H.S. is 60

(P. Di Francesco)

IPHT, 2021

1987

International Summer School on Conformal Invariance and String Theory

1-12 September 1987. Poiana Brasov, Romania (C87-09-01.3)

Two-dimensional critical models on a torus

H. Saleur (Saclay), P. di Francesco (Saclay) (1987)

Published in: In *Poiana Brasov 1987, Proceedings, Conformal invariance and string theory* 63-87. •

Contribution to: International Summer School on Conformal Invariance and String Theory, 63-87

1988

Fields, Strings, Critical Phenomena

28 June-5 August 1988.
Les Houches, France

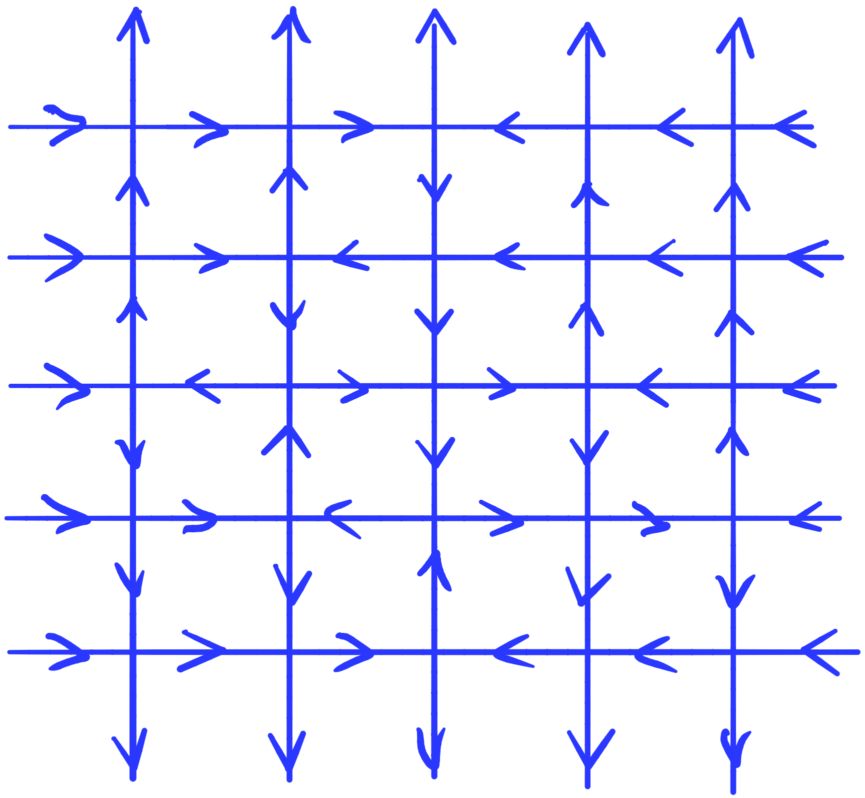
TRIANGULAR ICE: COMBINATORICS & LIMIT SHAPES

(PDF + E. Guitter IPHT Saclay)

+ B. Debin UC Louvain

1. square ice, 6V model, DWBC
2. Triangular ice, 20V model, DWBC
3. Domino Tilings of the Holey Aztec Square
4. Proof of the 20V – HAS DT correspondence
5. Limit shape / Arctic Phenomenon

1. 6V model combinatorics



Replace data by dipolar momenta $\{\rightarrow, \leftarrow, \downarrow, \uparrow\}$

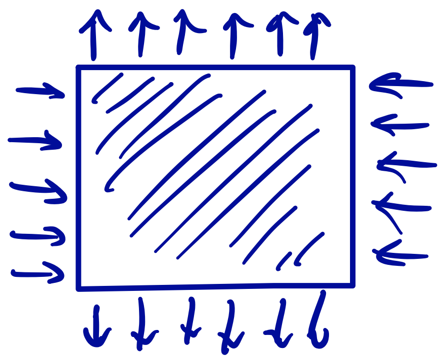
Ice Rule at each vertex

incoming arrows

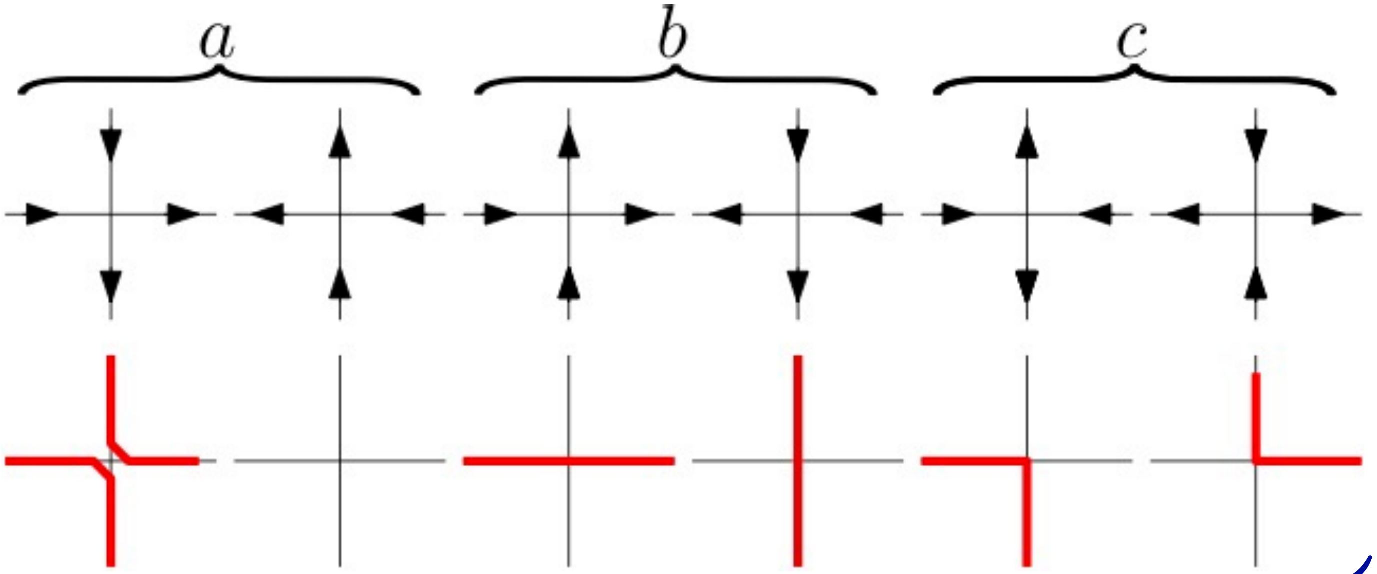
= # outgoing arrows $\Rightarrow$ 6V

+ Domain Wall Boundary Conditions

($n \times n$ square)



Bijections



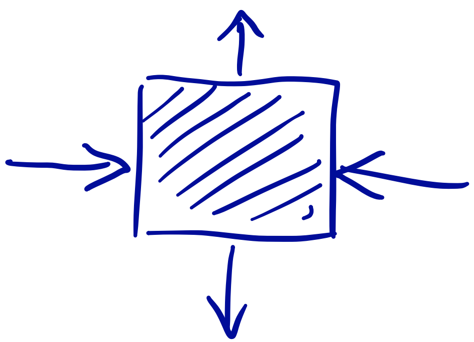
Transmittor vertices

Reflector vertices

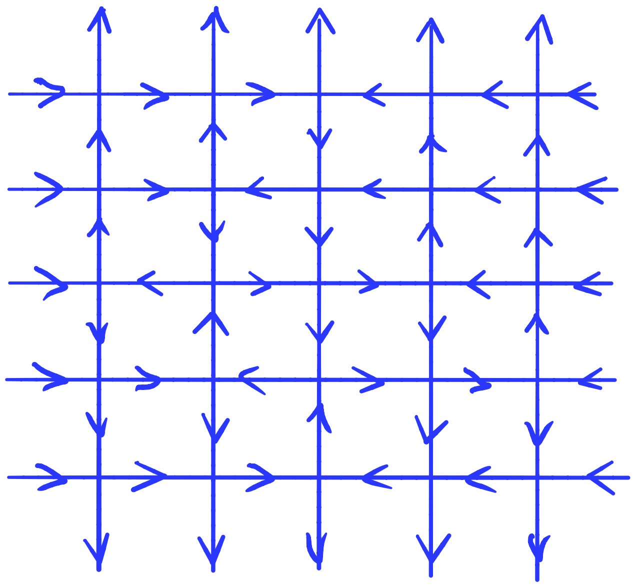
0	0	0	0	+1	−1

1. 6V configs
2. osculating paths
   (NW → SE)
3. ASM entries

Alternance conditions

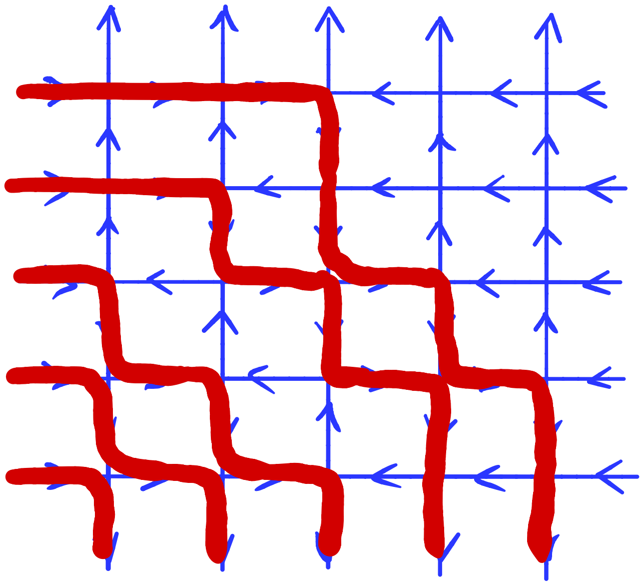


odd # of reflections!



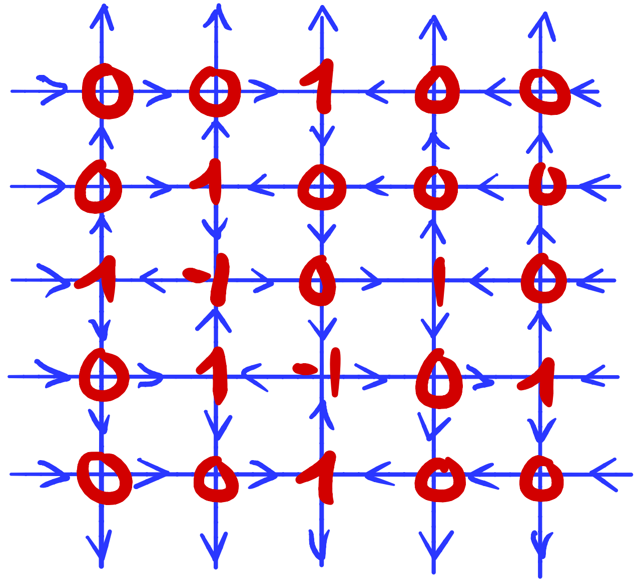
6V

1



Osculating Paths

2



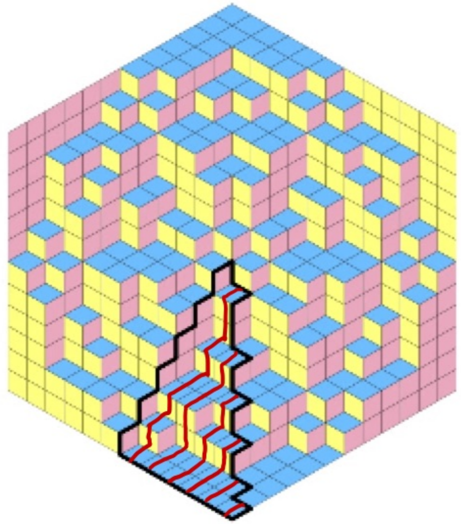
ASM

3

$$\overset{n}{\longleftrightarrow}$$

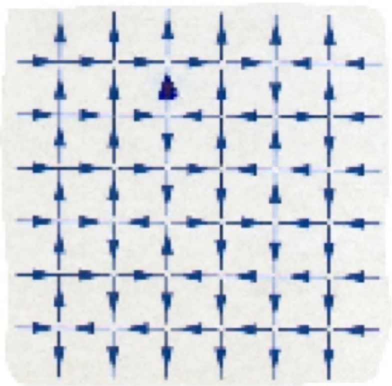
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad n$$

