Can the Macroscopic Fluctuation Theory be Quantized ?

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Back in the 80's…

Hopefully useful in a future collaboration… 🧐

Is there a Quantum Analogue of the Macroscopic Fluctuation Theory ?

— **Is there universality in the fluctuations** (of local observables, of entanglement, etc) in out-of-equilibrium diffusive many-body systems ?....

— **Which questions we would like to address ?**

—> Describe **diffusive** (not ballistic) **but coherent processes** (at mesoscopic scale) **and their fluctuations in many body systems out-of-equilibrium** …

—> Get **quantitative information on transport, interferences, entanglement or information statistics and dynamics**, on **monitoring**, etc, in those systems …

— **Nice, rich, quantum stochastic processes**, generalising well known classical processes (say SSEP, ASEP) with **unexpected connexions** …
(integrability, probability & group theory, combinatorics, …)

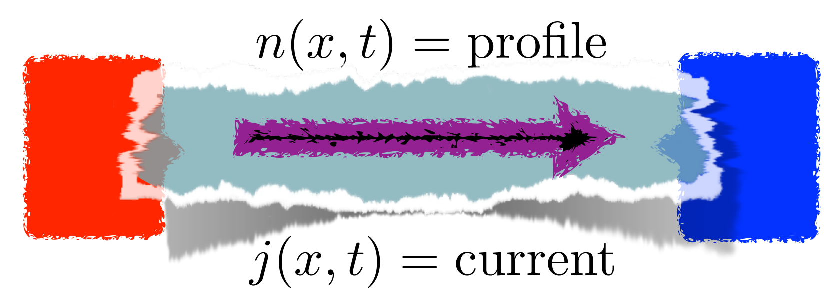
Classically : the Macroscopic Fluctuation Theory

— An effective theory coding for the **fluctuations in out-of-equilibrium classical systems** via a noisy Fourier-Fick's law

$$\partial_t \mathbf{n}(x,t) + \partial_x \mathbf{j}(x,t) = 0$$
$$\mathbf{j}(x,t) = -D(\mathbf{n})\partial_x \mathbf{n}(x,t) + \sqrt{L^{-1}\sigma(\mathbf{n})}\,\xi(x,t)$$

D(n) = diffusion constant
locally diffusive systems.

« noise »
(controlled by the system size)



[Bertini, Sole, Gabrielli, Jona-Lasinio, Landim, …]

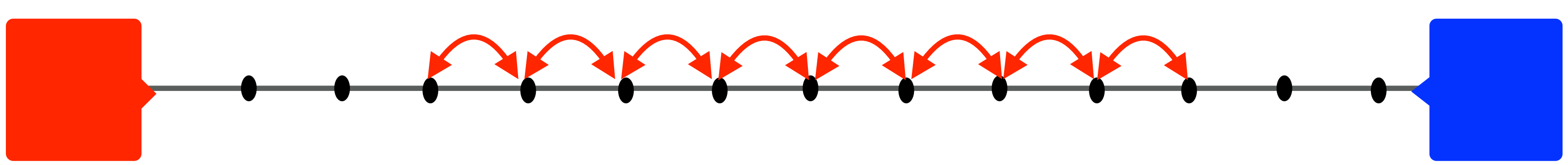
—> **Statistics on profile, transport and their fluctuations.**

Say, large deviation functions for fluctuations of the density profile :

$$\mathbb{P}\text{rob}\Big[\text{profile} = n(\cdot)\Big] \asymp e^{-(L/a_{uv})\,F[n(\cdot)]}$$

« analogue » of free energy out-of-equilibrium

— MFT emerged from studies of (stochastic) lattice out-of-equilibrium models…



SSEP : Markov chain model of hopping particles

[Kipnis, Landim, Liggett, Spohn, Derrida, Mallick, Evans, et al …]

A (Putative) Quantum Mesoscopic Fluctuation Theory ?

