined in () are usually written $K_{1,1+j}$ for reasons that will become clear later on. We can thus summarise our final result as

$$Z_{N,M}(n,\ell) = \sum_{j=0}^{L} [1+2j]_t\, K_{1,1+2j}(n)\,. \tag{9.55}$$

The point is that the $K_{1,1+2j}$ can be computed exactly in the continuum limit, using CFT techniques. The expression (9.55) will then give access to—among many other things—exact crossing formulae in percolation.

Finally one should also note the sum rule

$$\sum_{j=0}^{L} d_{2j} D_{2j} = \ell^N\,, \tag{9.56}$$

which expresses the fact that there are ℓ degrees of freedom living on each site.