The Majorana-Hubbard Model in 1D

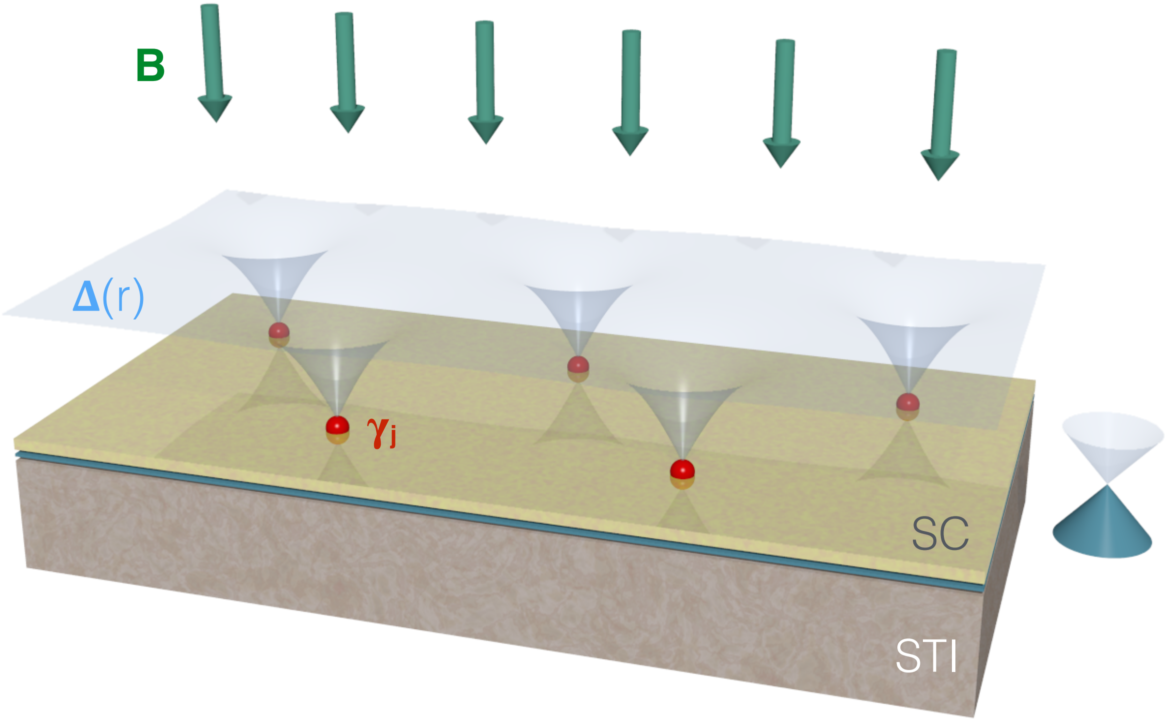
with Armin Rahmani, Marcel Franz, Xiaoyu Zhu

Outline

1) Motivation and Model
2) 1 Dimension
3) Conclusions

1) Motivation and Model

Macroscopic numbers of Majorana modes are predicted to occur if a layer of ordinary superconductor is placed on a strong topological insulator in a transverse magnetic field



Majorana fermions are a candidate for qubits. This has led to great interest in them. It has been the main focus of research at Microsoft Station Q. Charlie Marcus at the Niels Bohr Institute is one of the leading experimentalists working in this field. Various experimentalists have claimed to see Majorana fermions but this has remained controversial. Interaction effects involving Majorana fermions have not yet been studied experimentally but at least two experimental groups are interested in studying them to test our theoretical predictions.

-A MM is localized near the centre of each superconducting vortex
-MM's can tunnel between vortices and interact with each other with short-range interactions, $\propto e^{-r/\xi}$
-tunneling amplitude goes to zero if gate chemical potential of topological superconductor is tuned to a special value
-We have studied simplest possible version of this model with shortest possible range interactions- "Majorana-Hubbard Model",
hopping amplitude t, interactions g, g/t of either sign
So far: -1 dimensional case, 2 dimensional square lattice, square lattice ladders, triangular lattice ladders

1D Case

$$H=\sum_j \left[it\gamma_j\gamma_{j+1} + g\gamma_j\gamma_{j+1}\gamma_{j+2}\gamma_{j+3} \right]$$

$$\gamma_j^{+}=\gamma_j, \{\gamma_j, \gamma_k\} = 2\delta_{j,k}$$

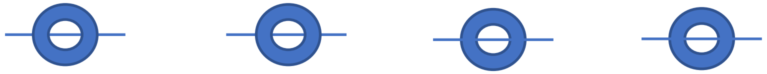
- No conserved particle number in this model but important discrete symmetries
- Can be studied by field theory and DMRG

-Majoranas ”like” to pair up and form complex “Dirac” fermions

-defining $c_j = \frac{(\gamma_{2j}+i\gamma_{2j+1})}{2}, i\gamma_{2j}\, \gamma_{2j+1} = 2c_j^+ c_j - 1$ so half the interactions terms become $-g \sum_j (2c_j^+ c_j - 1)(2c_{j+1}^+ c_{j+1} - 1)$

-note that g>0 is attractive interaction, g<0 repulsive

-no conserved charge so mean field density determined by interactions. 2 mean field ground states for g>0 (attractive interactions)