— Is quantizing (naively) the MFT an option ?

$$\partial_t \mathrm{n}(x,t) + \partial_x \mathrm{j}(x,t) = 0$$

$$\mathrm{j}(x,t) = \underbrace{-D(\mathrm{n})\partial_x \mathrm{n}(x,t)}_{\text{Fourier law}} + \sqrt{L^{-1}\underbrace{\sigma(\mathrm{n})\,\xi(x,t)}_{\text{noise}}}$$

—> Other options …

—> Simple (realistic ?) toy models…

What / how for Q-MFT ?

—> Deal with **diffusive** (not ballistic), **noisy but coherent quantum processes** …
in many-body systems (out-of-equilibrium) …

—> **Transport but beyond « hydrodynamics »**
(off-diagonal contribution)

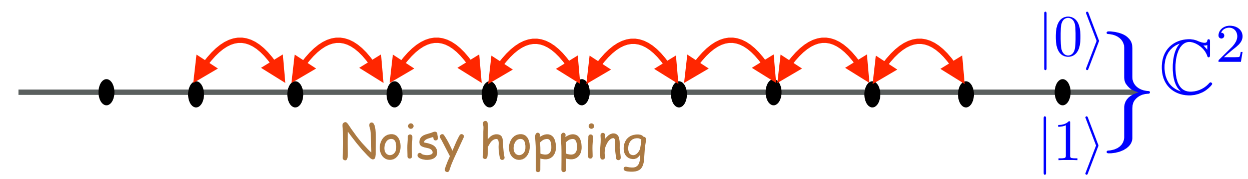
- structure of coherences ;
- structure of entanglement ;
- dynamics/fluctuations ;
- etc.

—> **Not yet fully available… Look for simple models** (if universal enough)…

The Quantum SSEP:

« Quantum Symmetric Simple Exclusion Process »

— Fermions hopping on a lattice with **Brownian amplitudes**



$$dH_t = \sqrt{D}\, \sum_j \left(c^\dagger_{j+1} c_j\, dW^j_t + c^\dagger_j c_{j+1}\, d\overline{W^j_t}\right)$$ + boundary terms…

— One of the simplest model of **quantum (fluctuating) diffusion**

$$d\hat{n}_j = D\ (\Delta \hat{n})_j\, dt + d[\mathrm{Qu-Noise}]$$ for $\hat{n}_j = c^\dagger_j c_j$ = number operators.

— Quadratic but noisy model (—> free fermion techniques) :

The evolution is stochastic, so is the quantum state and hence : $G_{ij} = \langle c^\dagger_j c_i \rangle_t$

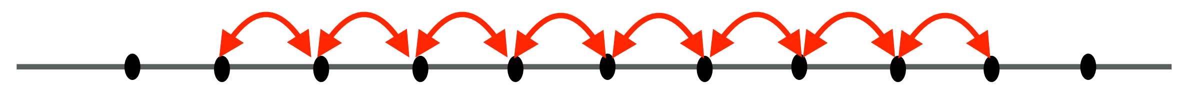
Stochastic process on G : $G_{t+dt} = e^{-i dh_t} G_t e^{+i dh_t}$

with \`dh\` the one-particle hamiltonian

— Closed versus Open systems

- (A) : **closed** —> equilibrium
- (B) : **open** —> driven out of equilibrium

The Quantum SSEP (II) :



— **Relation to SSEP ?** In mean, for $\bar{\rho}_t := \mathbb{E}[\rho_t]$

For mean density matrices, diagonal in the occupation number basis,

the dynamical equation $\partial_t \bar{\rho}_t = \mathcal{L}_{\rm ssep}(\bar{\rho}_t)$ is the SSEP master equation.