ASM



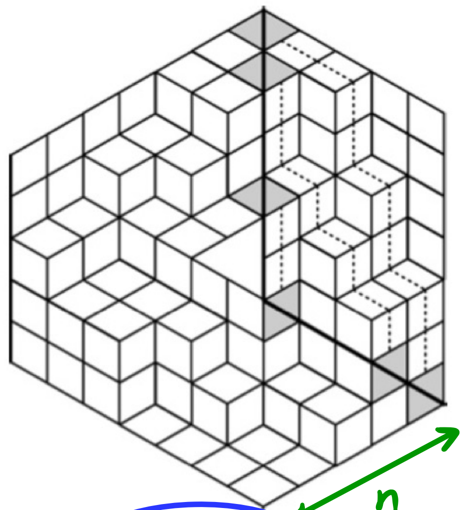
TSSC PP

$$ASM_n = \frac{\prod_{i=0}^{n-1} (3i+1)!}{\prod_{i=0}^{n-1} (n+i)!}$$



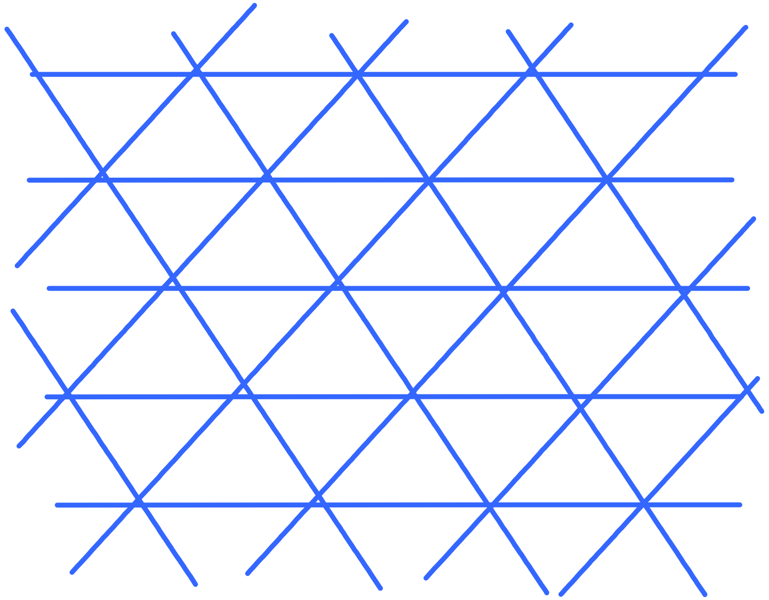
6V DWBC

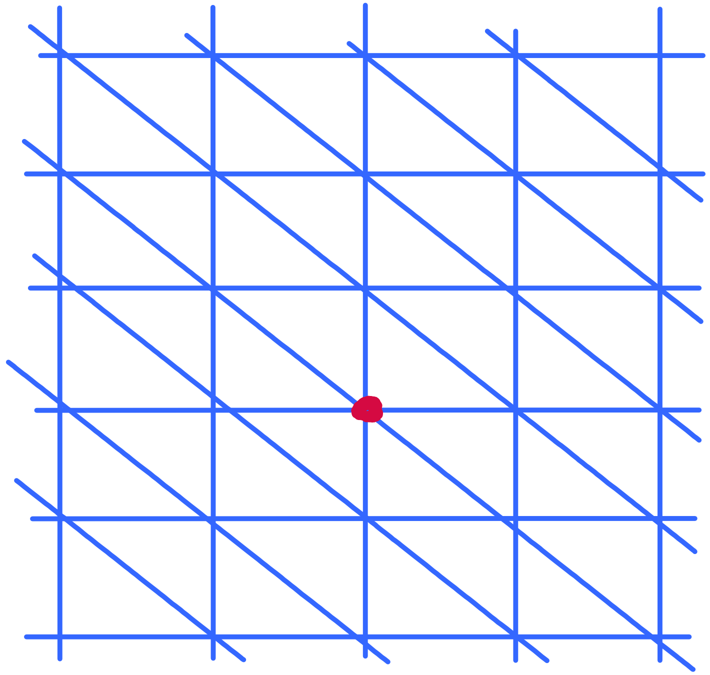
Korepin, Izergin
Andrews
Kuperberg, Zeilberger
Razumov, Stroganov
Cantini Sportiello
Zinn-Justin, PDF, Behrend
DeGier, Nienhuis
Knutson, Krattenthaler, Fisher



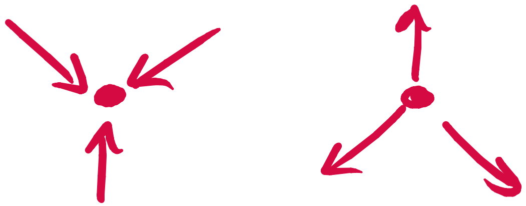
DPP

2. TRIANGULAR ICE (20V model)





ice rule



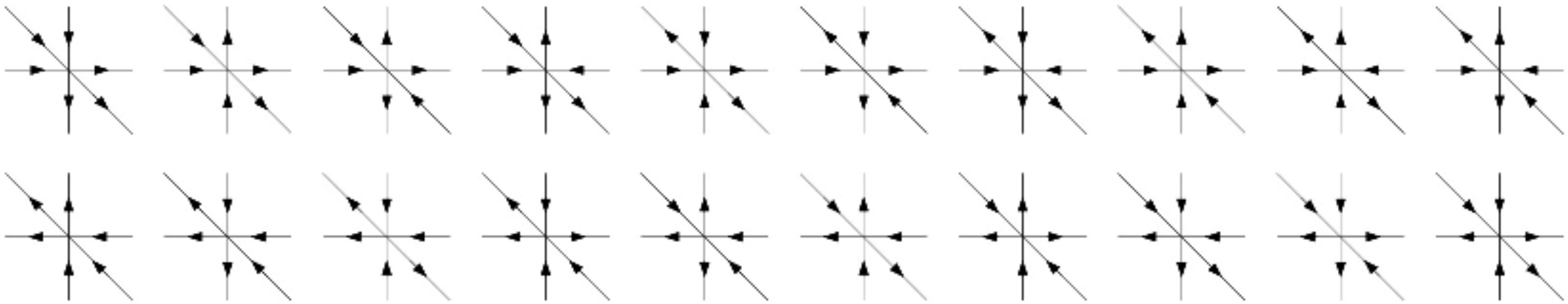
at each vertex

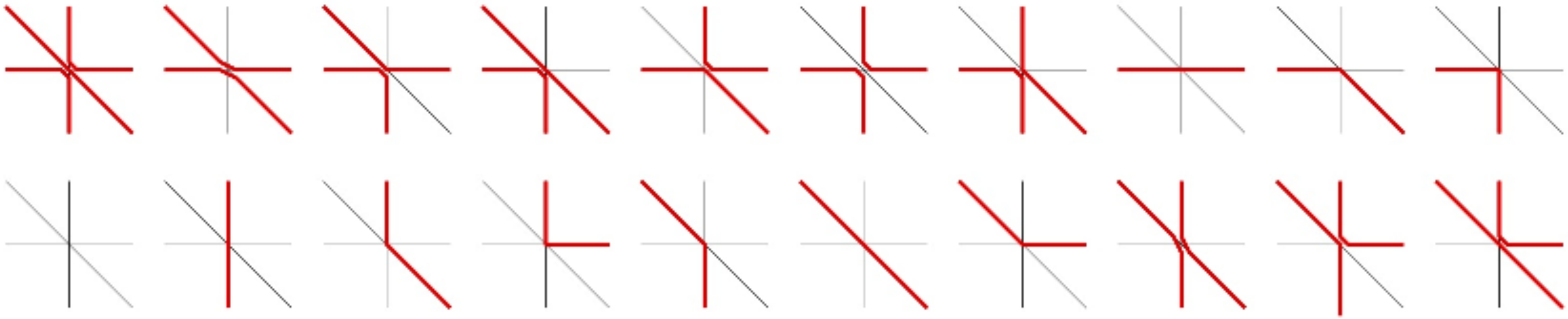
$$\binom{6}{3} = 20$$

[Kelland, Baxter]

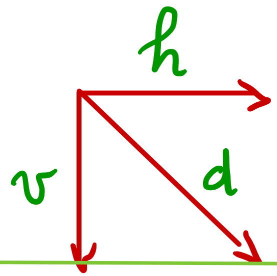
TRIANGULAR ICE (20V model)

Twenty vertices:





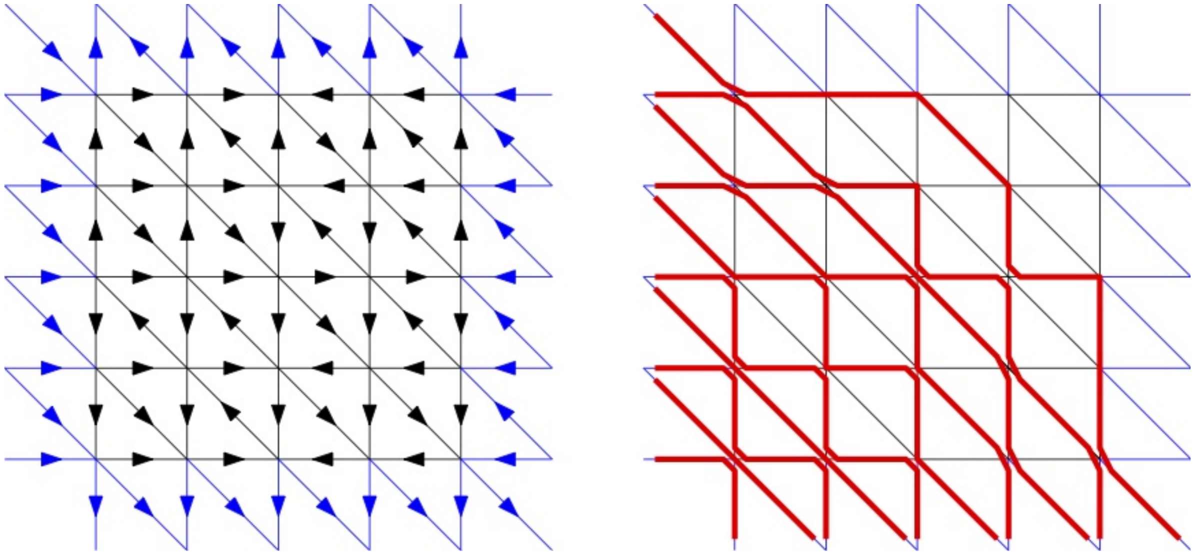
Osculating Schröder paths



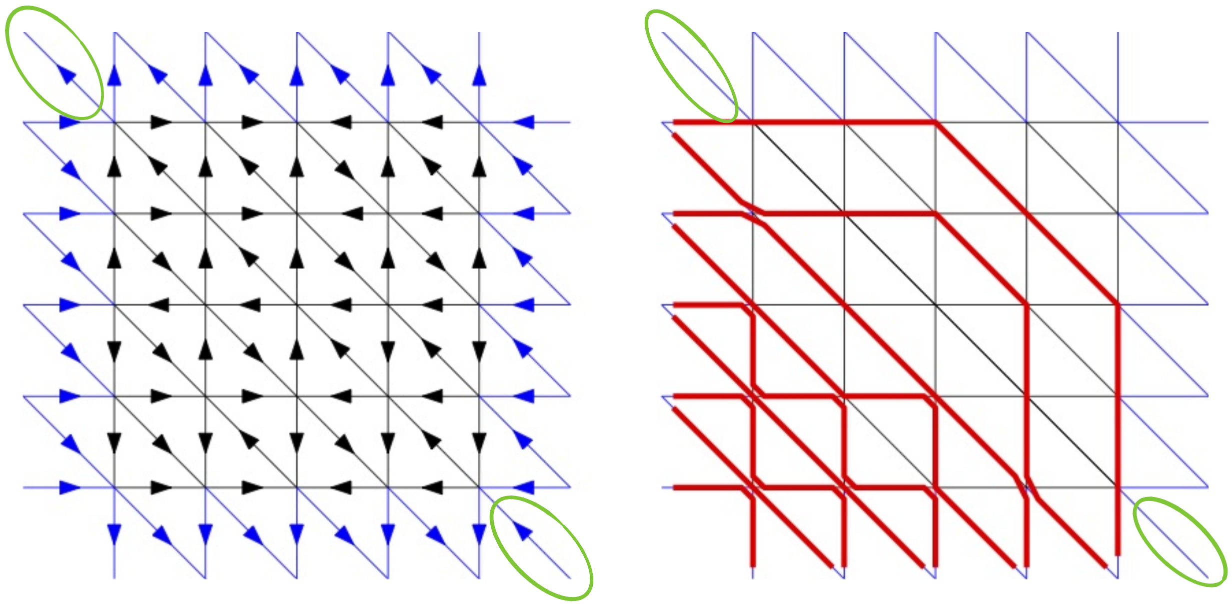
h, v, d steps

DOMAIN WALL BOUNDARY CONDITIONS

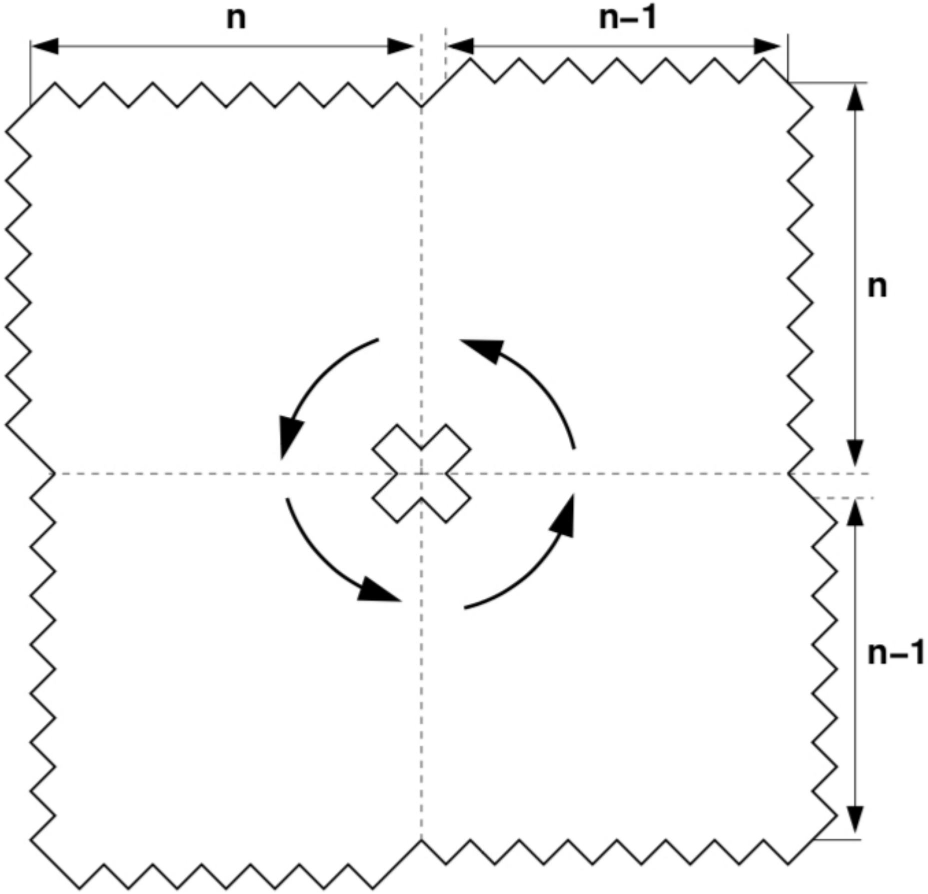
DWBC1



DWBC2



3. DOMINO TILINGS OF THE HOLEY AZTEC SQUARE WITH QUARTER-TURN SYMMETRY



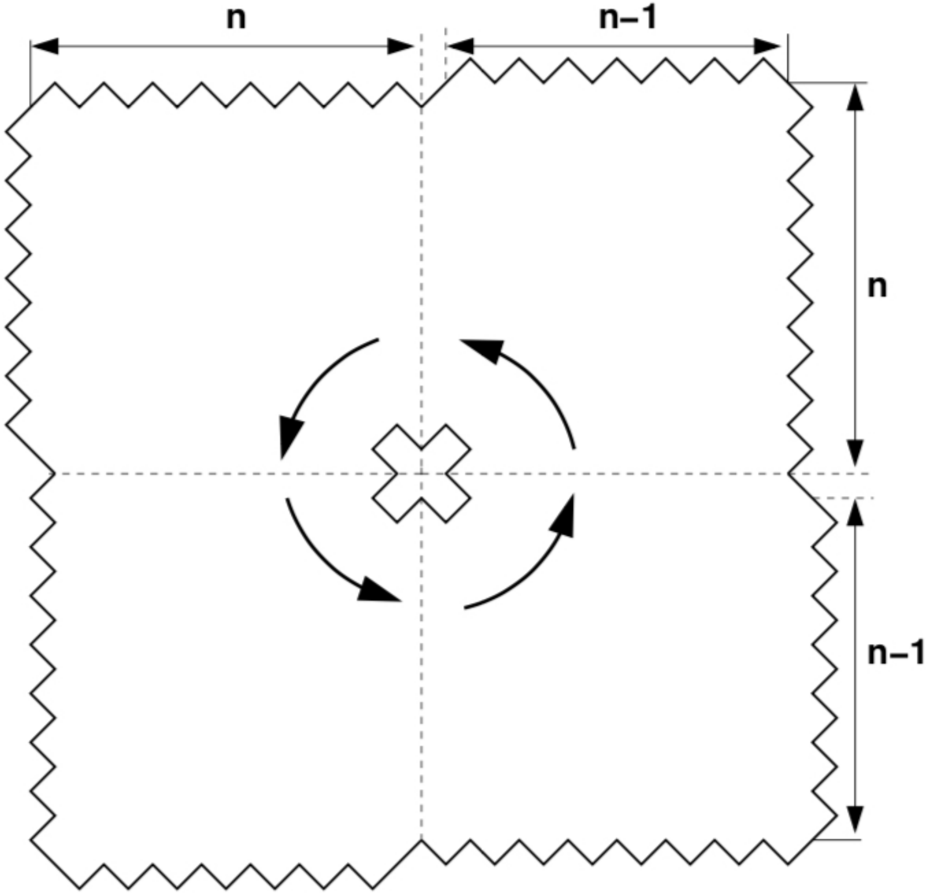
$\mathcal{S}_n$

Domino Tilings: use ◇ and ◇ 2×1 dominos

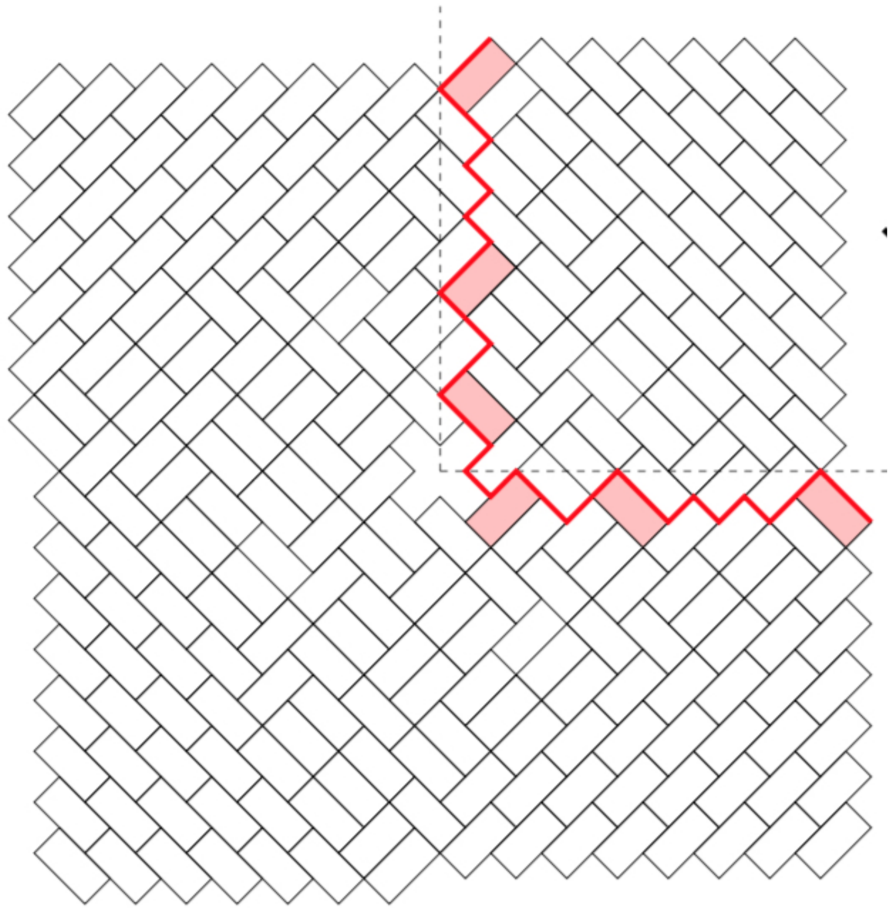
Rotational symmetry by $\frac{\pi}{2}$

NB: the hole makes it tileable!

DOMINO TILINGS OF THE HOLEY SQUARE WITH QUARTER-TURN SYMMETRY

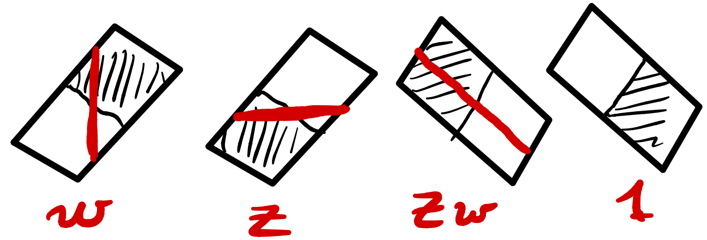


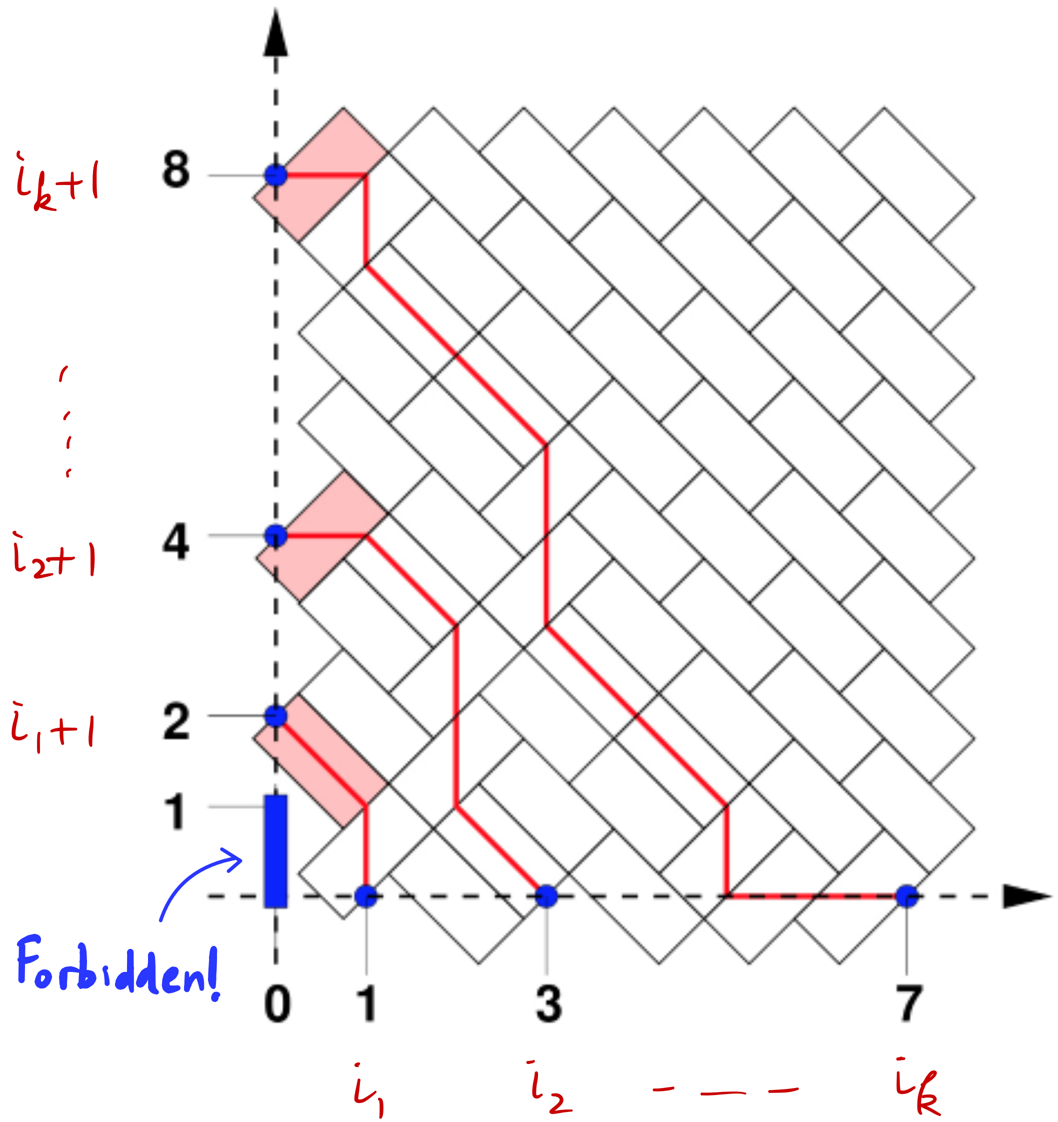
$\mathcal{S}_n$



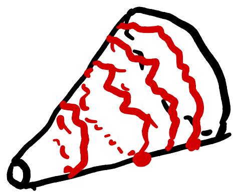
sample tiling

Counting Configurations





- Non-intersecting Schröder paths w fixed ends
- first step cannot be |
- start and ends identified (cone).