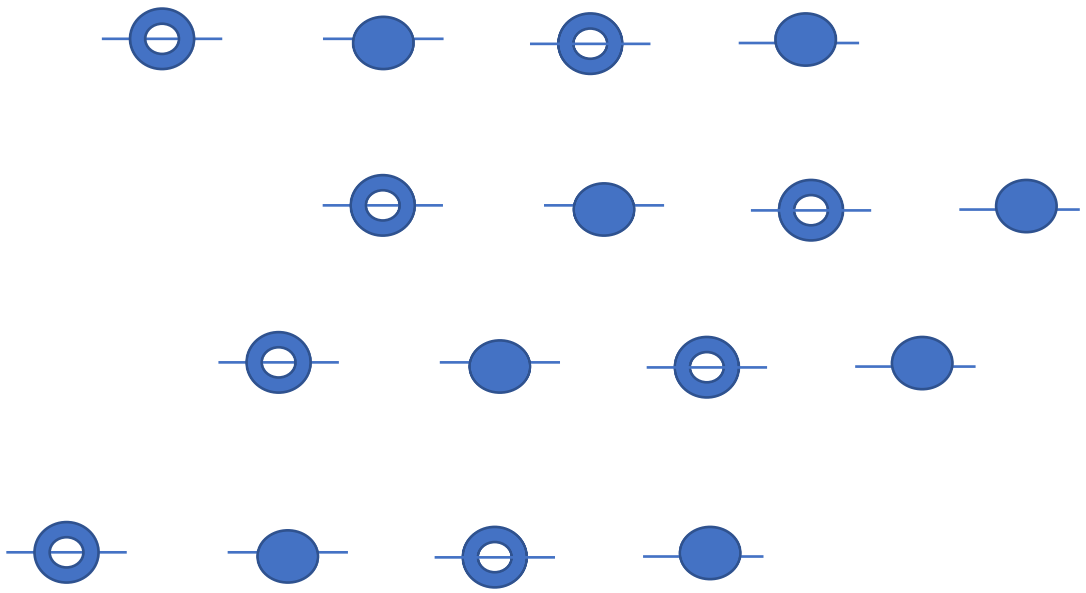
-2 ways of pairing up MM’s

-resulting Dirac levels are empty (for t>0)

We may rewrite full Hamiltonian in complex fermion basis defining $\hat{p}_j \equiv 2c_j^+ c_j - 1$ as:

$$H = \sum_j \{t\hat{p}_j - t(c_j^+ - c_j)(c_{j+1}^+ + c_{j+1}) + g[-\hat{p}_j\hat{p}_{j+1} + (c_j^+ - c_j)\hat{p}_{j+1}(c_{j+2}^+ + c_{j+2})]\}$$

Keeping only 1st t term and 1st g term is mean field theory -remarkably, this turns out to be qualitatively correct at large |g|



4 mean field ground states for g<0 (repulsive interactions)

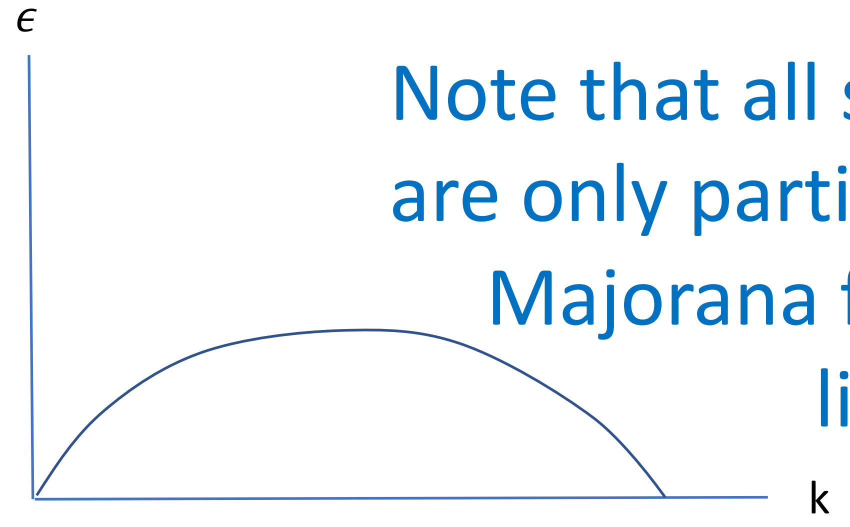
-charge density wave

-we verified that these phases occur at strong coupling numerically - DMRG

Non-interacting model we solve by Fourier transforming:

$$\gamma_k = \sqrt{\frac{1}{2L}} \sum_{j=0}^{L} \gamma_j e^{ikj}$$

Note that $\gamma_{-k} = \gamma_k^+$. We can diagonalize: $H = 2t \sum_{0<k<\pi} \gamma_k^+ \gamma_k \sin k$



Note that all states are empty in ground state. There are only particle excitations, not holes – signature of Majorana fermions. Low energy excitations have linear dispersion: relativistic

Low energy effective Hamiltonian is relativistic Majorana model:

Let $\gamma_j \approx 2\gamma_R(vt - aj) + (-1)^j 2\gamma_L(vt + aj)$

$$H = iv\int dx[\gamma_R\partial_x\gamma_R - \gamma_L\partial_x\gamma_L]$$

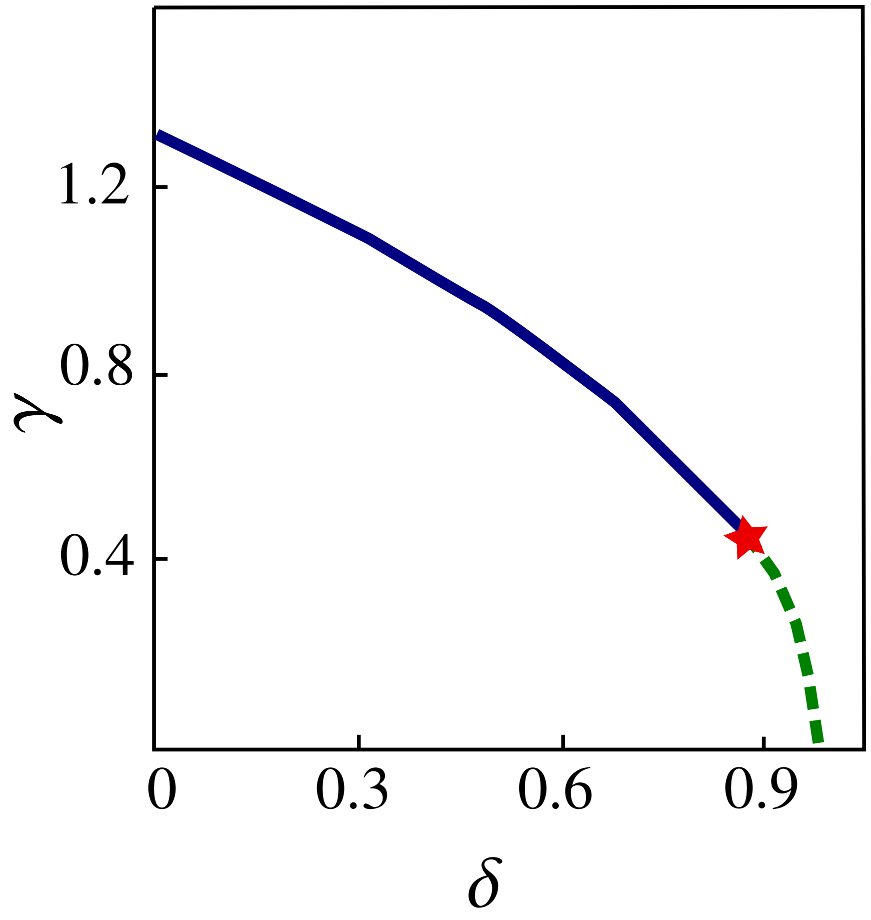
where $\gamma_{R/L}$ is Hermitean, v=4t

These operators have RG scaling dimension ½ so that H has dimension 1 (energy). Lowest dimension continuum interaction term is $H_{int} = -256g\int dx\gamma_R\partial_x\gamma_R\gamma_L\partial_x\gamma_L$ of dimension 4, highly irrelevant at weak coupling. (Derivatives needed since $\gamma_{R/L}^2$ =constant.) So, we expect to get free Majorana dispersion at least for small enough g.

At sufficiently large g we find broken symmetry phases predicted by Mean Field Theory. Remarkably, transition at g>0 occurs at g≈256 and for g<0 at g≈ -3.0. This made numerics extremely challenging. Correlation length is ∞ for 0<g<256 and only comes down to perhaps a few hundred at g= ∞. Nature of 2nd order phase transition for g=256 is interesting. The critical line with g<256 corresponds to the 1D quantum Ising model: $H = \sum_j [-\sigma_j^z \sigma_{j+1}^z + h\sigma_j^x]$ at critical point h=1. The symmetry that keeps model critical is translation by 1 site, which takes $\gamma_R \to -\gamma_R$ forbidding the mass term $m\, \gamma_R\, \gamma_L$. The corresponding symmetry in

Ising model is called Kramers-Wannier duality. This is a symmetry which takes $h-1 \to -(h-1)$ for h near 1. It switches broken and unbroken symmetry phases. Ising transition becomes 1st order if we insert a high enough density of randomly located vacancies. This transition can be realized By using s=1 spins, instead of s=1/2 and inserting a $(S^z)^2$ term which favours $S^z=0$.