— The quantum model codes for **quantum coherent effects.**

classical | quantum

More than in the classical analogue (due to coherences/interferences)

— **Higher moments and fluctuations :** $\bar{\rho}_t^{(n)} := \mathbb{E}[\rho_t \otimes \cdots \otimes \rho_t]$

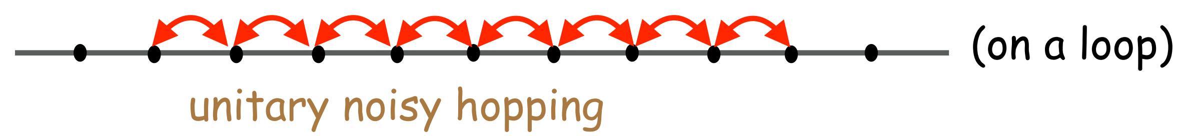
Their dynamics is also Lindblad like
(but at higher rank)

$$\partial_t \bar{\rho}_t^{(n)} = \mathcal{L}_{\rm ssep}^{(n)}(\bar{\rho}_t^{(n)})$$

spin chain
dynamics ...

(A) Closed model : Fluctuations at Equilibrium

— **Periodic boundary condition —> Equilibrium State** (at large time)



— **Steady/Invariant measure :** $G_{ij} = \langle c_j^\dagger c_i \rangle$ ⟶ mapping to RMT (via ergodicity)

- 1 point functions (mean decoherence):

$$\mathbb{E}[G_{ii}] = N_1/L \equiv \bar{n} \qquad \textbf{(equilibrium)}$$
$$\mathbb{E}[G_{ij}] = 0 \quad (i \neq j) \qquad \textbf{(decoherence)}$$

- 2 point functions (**fluctuating coherences**): ($\Delta\bar{n} =$ initial density variance, $N_k = \mathrm{Tr}(G^k)$)

$$\mathbb{E}[|G_{ij}|^2] = \frac{LN_2 - N_1^2}{L(L^2-1)} \equiv \frac{(\Delta\bar{n})^2}{L}$$

i.e. $\langle c_j^\dagger c_i \rangle \simeq O(\frac{1}{\sqrt{L}})$

—> **Fluctuating coherences** are generated from the inhomogeneities of the profile.

— Steady state = **non random equilibrium state + fluctuations**

$$\lim_{t\to\infty} \rho_t \equiv \rho_{\mathrm{eq.}} + (\delta\rho)$$

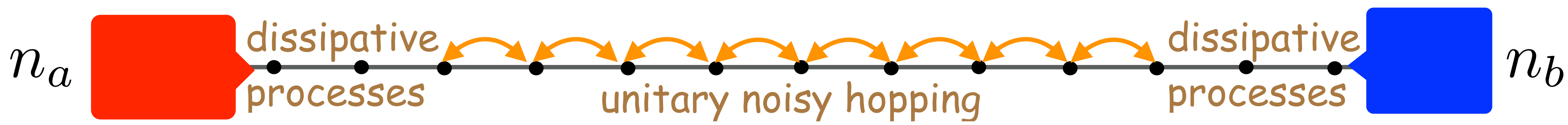
with $\rho_{\mathrm{eq.}} = Z^{-1}\, e^{-\mu \hat{N}_{\mathrm{tot}}}$

plus fluctuations are of order 1/L^{1/2}

—> **Large deviation function.**

(B) Open model : Non-Equilibrium Fluctuations

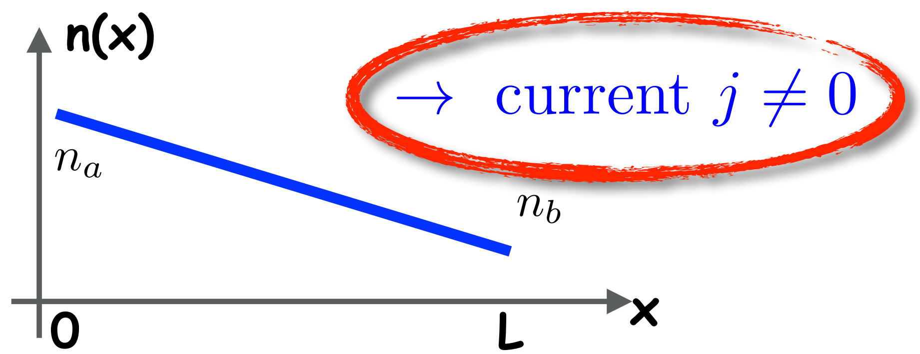
— **Open boundary condition** —> System is driven **out-of-equilibrium**