Counting Configurations

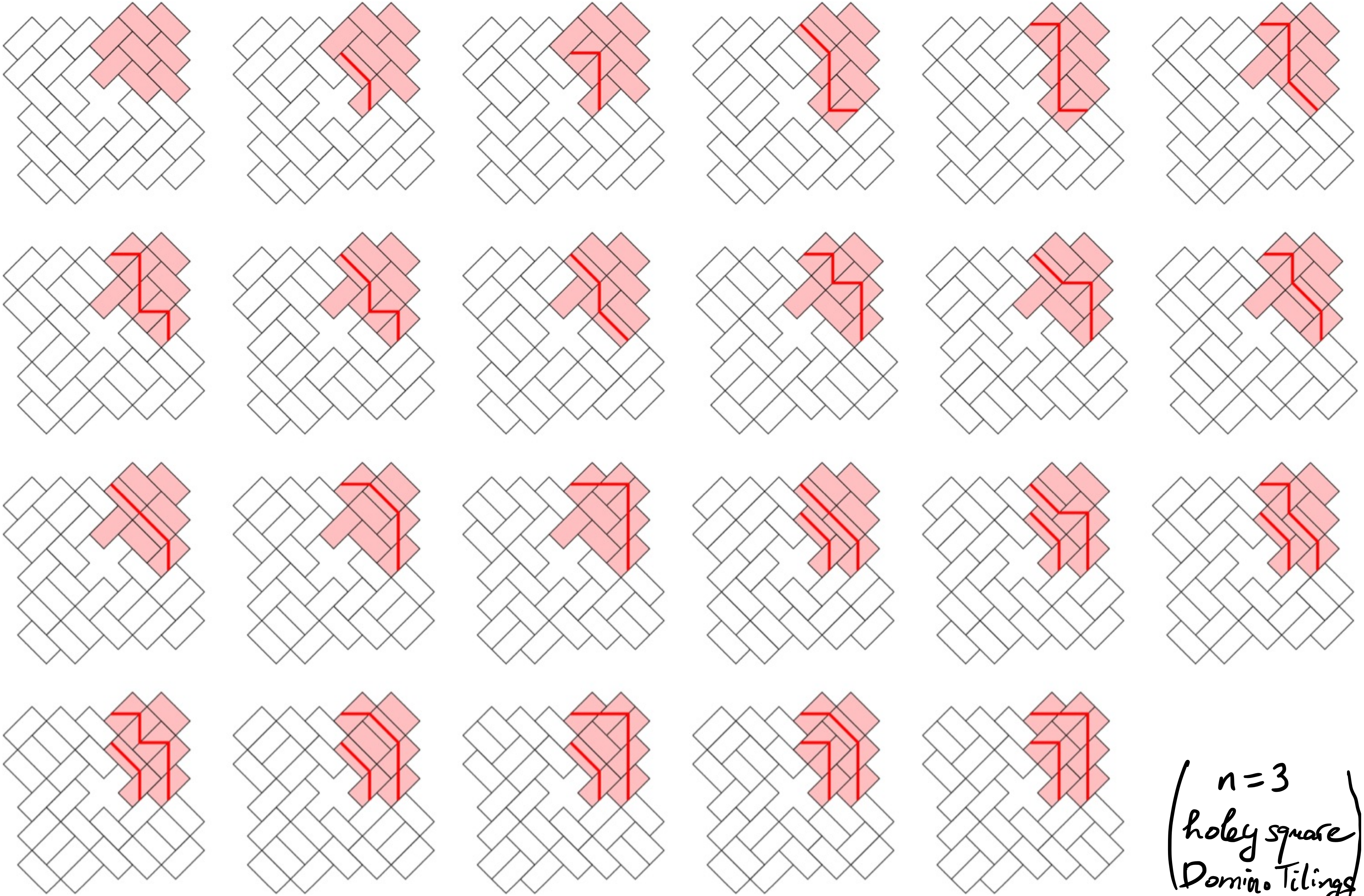
Thm [PDF-Guitter 19]

$$T_4(\mathcal{S}_n) = \det_{0\le i,j\le n-1}\left(\left\{ \frac{1}{1-zw} + \frac{2z}{(1-z)(1-z-w-zw)} \right\}_{z^i w^j} \right)$$

Proof: (Cauchy Binet) $\det(\mathrm{Id} + M) = \sum_{i_1<\dots<i_k} \underbrace{\left| M^{i_1\dots i_k}_{i_1\dots i_k} \right|}$ (Gessel-Viennot)

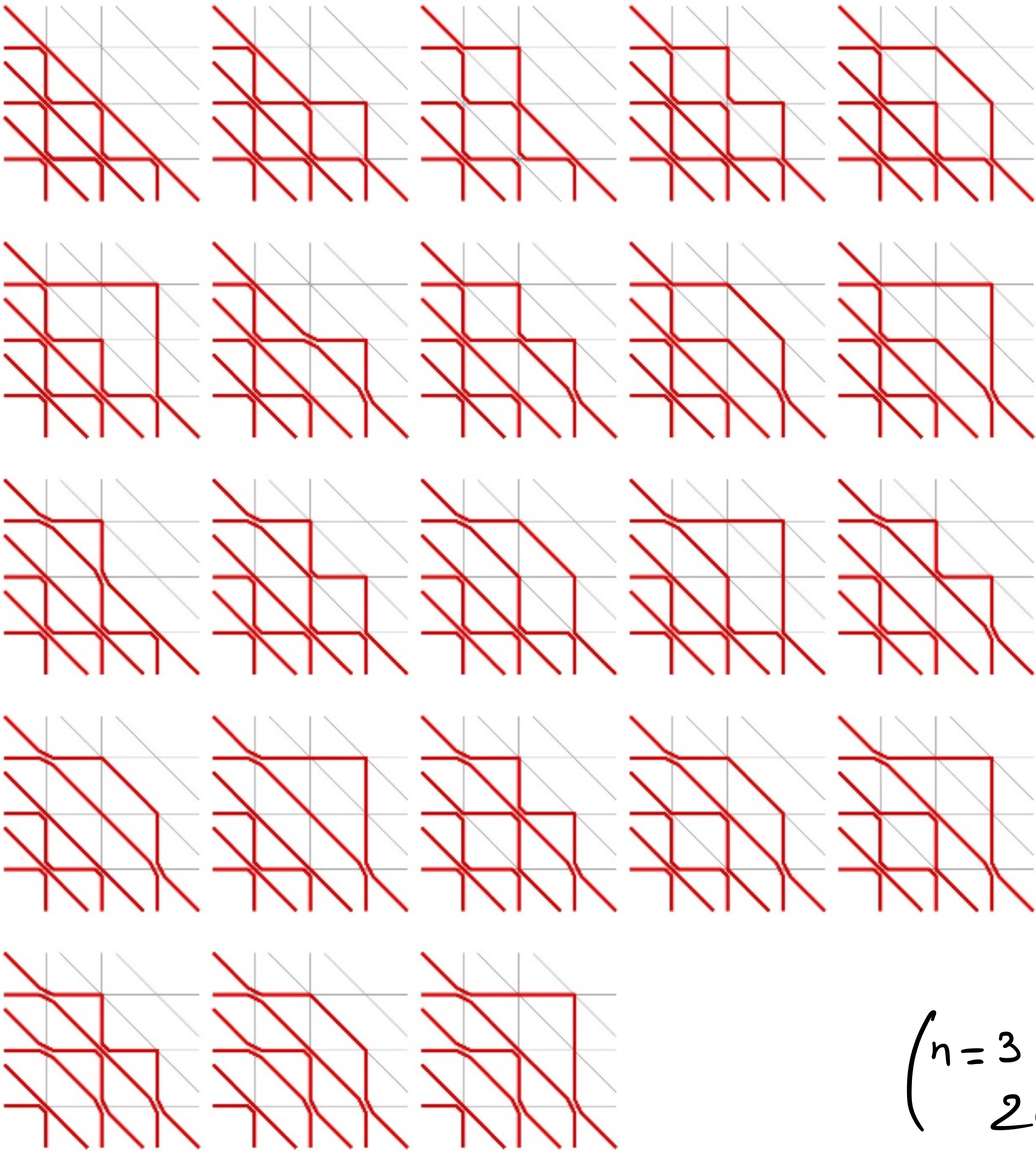
$$T_4(\mathcal{S}_n) = 1,\ 3,\ 23,\ 433,\ 19705,\ 2151843,\ \dots$$

Ex: $n=3$ $\det\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 4 & 8 & 12 \end{pmatrix}\right] = 23$ Domino Tiling configurations $\rightarrow$



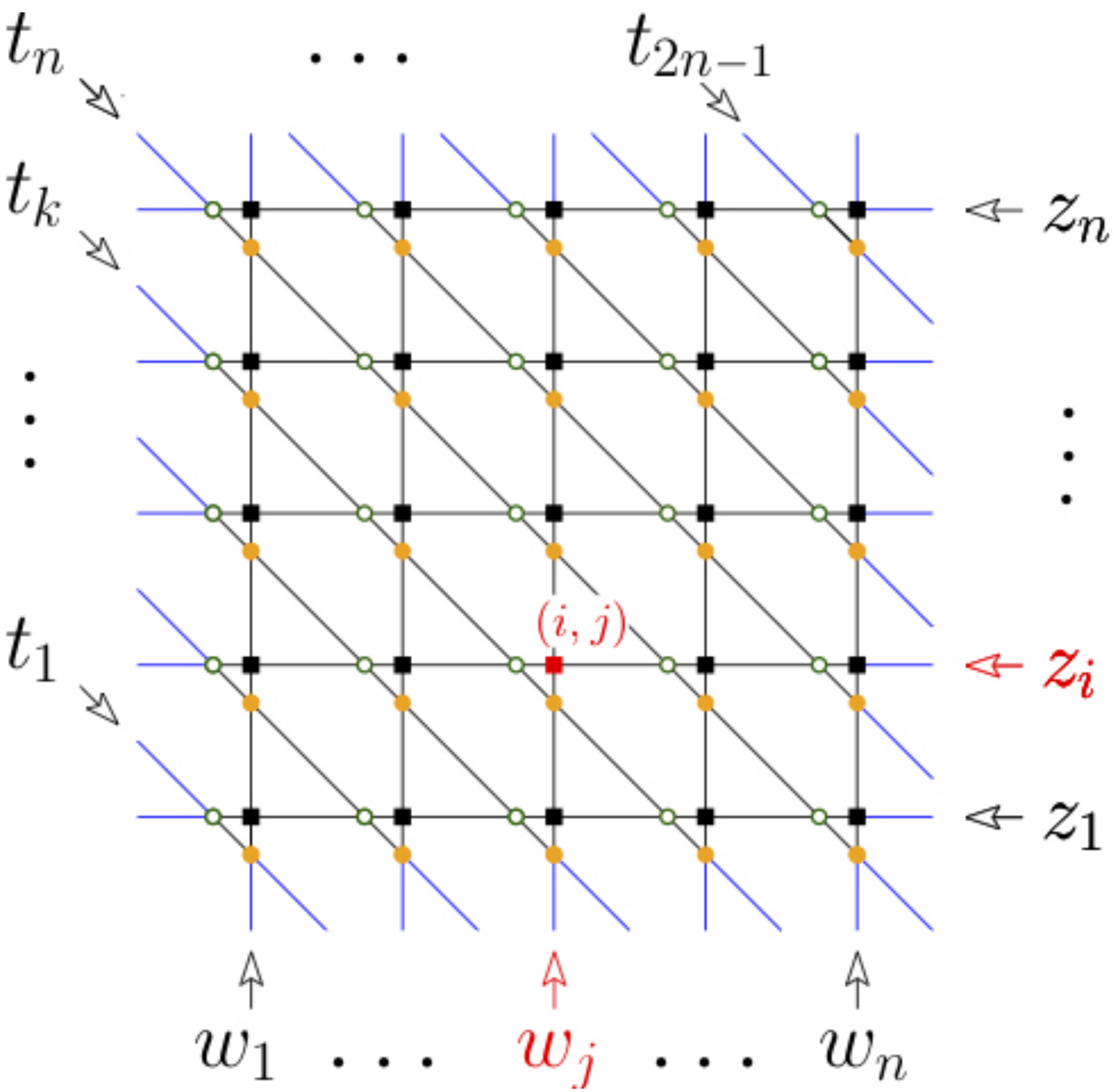
$n=3$

(holey square
Domino Tilings)