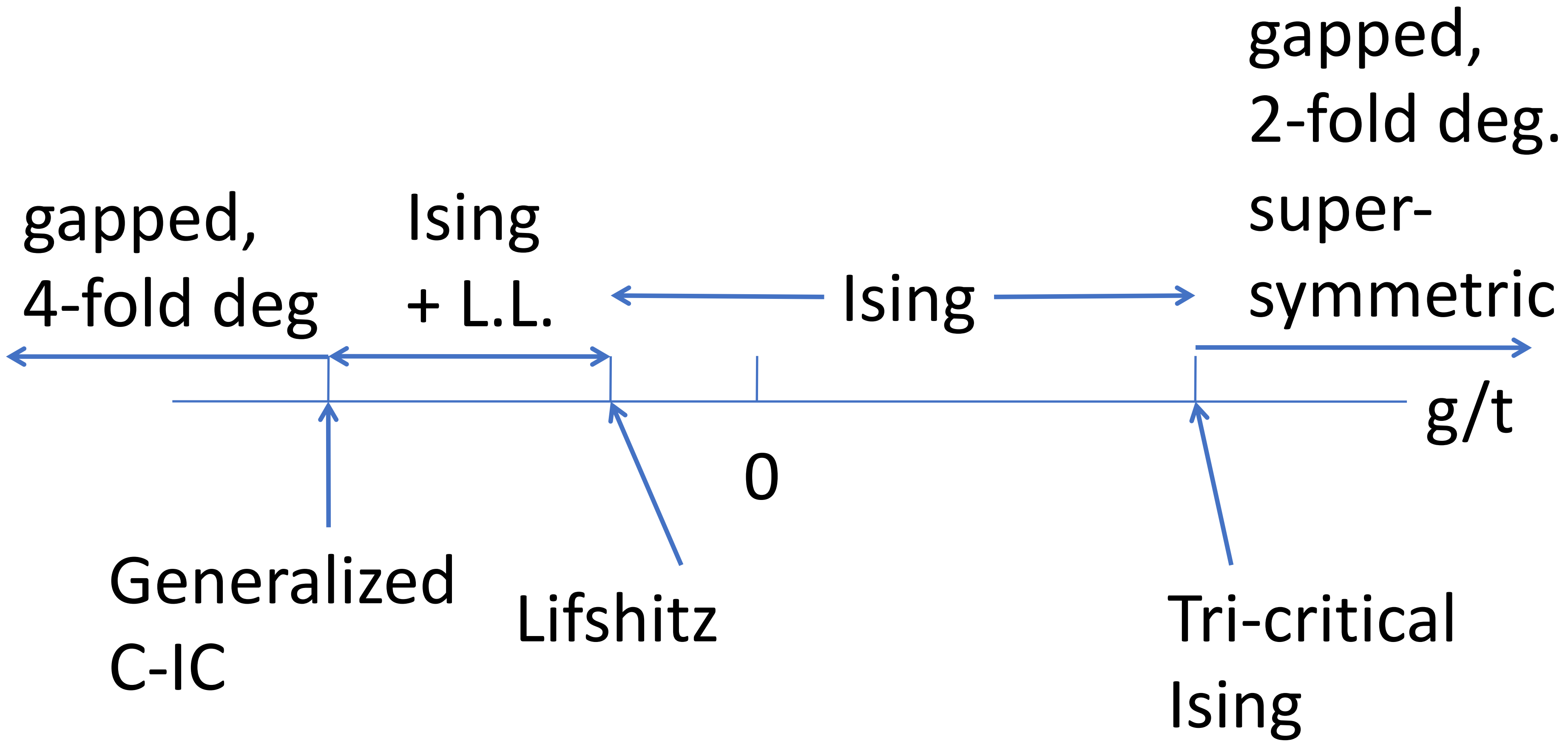
$$H = -\sum_j [S_j^z S_{j+1}^z - \gamma S_j^x - \delta (S_j^z)^2]$$



Solid line is 2nd order transition dashed line is 1st order. Star marks transition from 2nd to 1st order: tricritical Ising model. This model is a Supersymmetric conformal field theory. For

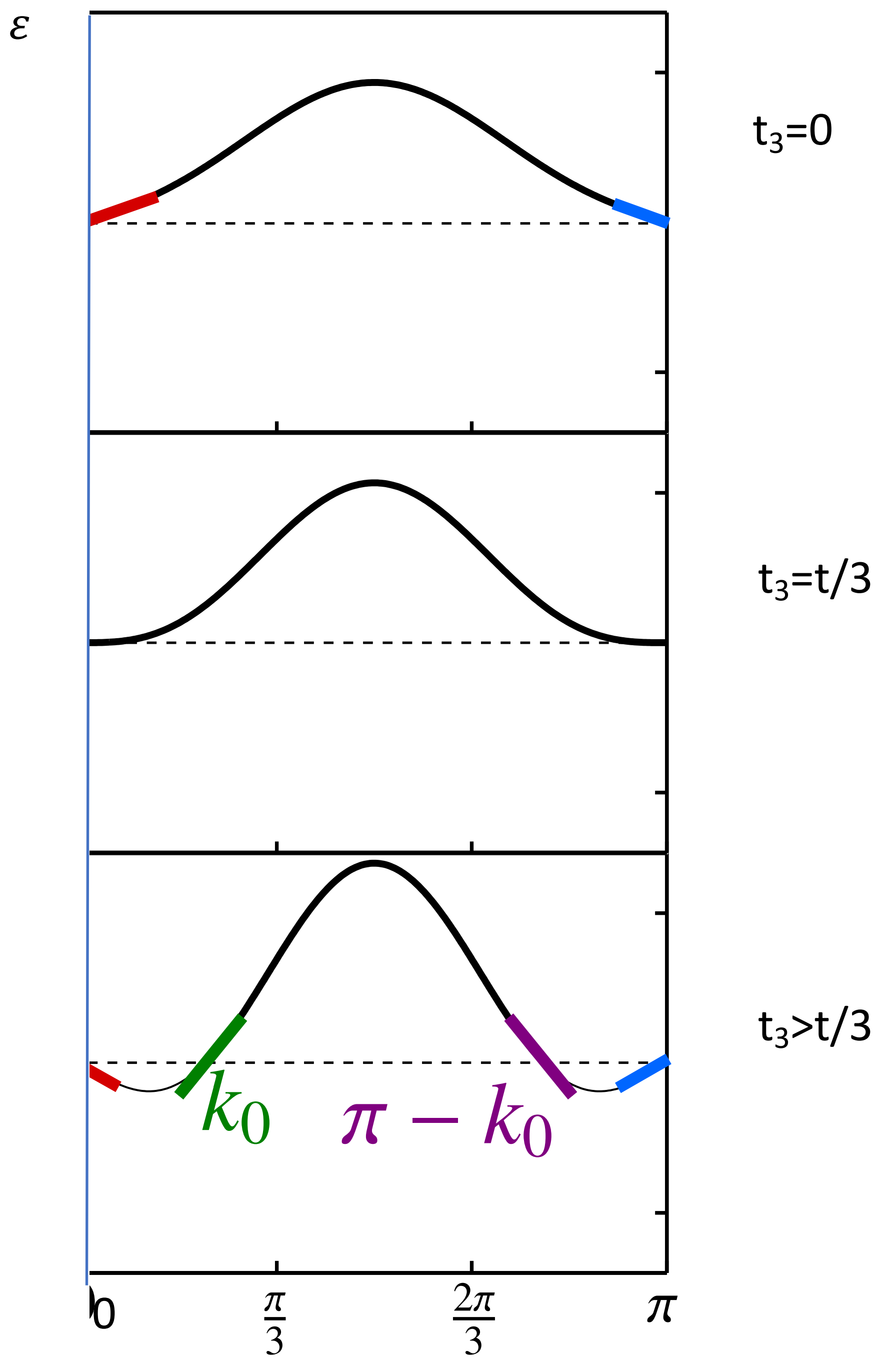
g slightly bigger than 256 we predict fermions and bosons (bound states) of same mass.