— **Mean profile** (x=j/L) —> out-of-equilibrium:

$$[n_j] := \mathbb{E}[\langle c_j^\dagger c_j \rangle] = n_a + x(n_b - n_a)$$

(x=j/L, at large system size L)



— **Fluctuations & coherences** —> Steady statistics of (coherent) fluctuations.

$$\mathbb{E}[G_{ij}G_{ji}] = \frac{1}{L}(\Delta n)^2 x(1-y) + O(L^{-2}) \qquad \text{for } G_{ij} = \langle c_j^\dagger c_i \rangle$$

More generally, the scaling with L of the higher cumulants ensures the (formal) existence of large deviation function

recursively known

$$\mathbb{E}[G_{i_1 i_N} \cdots G_{i_3 i_2} G_{i_2 i_1}]^c = \frac{1}{L^{N-1}} g_N(x_1, \cdots, x_N) + O(L^{-N})$$

Where are we (with Q-SSEP & Co) ?

— Invariant measure (large time statistics)

—> Closed Q-SSEP : Mapping to RMT and to the HSIZ integral. ✓
—> Open Q-SSEP : Combinatorial results… (« need a better description »)

— Large deviation for state fluctuations

—> Closed Q-SSEP : Formal power series (only). ✓
—> Open Q-SSEP : Better characterization ?… (« missing … »)

— Entanglement (steady) statistics (dynamics ??)

—> Closed Q-SSEP : Large deviation function for sub-system entanglement ✓
—> Open Q-SSEP : (« missing … »)

— Fluctuation dynamics (scaling/continuous theory)

—> Closed Q-SSEP : Hydrodynamics like description of the large time dynamics of fluctuations (—> relevant corrections to diffusion)…. ✓ ✓
—> Open Q-SSEP : Hydrodynamics… (« in progress »)

—> Extension of the domain of validity of those hydrodynamics equations « à la MFT »….

(Un-reasonable) connexions to combinatorics

- Recall $$\left[G_{i_1 i_N} \cdots G_{i_3 i_2} G_{i_2 i_1}\right]^c = \frac{1}{L^{N-1}}\, g_N(x_1, \cdots, x_N) + O(L^{-N})$$

- Take all points to be equal : $x_j = -t$

  Then : $$g_N(\mathbf{x})|_{x=-t} = t(1+t)\Phi_{N-1}(t)$$

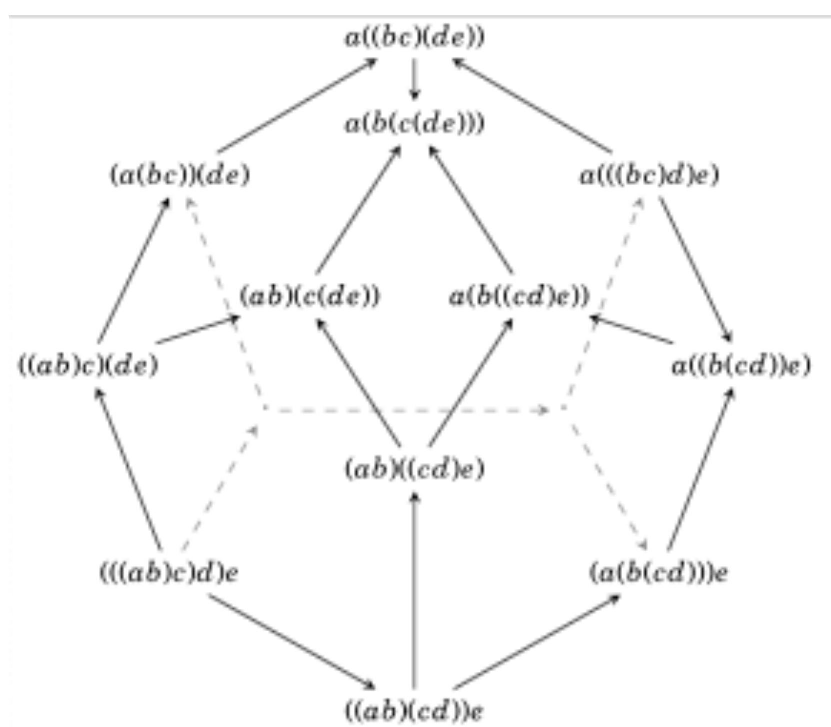
$$\begin{aligned}
\Phi_2(t) &= 1 + 2t, \\
\Phi_3(t) &= 1 + 5t + 5t^2, \\
\Phi_4(t) &= 1 + 9t + 21t^2 + 14t^3, \\
\Phi_5(t) &= 1 + 14t + 56t^2 + 84t^3 + 42t^4.
\end{aligned}$$