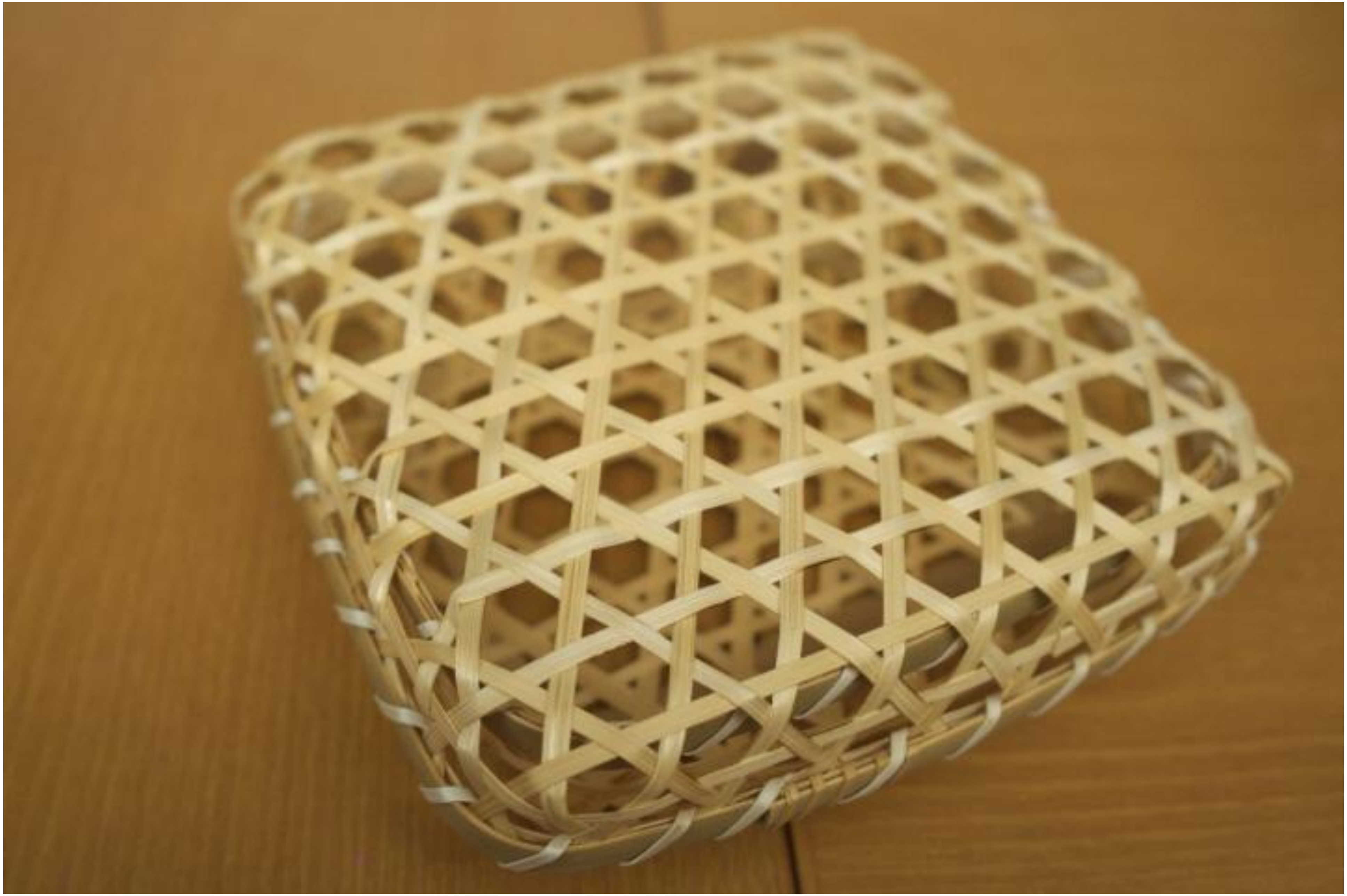


(n = 3 DWBC1
20V configurations)

ICE ON THE KAGOME LATTICE

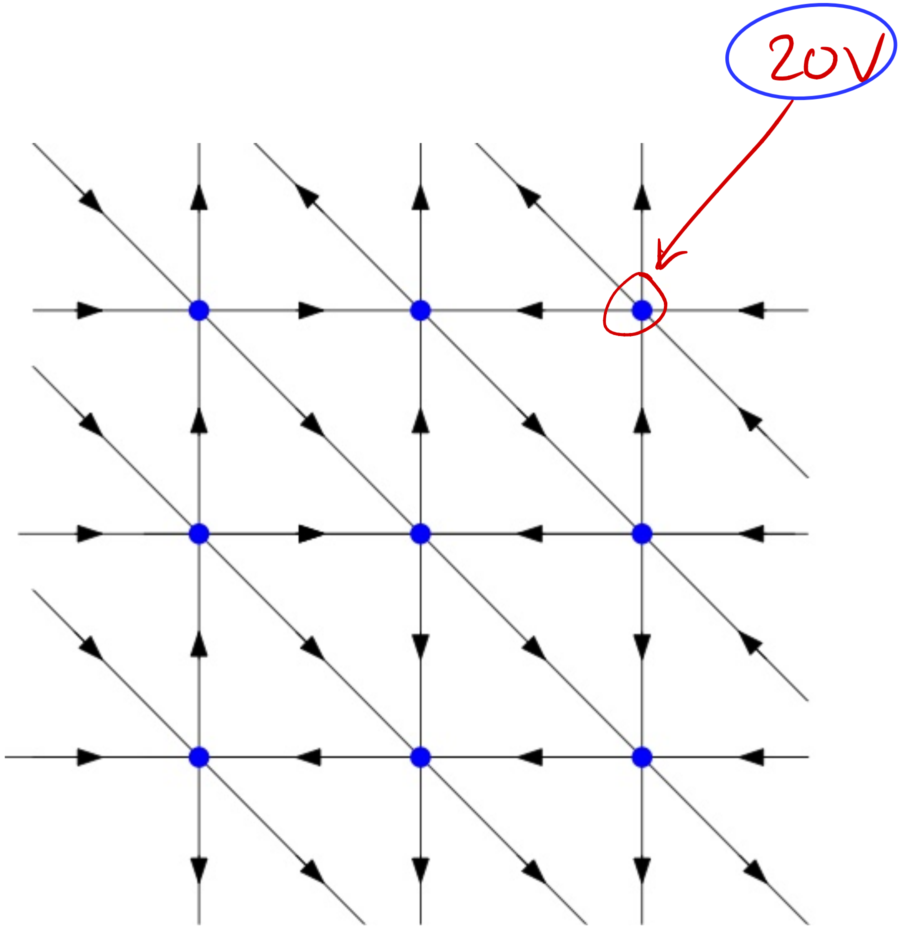


- z_i, w_j, t_k are complex (spectral) parameters
- The weights are functions of a pair of spectral parameters and obey the Yang-Baxter eqn

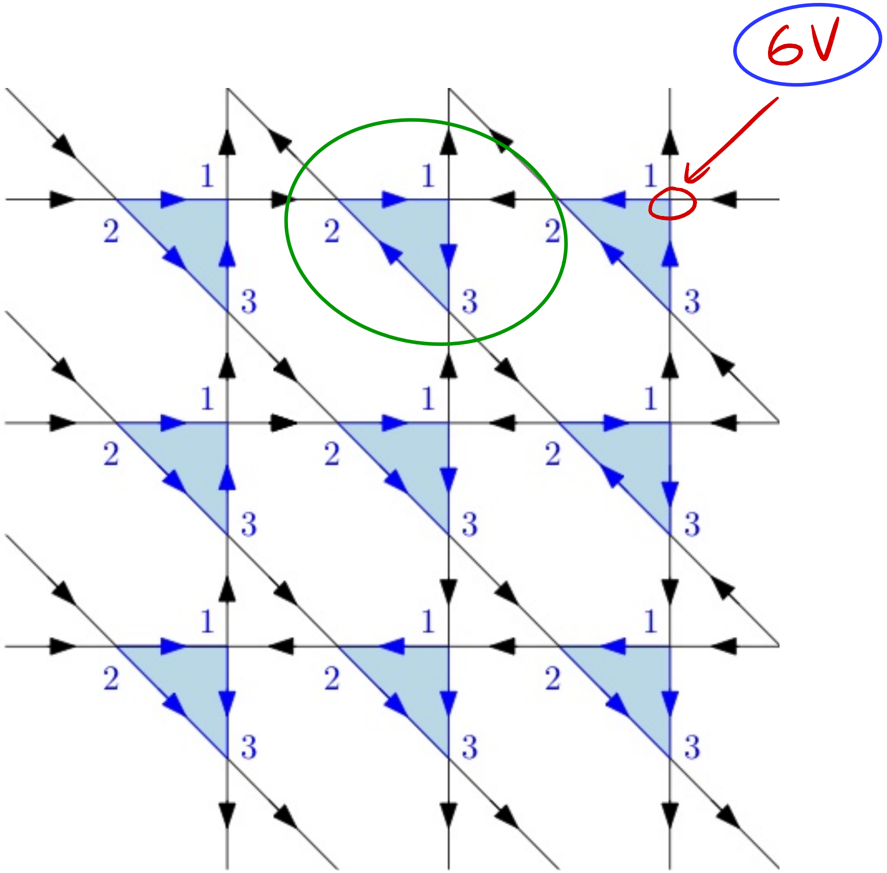


籠 (*kago*, “basket”) + 目 (*me*, “eye, hole”)

A lattice of KAGOME (Daikokuya, Kitashirakawa)