Phase Diagram for 1D Case



g/t=±∞ are equivalent, for g/t>>1 phase, low-lying excited doublet has energy $\propto |t|$ (1st order transition)

There is also a remarkable intermediate phase for g<0 (repulsive interactions). This can be understand from interactions modifying the dispersion relation for the free Majorana fermions. A 2nd neighbour hopping term is not allowed by symmetry. The symmetry is time reversal: $i \rightarrow -i, \gamma_j \rightarrow (-1)^j \gamma_j$. But a 3rd neighbour hopping term Is allowed and gets generated by interactions. This modifies the dispersion relation to $\varepsilon = 2t \sin k + 2t_3 \sin(3k)$



For $t_3 > t/3$, low energy theory has both relativistic Majorana fermions and complex fermions. Additional interaction terms allowed by symmetry and non-oscillating are:

$$H_{int} = \int dx[g_1 \psi_R^+ \psi_R \psi_L^+ \psi_L + g_2 \gamma_R \gamma_L (\psi_R \psi_L + \psi_R^+ \psi_L^+]$$

1st term is standard Luttinger liquid interaction- continuously changes “Luttinger parameter” which determines critical exponents.

2nd term is irrelevant for repulsive interactions (g<0).
There is also a term that alternates with a wave-vector determined by k_0:

$$H_{int} = \int dx \gamma_R \gamma_L [e^{i(4k_0-\pi)x} \psi_R^+ \partial_x \psi_R^+ \psi_L \partial_x \psi_L - h.c.]$$

This is irrelevant unless repulsive interactions are very strong, K<1/4, and oscillates unless $k_0 = \pi/4$.
The transition into this "c=3/2 phase", at g=-0.3, is a "Lifshitz transition". Not relativistic, cubic dispersion relation.

The transition out of the c=3/2 phase into the strong coupling broken symmetry phase occurs at g=-3.0. It occurs because K goes to ¼ and k_0 goes to $\pi\ /4$ simultaneously! We calculate K and k_0 using DMRG to get the finite size spectrum and came to this remarkable conclusion.

Conclusions

-the Majorana-Hubbard model on various lattices has rich phase diagrams
-Majoranas like to pair up to form complex fermions for strong enough coupling, breaking discrete symmetries
-Supersymmetry can be realized in both 1 and 2 dimensions

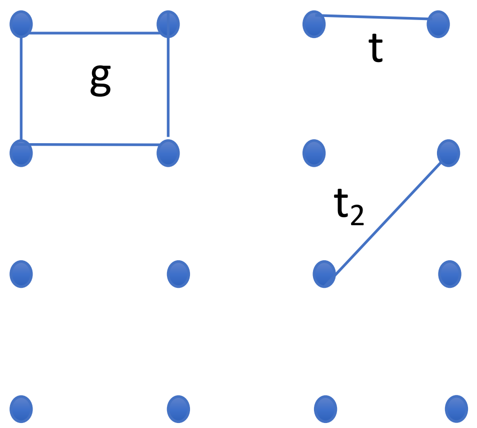
2 Dimensional Square Lattice Case

$$H = H_0 + H_{int} + H_2$$

$$H_0 = it \sum_{m,n} \gamma_{m,n}\left[(-1)^n \gamma_{m+1,n} + \gamma_{m,n+1}\right]$$

$$H_{int} = g \sum_{m,n} \gamma_{m,n}\gamma_{m+1,n}\gamma_{m+1,n+1}\gamma_{m,n+1}$$

$$H_2 = it_2 \sum_{m,n,s,s'} \gamma_{m,2n}\gamma_{m+s,2n+s'} \qquad s,s' = \pm 1$$



Signs of hopping terms are determined by 1 flux quantum per plaquette.

Interactions on plaquettes.

g>0 is attractive interaction

$-t_2$ term breaks T-reversal and parity symmetry and changes phase diagram significantly

-phases are characterized by spontaneously broken symmetries

-what are symmetries of H which might get broken?