—> These are the generating functions counting the number of (n-k) dimensional faces in the **associahedron** of order n.

- vertices = different parenthesis
- edges = associativity rule

$$(a(bc)) \longleftrightarrow ((ab)c)$$

- this graph is a polytope.



Conclusion :

Many open questions related to quantum stochastic processes and to constructing a Quantum Mesoscopic Fluctuation Theory

… … …

Thank you !!

Bon Anniversaire !!
Hubert !!

Closed Q-SSEP invariant measure (ergodicity)

Theorem: [Bauer-Bernard-Jin]

Let G_0 be the initial matrix of 2-point functions

Let Z(A) be the stationary generating function

$$Z(A) = \mathbb{E}[e^{\mathrm{tr}(AG)}]$$

Then :

Z(A) is U(L) invariant and represented by the Harish-Chandra-Itzykson-Zuber integral from random matrix theory:

$$Z(A) = \int_{U(L)} dV\, e^{\mathrm{tr}(VAV^\dagger G_0)}$$

—> Large deviation function for G.

— **Why**: - The U(L) invariance emerges from the dynamics (stationarity condition).

- Consequence from Hormander's theorem (via iteration of small generators)

$$e^{-idh_{t_1}} \cdot e^{-idh_{t_2}} \cdots e^{-idh_{t_m}}$$

—> A kind of ergodicity.

- If U(L) invariant

$$Z(A) = \int_{U(L)} dV\, Z(VAV^\dagger) = \int_{U(L)} dV\, \mathbb{E}[e^{\mathrm{tr}(AV^\dagger GV)}] = \int_{U(L)} dV\, e^{\mathrm{tr}(VAV^\dagger G_0)}$$

Closed Q-SSEP : dynamics of fluctuations (i)

[Bernard-Essler-Hruza-Medenjak]

— **In the scaling limit :**

$$a_{UV} \to 0,\ N \to \infty, L = N a_{UV} = \text{fixed}, \quad J \to \infty, D = J a_{UV}^2 = \text{fixed}$$

— **Mean density is diffusive. Higher diffusive but with relevant corrections.**

$$\mathfrak{g}_+(x,y,t) := \lim_{\text{scaling}} \mathbb{E}[G_{jj}(t) G_{kk}(t)] \ , \qquad \text{(density correlations)}$$

$$\mathfrak{g}_-(x,y,t) := \lim_{\text{scaling}} \mathbb{E}[G_{jk}(t) G_{kj}(t)] \ . \qquad \text{(coherence correlations)}$$

with $\mathfrak{g}_\sigma(x,y,t) = \mathfrak{g}_\sigma^{(0)}(x,y,t) + \varepsilon\, \mathfrak{g}_\sigma^{(1)}(x,y,t) + O(\varepsilon^2)$. $\qquad \varepsilon = a_{UV}/L = 1/N$

- diffusion of densities

$$\partial_t \mathfrak{g}_\sigma^{(0)}(x,y,t) = D(\nabla_x^2 + \nabla_y^2)\mathfrak{g}_\sigma^{(0)}(x,y,t) \ .$$