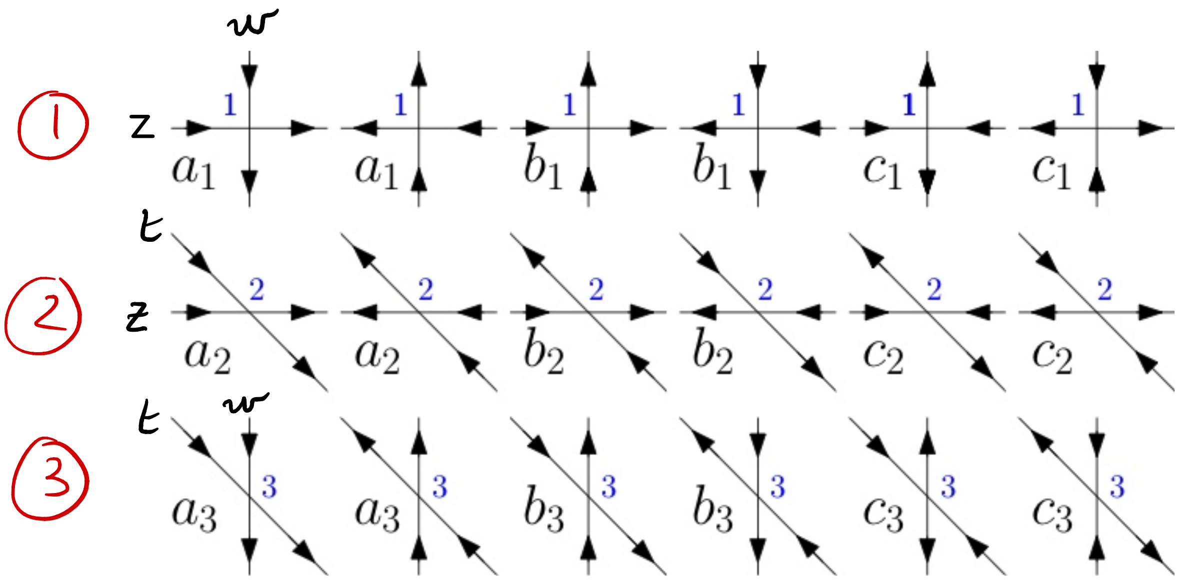


Triangular lattice ice



Kagome Lattice ice

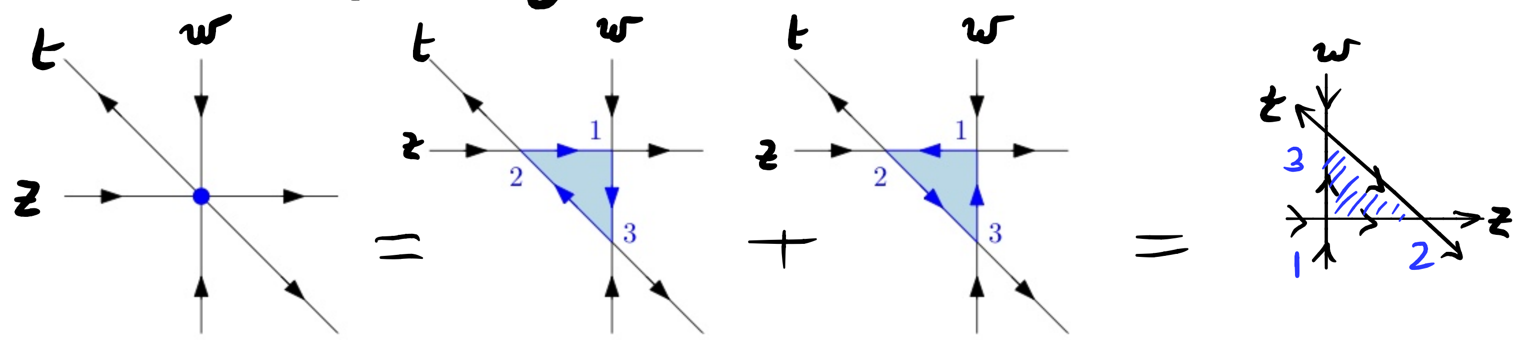
BOLTZMANN WEIGHTS



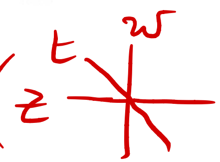
weights of the 6V models on the 3 sublattices

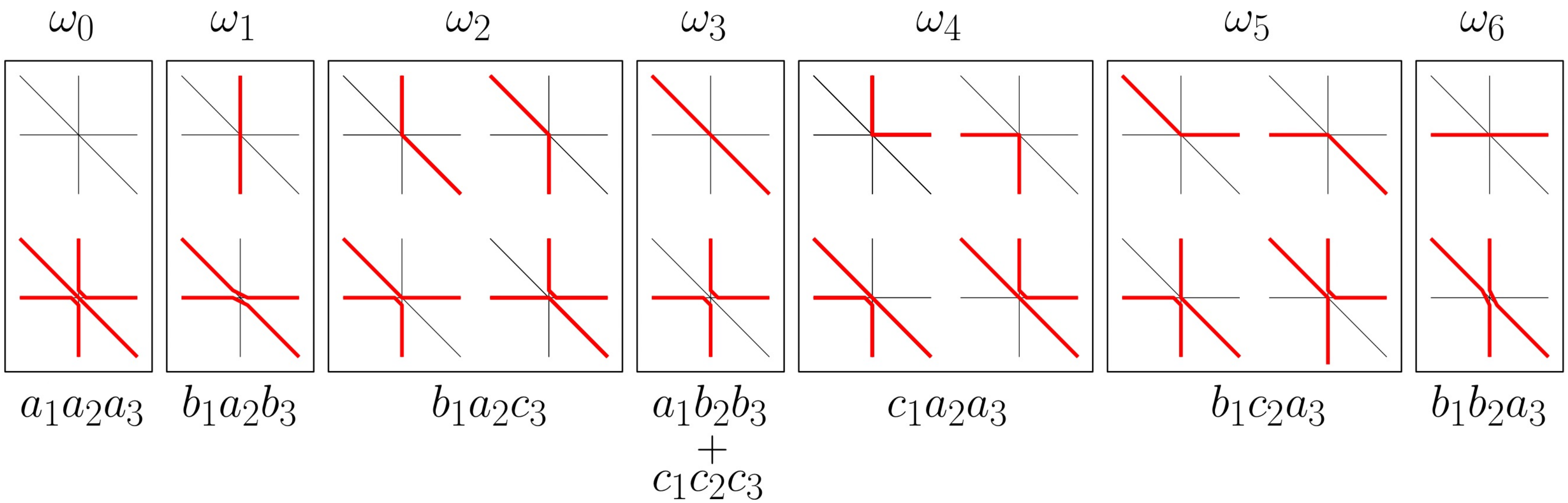
- 20V weights are given by sums over inner triangle configs

Example:



$$w = a_1 b_2 c_3 + c_1 c_2 b_3 = b_1 a_2 c_3$$

- Homogeneous case: 3 parameter family: ( , q)



$$\omega_0 = \sin(\lambda+\eta)\sin\left(\frac{\lambda+3\eta+\mu}{2}\right)\sin\left(\frac{\lambda+3\eta-\mu}{2}\right)$$

$$\omega_1 = \sin(\lambda-\eta)\sin\left(\frac{\lambda-\eta+\mu}{2}\right)\sin\left(\frac{\lambda+3\eta-\mu}{2}\right)$$

$$\omega_2 = \sin(2\eta)\sin(\lambda-\eta)\sin\left(\frac{\lambda+3\eta-\mu}{2}\right)$$

$$\omega_3 = \sin(2\eta)^3 + \sin(\lambda+\eta)\sin\left(\frac{\lambda-\eta+\mu}{2}\right)\sin\left(\frac{\lambda-\eta-\mu}{2}\right)$$

$$\omega_4 = \sin(2\eta)\sin\left(\frac{\lambda+3\eta+\mu}{2}\right)\sin\left(\frac{\lambda+3\eta-\mu}{2}\right)$$

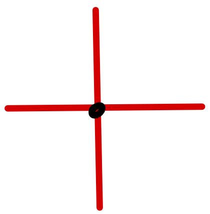
$$\omega_5 = \sin(2\eta)\sin(\lambda-\eta)\sin\left(\frac{\lambda+3\eta+\mu}{2}\right)$$

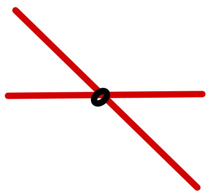
$$\omega_6 = \sin(\lambda-\eta)\sin\left(\frac{\lambda+3\eta+\mu}{2}\right)\sin\left(\frac{\lambda-\eta-\mu}{2}\right),$$

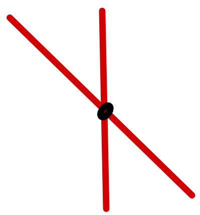
$$\begin{cases} q = e^{i\eta} \\ z = e^{i(\eta+\lambda)} \\ w = e^{-i(\eta+\lambda)} \\ t = e^{i\mu} \end{cases}$$

<u>Remark</u>: uniform weights $\omega_i = 1 \ \forall i$ are obtained for: $\boxed{\eta = \frac{\pi}{8} \quad \lambda = \frac{5\pi}{8} \quad \mu = 0}$

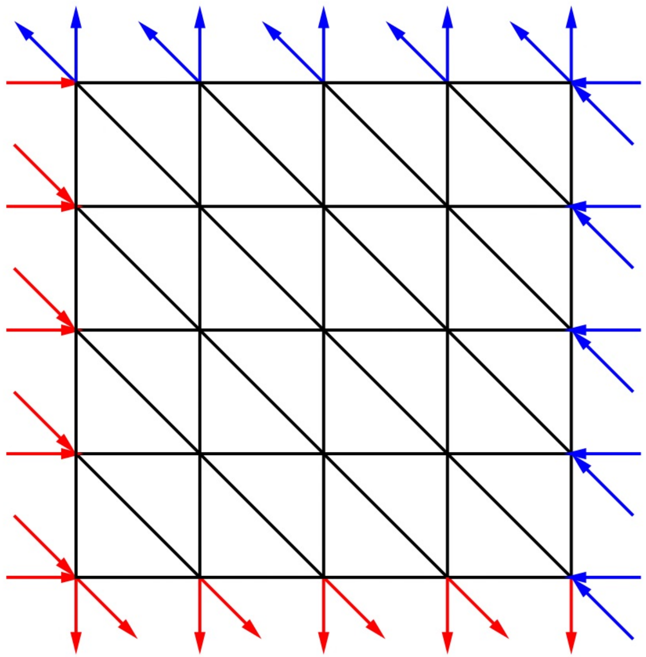
This corresponds to non-uniform weights on the 3 sublattice 6V models (up to an overall factor).

①  $\boxed{a_1 = 1 \quad b_1 = \sqrt{2} \quad c_1 = 1}$

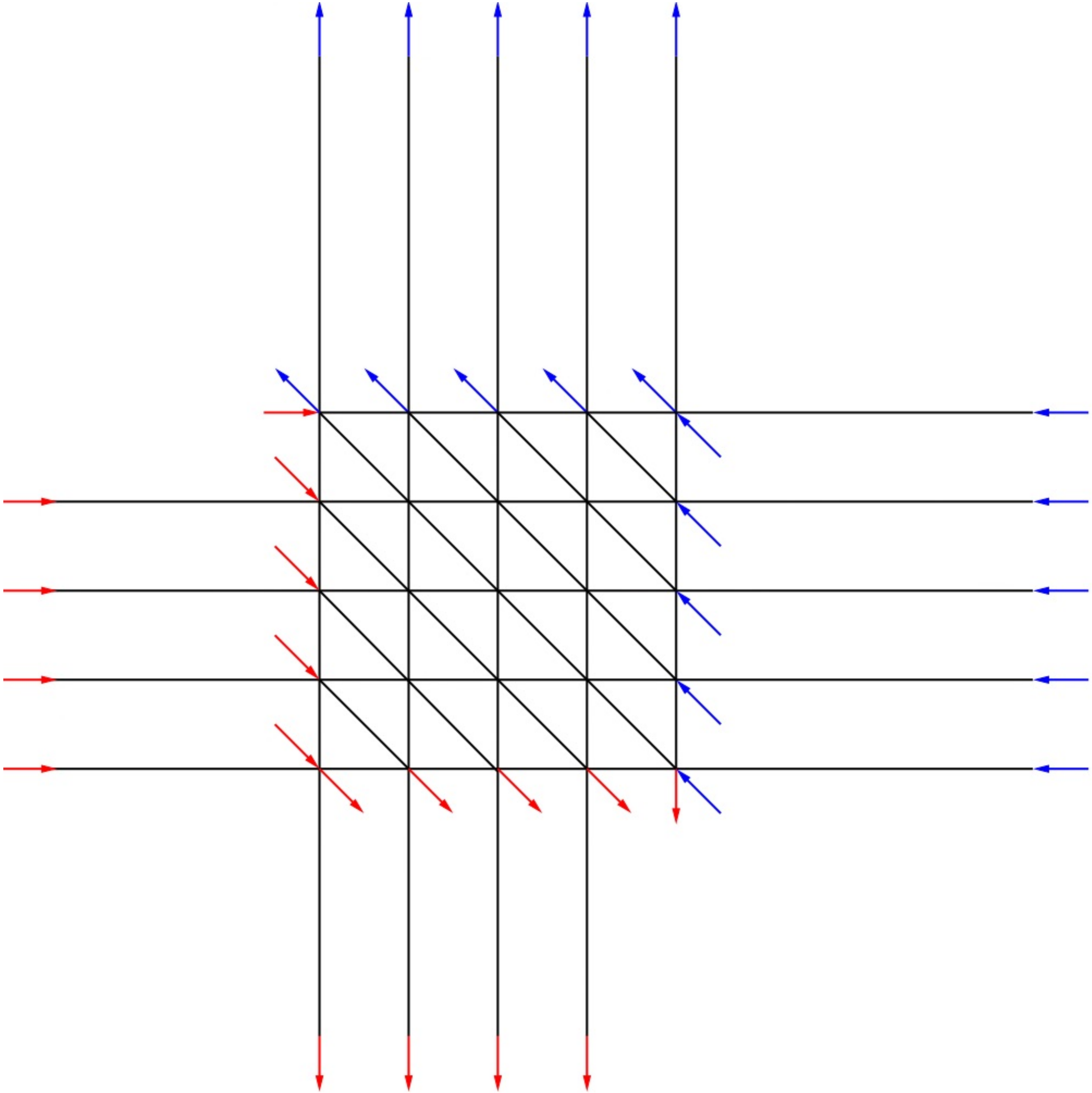
②  $a_2 = \sqrt{2} \quad b_2 = 1 \quad c_2 = 1$