1) Translation by 1 site in x or y direction
2) $\pi/2$ rotation symmetry

(if $t_2=0$ only)

3) Time reversal
4) Parity (spatial reflection)

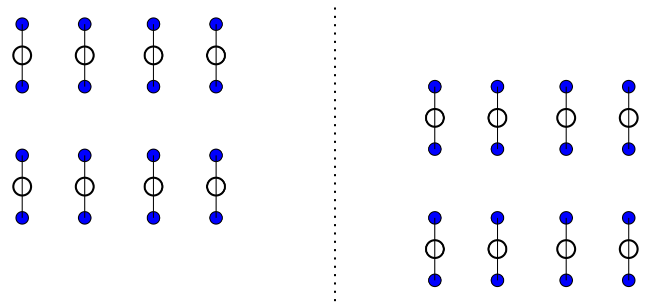
(PT a symmetry even for t_2 non-zero)

In addition there are "emergent symmetries" in low energy effective field theory

-these pairing phases also break rotational symmetry and parity symmetry (t_2=0)

-depending on sign of g, we can obtain “ferromagnetic” or “antiferromagnetic” pairing phases:

$\langle i\gamma_{m,n}\gamma_{m,n+1}\rangle = A+B(-1)^n$



-ferromagnetic favoured for g>0

-pairs of Majoranas form Dirac fermions, all energy levels empty

-4 ground states

$<i\gamma_{m,n}\gamma_{m,n+1}>=C(-1)^{m}+D(-1)^{m+n}$

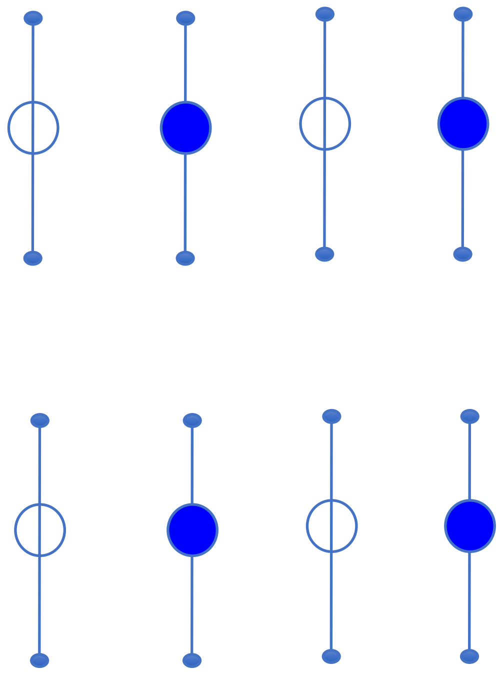
-antiferromagnetic

-favoured for $g<0$

-pairs of Majoranas form Dirac fermions: alternating filled or empty

-8 ground states in this case:

-Translate by 1 site in x or y direction



-these pairing phases also break rotational symmetry and parity symmetry ($t_2=0$)

-Time Reversal:

takes $\gamma_{m,n} \rightarrow (-1)^{m+n}\gamma_{m,n}$, $i \rightarrow -i$ (anti-unitary)

-broken by $it_2\gamma_{m,n}\gamma_{m+1,n+1}$

Field Theory/Renormalization Group Approach and Nature of Critical Points

-exact dispersion relation for non-interacting model:

$$E_{\pm} = \pm\sqrt{(4t sin\, k_x)^2 + (4t sin\, k_y)^2 + (8t_2 cos k_x \cos k_y)^2}$$

with $0 \leq k_x < \pi$, $-\pi/2 \leq k_y < \pi/2$,

-for $t_2 << t$, low energy excitations near 2 points in k-space: (0,0) and (π,0) where Lorentz-invariant dispersion relation occurs: $E_{\pm} \sim \pm\sqrt{16t^2|\vec{k}|^2 + 64(t_2)^2}$

-2 "valleys" like in graphene but Majorana modes

-low energy field theory has 2 species of 2-component Majorana fermions which can be combined into a single species of Dirac fermions

-ignoring higher derivative terms, Lagrangian density, including interactions, is Lorentz invariant

$$L = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi - 32\, g\, (\bar{\psi}\psi)^2$$

Here the γ^{μ} are 3 2-dimensional gamma matrices, I set v=1 and $\bar{\psi} \equiv \psi^{+}\gamma^{0}$

-2 components from inequivalent even and odd rows

-in addition to emergent Lorentz invariant there is an emergent U(1) (particle number conservation) symmetry!

-Interactions are “irrelevant” in RG sense
-if bare g is small enough, it renormalizes to 0 giving an effective free fermion phase
The interaction term in the low energy theory is

$$H_{\text{int}} = -64\, g\, \psi_1^{+}\psi_2^{+}\psi_2\psi_1 = 32\, g\, (\overline{\psi}\psi)^2$$

For g>0 this is an attractive pairing interaction, so we get a transition to a superfluid phase for strong enough positive g – corresponds to ferromagnetic pairing phase
-also higher dimension U(1) breaking terms

-for t_2=0, fermions are massless in $g<g_c$ phase
-at superfluid transition we get a massless (Goldstone) boson as well as massess fermions
-this transition is Supersymmetric – equivalent fermions and bosons
-this has been studied in other condensed matter contexts but it is remarkable here that U(1) symmetry is emergent
-a non-zero t_2 gives fermions a mass
-now superfluid transition is in usual U(1)-breaking (2+1) dimensional universality class

There is a U(1) breaking term in the effective Hamiltonian:
$H_{int} = g' \int dx[\psi_1 \partial_x \psi_1 \, \psi_2 \partial_x \psi_2$-$\psi_1 \partial_y \psi_1 \psi_2 \partial_y \psi_2$+h.c.].
With Kyle Wamer we showed that this remains irrelevant at critical point, using the ε- expansion.