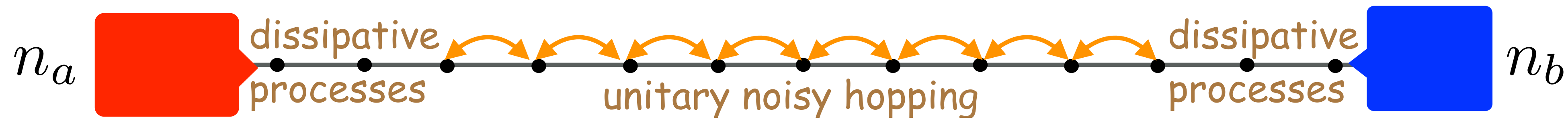
$$\mathfrak{g}_+^{(0)}(x,y,t) = \mathfrak{n}(x,t)\mathfrak{n}(y,t) \ , \quad \mathfrak{g}_-^{(0)}(x,y,t) = 0 \ .$$

- correction to diffusion of coherences

$$\partial_t \mathfrak{g}_-^{(1)}(x,y,t) = D(\nabla_x^2 + \nabla_y^2)\mathfrak{g}_-^{(1)}(x,y,t) + \mathfrak{h}_+(x,y,t) \ ,$$

$$\text{with } h_+(x,y,t) = 2\, DL\, \nabla_x \nabla_y \big(\delta(x-y)\mathfrak{n}(x,t)\mathfrak{n}(y,t)\big) \ .$$

(B) Open : Non-Equilibrium Steady Fluctuations



— Steady statistics of (coherent) fluctuations : for $G_{ij} = \langle c_j^\dagger c_i \rangle$

Example: $[G_{ij} G_{ji}] = \frac{1}{L} (\Delta n)^2 x(1-y) + O(L^{-2})$ (with x=i/L, y=j/L, i<j)

— Products of G's are represented by graphs. (vertices = points, edges = G-insertions)
The steady measure is then a measure on labeled graphs.

Claim:

(i) The single G-loops are the **dominant cumulants** (of G's).

(ii) $\left[G_{i_1 i_N} \cdots G_{i_3 i_2} G_{i_2 i_1}\right]^c = \frac{1}{L^{N-1}} g_N(x_1, \cdots, x_N) + O(L^{-N})$

(iii) The g_N's are **recursively explicitly known** (polynomials in the x's) and can be computed from generating functions.

(iv) The scaling with L ensures the existence of a **large deviation function**.

—> Ad-hoc techniques (using the permutation group and formal power series)
Connexion with combinatorics (Catalan number, etc). (polynomials with integer coefficients)

Low order cumulants (Open Q-SSEP):

— Cumulants of order two at finite system size:

For i<j:

$$[G_{ij}G_{ji}]^c = \frac{(\Delta n)^2\,(i+a)(L-j+b)}{(L+a+b-1)(a+b+L)(a+b+L+1)},$$

$$[G_{ii}G_{jj}]^c = -\frac{(\Delta n)^2\,(i+a)(L-j+b)}{(L+a+b-1)(a+b+L)^2(a+b+L+1)},$$

$$[G_{ii}^2]^c = \frac{(\Delta n)^2\,\big(2(i+a)(L-i+b)-(L+a+b)\big)}{2(a+b+L)^2(a+b+L+1)},$$

— Cumulants of order two at large system size:

off-diagonal fluctuations (coherences)

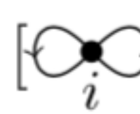
For x<y:

$$[G_{ij}G_{ji}]^c = \frac{1}{L}(\Delta n)^2\,x(1-y) + O(L^{-2}),$$

$$[G_{ii}G_{jj}]^c = -\frac{1}{L^2}(\Delta n)^2\,x(1-y) + O(L^{-3}),$$

$$[G_{ii}^2]^c = \tfrac{1}{L}(\Delta n)^2\,x(1-x) + O(L^{-2})$$

diagonal fluctuations

— $[G_{ii}G_{jj}]$ for $i \neq j$ by [], $[G_{ij}G_{ji}]$ for $i \neq j$ by [] and $[G_{ii}^2]$ by [].