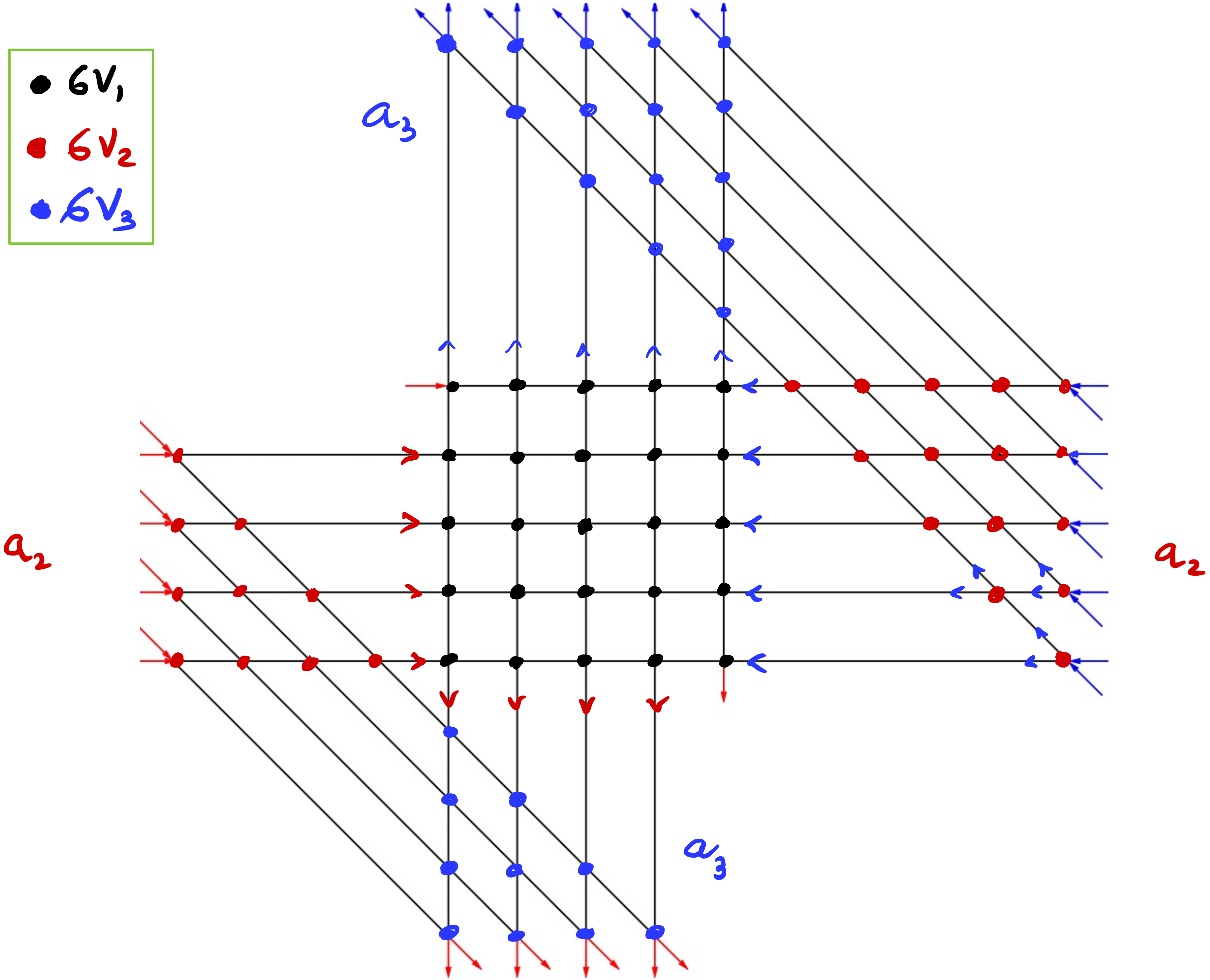
③  $a_3 = \sqrt{2} \quad b_3 = 1 \quad c_3 = 1$

TRANSFORMATION INTO a 6V MODEL

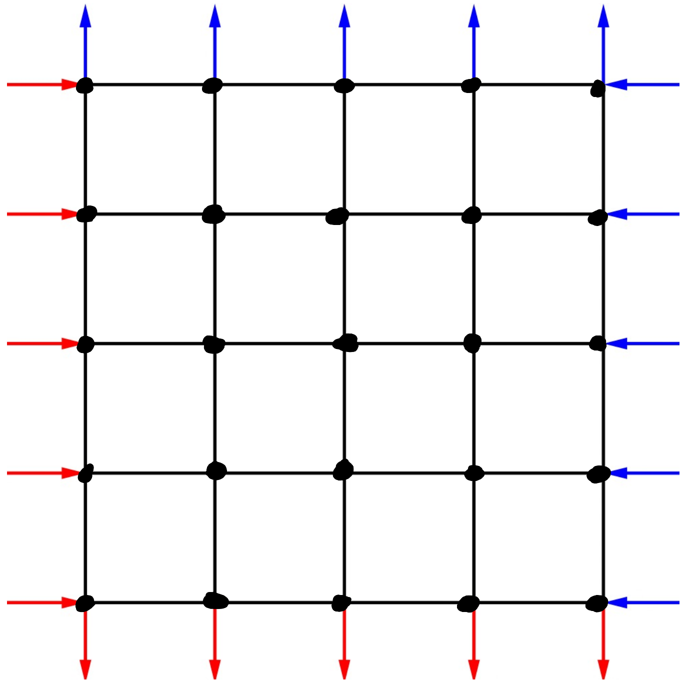


20V DWBC-2 (integrable weights).





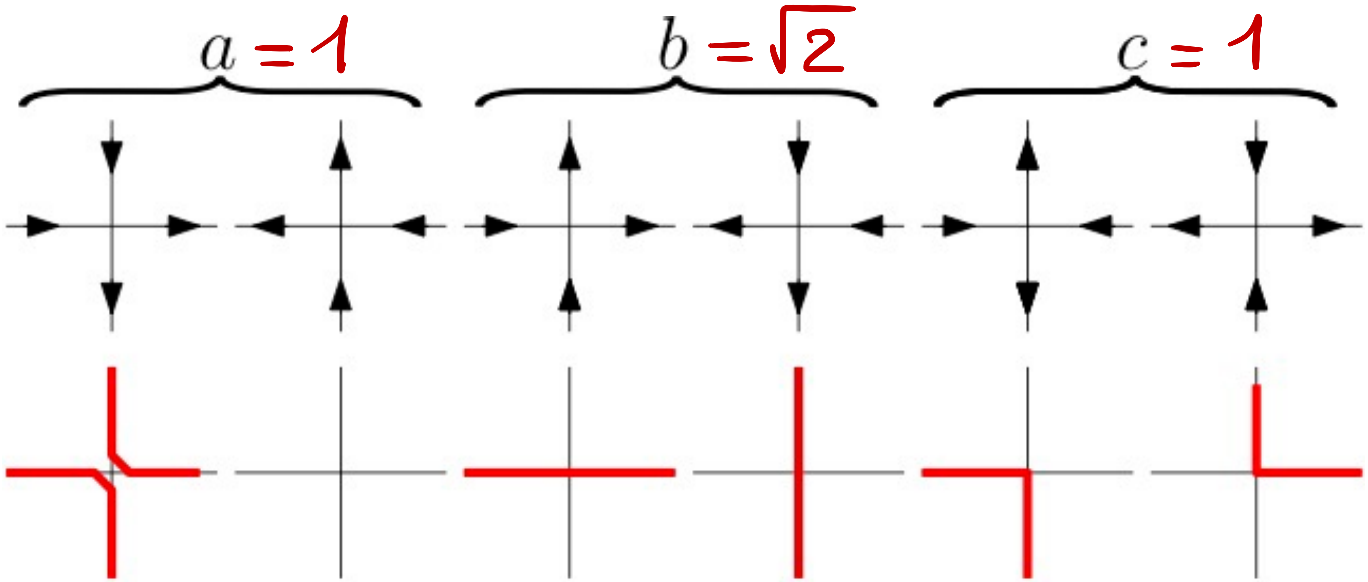
6V1
6V2
6V3
a3
a2
a2
a3



6V DWBC (sublattice 1 only).


20種の野菜と6種の果実
野菜汁60%+果汁40%=100%
KAGOME
野菜生活
100
ベリーサラダ
1食分の野菜
20
20種の野菜と6種の果実
野菜汁60%+果汁40%=100%
KAGOME
野菜生活
100
ベリーサラダ
1食分の野菜
6

Thm [PDF, E. Guitter 19] The partition function of the 20V model with all weights $= 1$ is equal to that of the 6V model with weights $(a, b, c) = (1, \sqrt{2}, 1)$ and DWBC

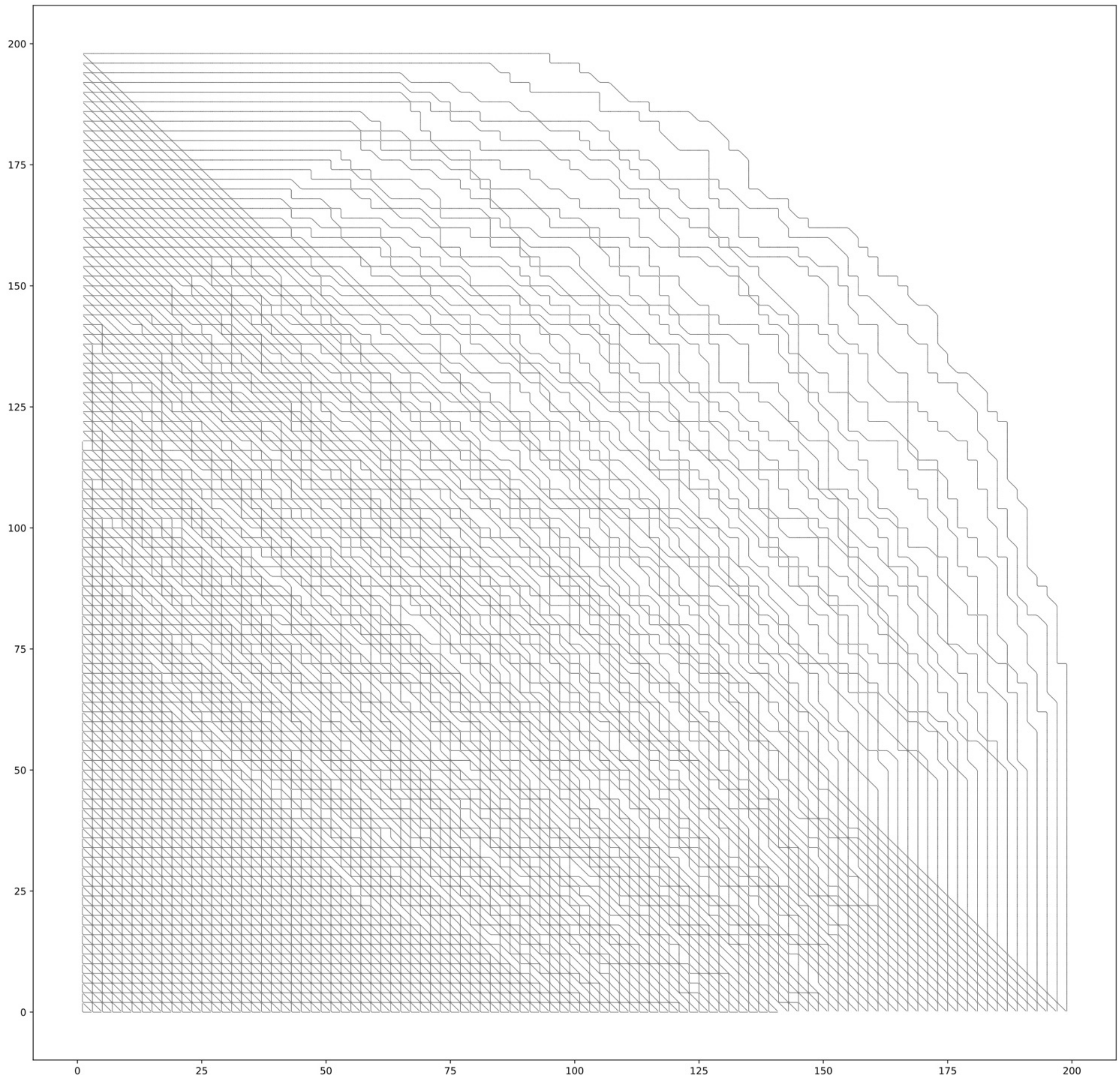


5. LIMIT SHAPE : THE ARCTIC PHENOMENON

- large size N ; typical configuration exhibits "frozen" domains / "liquid" domains