Note that this lattice model evades the Nielsen-Ninomiya theorem. We only get 1 channel of Lorentz invariant fermions in the low energy model. The reason is that U(1) symmetry is broken in the lattice model.

Results on Ladders

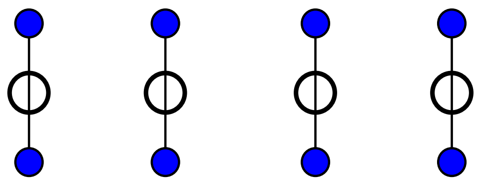
We have studied 2 and 4-leg ladders, with square lattice geometry by a combination of analytic and DMRG methods. In the 2-leg case, there is an exact U(1) symmetry, as we see by defining complex fermions: $c_m = (\gamma_{m,0} + i(-1)^m \gamma_{m,1})/2$. Then, the horizontal hopping term becomes: $2it \sum_m [c_m^+ c_{m+1} - h.c.]$ and the vertical hopping term $2t \sum_m (-1)^m c_m^+ c_m$.

If we impose periodic boundary conditions in the vertical direction, the vertical hopping term vanishes. Then, by Jordan-Wigner transformation, H maps into the xxz model:

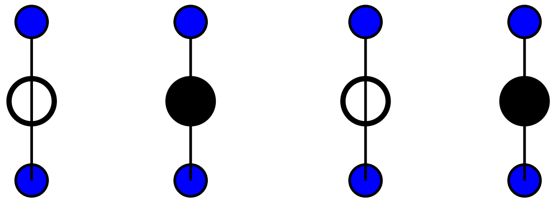
$$H = \sum_j [t(\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y) + 2g\sigma_m^z \sigma_{m+1}^z].$$

Noting that $i\gamma_{m,o}\,\gamma_{m,1} = (-1)^m(2c_m^+ c_m - 1) \rightarrow (-1)^m \sigma_m^z$,

We see that the antiferromagnetic order which occurs for g>1/2 corresponds to the mean field ground state:



and the ferromagnetic order which occurs for g<-1/2 corresponds to the mean field state:



For the 4-leg case there is no exact U(1) symmetry and the model is not analytically solvable except at infinite coupling. We analyzed it with DMRG. To analyze the infinite coupling limit we define complex fermions:

$c_{m,1} \equiv (\gamma_{m,0} + i\gamma_{m,1})/2, c_{m,2} \equiv (\gamma_{m,2} + i\gamma_{m,3})/2$. Then it can be seen that the interaction term preserves fermion parity on each rung (unlike the horizontal hopping term).

So, at strong coupling we can restrict ourselves to only 2 states (with fixed fermion parity) on each rung. We can map these 2 states into the xy model with Hamiltonian: $H = -2g\sum_m [\sigma_m^z \sigma_{m+1}^z + \sigma_m^x \sigma_{m+1}^x]$. This is the gapless xy (or xz) model. Unlike the 2-leg case, this model is gapless with no broken symmetry (except for fermion parity). Adding a small hopping term produces a gap. For small t/g we can ignore horizontal hopping since it changes the fermion parity on each rung. This can be shown to increase the energy by an amount of O(g).

On the other hand, vertical hopping just gives a perturbation:

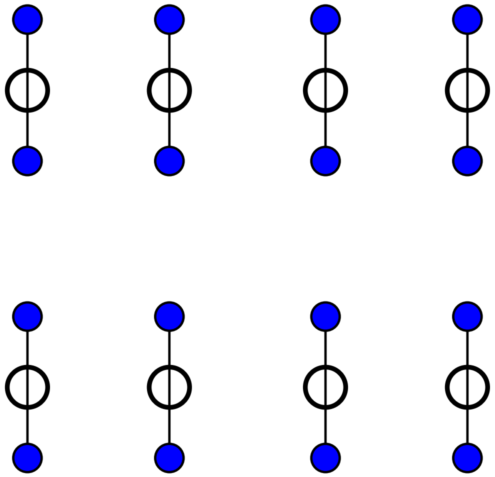
$$H = \sum_m [-2g(\sigma_m^z \sigma_{m+1}^z + \sigma_m^x \sigma_{m+1}^x) + 2t\sigma_m^z]$$

for even fermion parity and

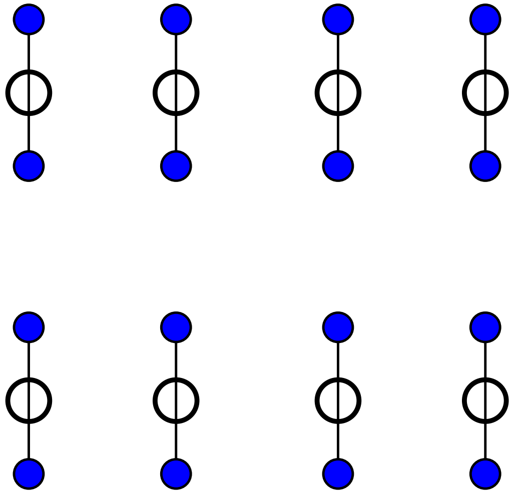
$$H = \sum_m [-2g(\sigma_m^z \sigma_{m+1}^z + \sigma_m^x \sigma_{m+1}^x) - 2t\sigma_m^x]$$

for odd fermion parity. These lead to gapped states with the spins ordering in the z or +x direction for even or odd fermion parity, for g>0. For even fermion parity $i\gamma_{m,0}\gamma_{m,1}, i\gamma_{m,2}\gamma_{m,3} \rightarrow \sigma_m^z$ so the ferromagnetic order corresponds to Majoranas pairing up to form Dirac fermions

On the other hand, for odd fermion parity $i\gamma_{m,1}\gamma_{m,2}, i\gamma_{m,3}\gamma_{m,0} \to -\sigma_m^x$ so we get the other mean field state:



even fermion parity



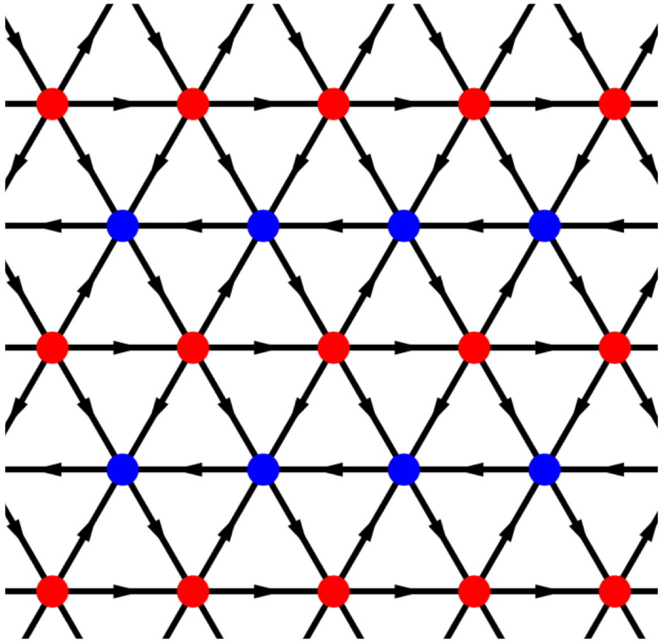
odd fermion parity

For g>0 we only find one transition. Mapping into complex fermions and ignoring interactions, we get 1 gapped complex fermion and 1 gapless complex fermion. At small g, we may integrate out the gapped mode. Then we find the U(1) breaking interactions are irrelevant and the U(1) preserving interactions are of standard spinless Luttinger liquid form: Umklapp term.

We thus predict a Kosterlitz-Thouless transition into a gapped phase at sufficiently strong g. This agrees well with DMRG results. In the 2D limit we expect this transition to become SUSY.

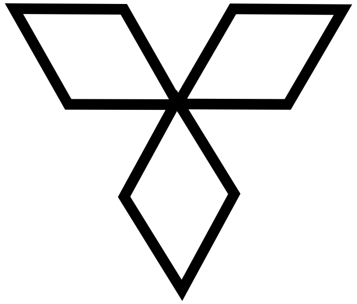
There are 3 additional transitions in the 4-leg ladder for negative g with stable phases with central charges 3/2, 2 and 1 which we haven't fully understood

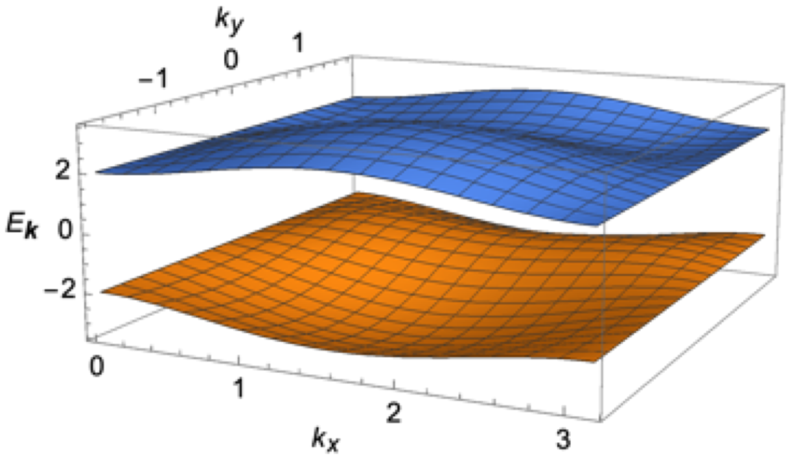
Triangular Lattice Case



$$\mathcal{H}_0 = \imath t \sum_{\langle ij \rangle} \eta_{ij} \gamma_i \gamma_j \quad \text{where} \quad \eta_{ij} \in \{+1, -1\}$$

$$\mathcal{H}_I = g \sum_i$$





-Now 3 types of plaquettes. The mean field states only minimizes the energy on 1/6 of the plaquettes.
So, mean field theory doesn’t predict Majorana pairing to form Dirac fermions.
-DMRG results on the 4-leg ladder do not find broken symmetries even at strong coupling, consistent with the mean field theory.
-There is a gapless phase for $g<-0.55$ which is rather mysterious

Conclusions

-the Majorana-Hubbard model on various lattices has rich phase diagrams
-Majoranas like to pair up to form complex fermions for strong enough coupling, breaking discrete symmetries
-Supersymmetry can be realized in both 1 and 2 dimensions