"frozen" domains $\downarrow$ regularly ordered paths

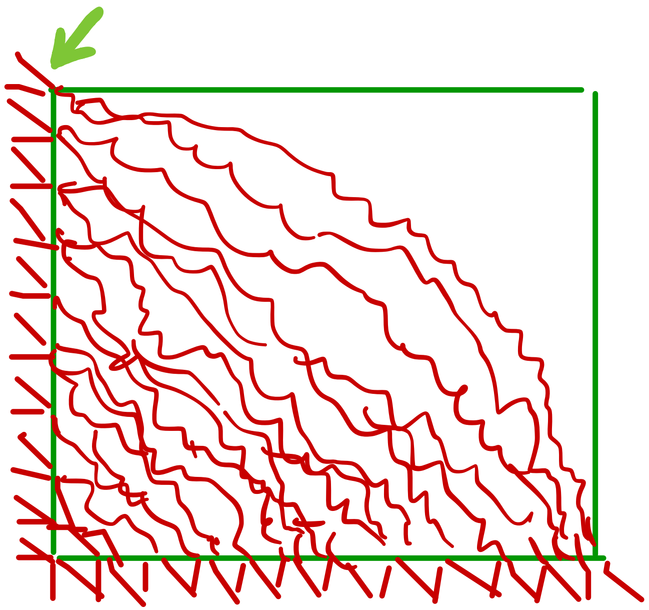
"liquid" domains $\downarrow$ disordered paths



DWBC1
uniform
weights
N= 200

ARCTIC PHENOMENON (20V DWBC1)

- Typical shape of a large configuration
  → use "tangent method"

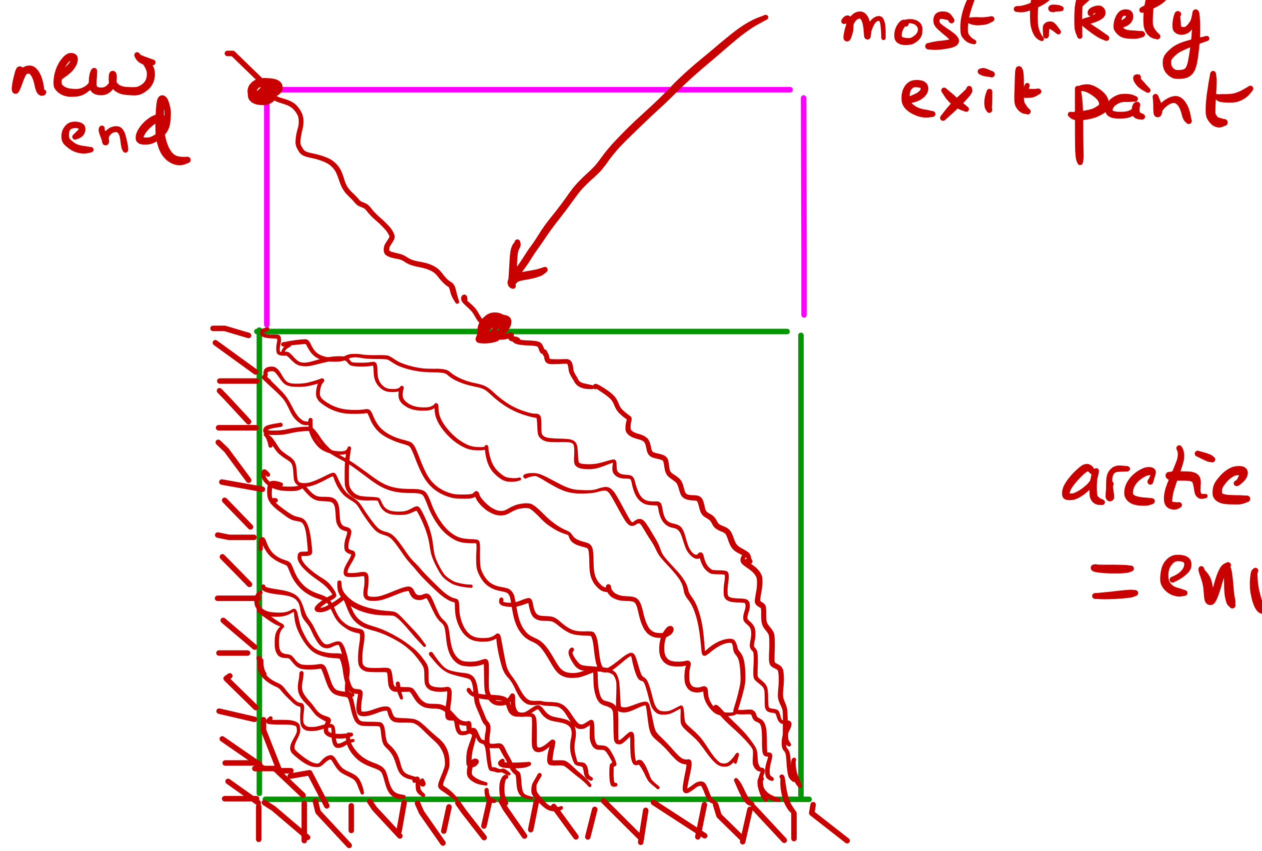


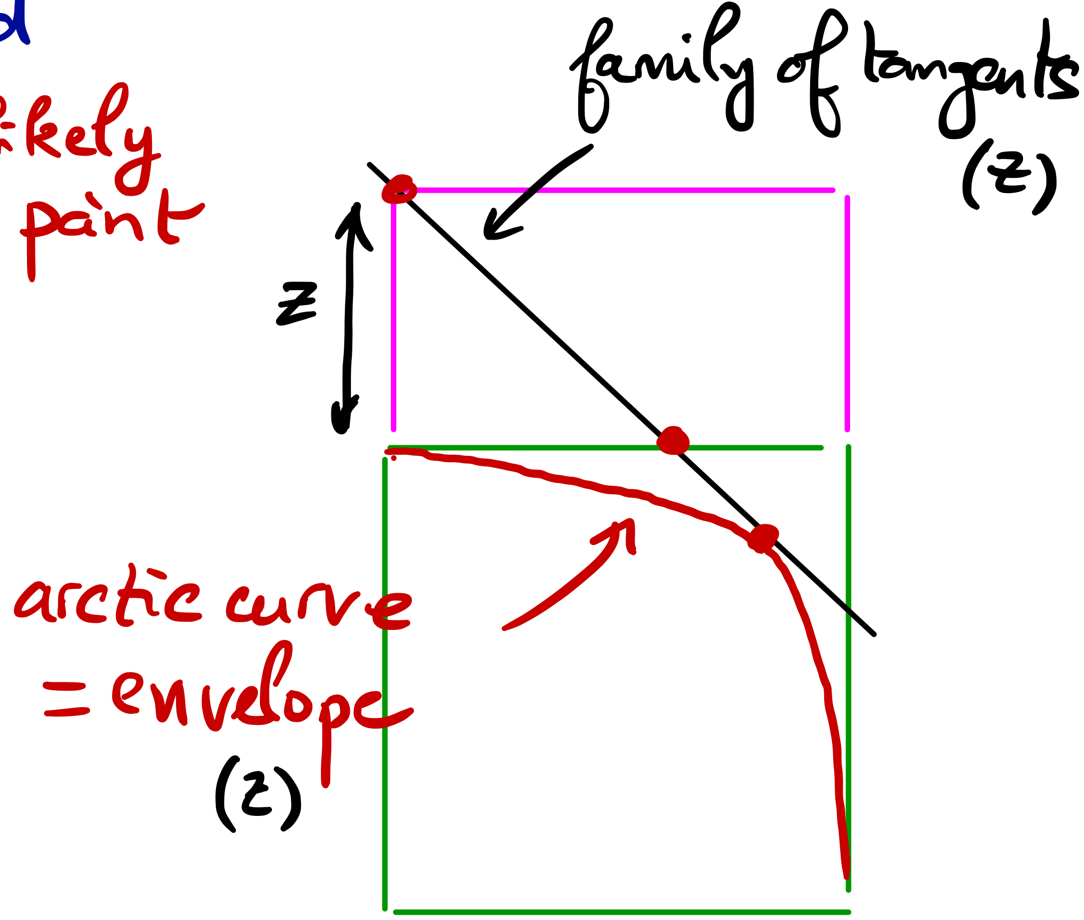
- modify last path exit paint
- use this new path as probe for the limit shape

ARCTIC PHENOMENON (20V DWBC1)

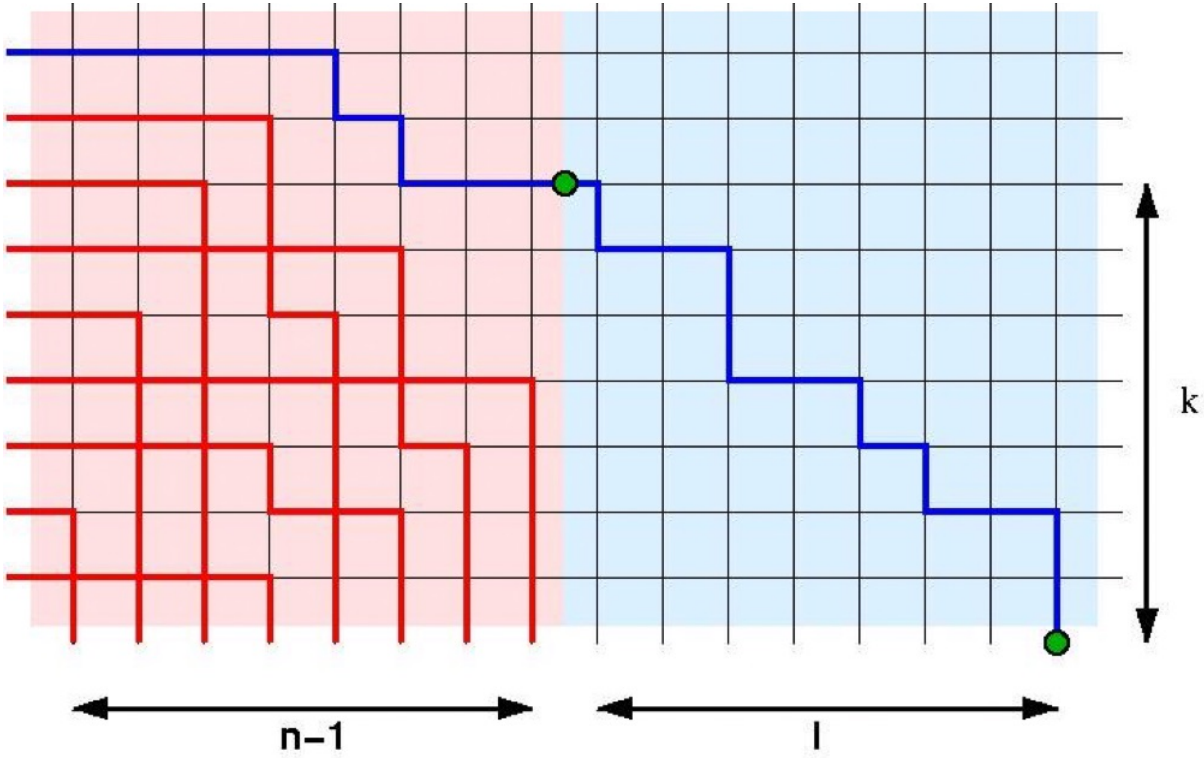
- Typical shape of a large configuration

→ use "tangent method"

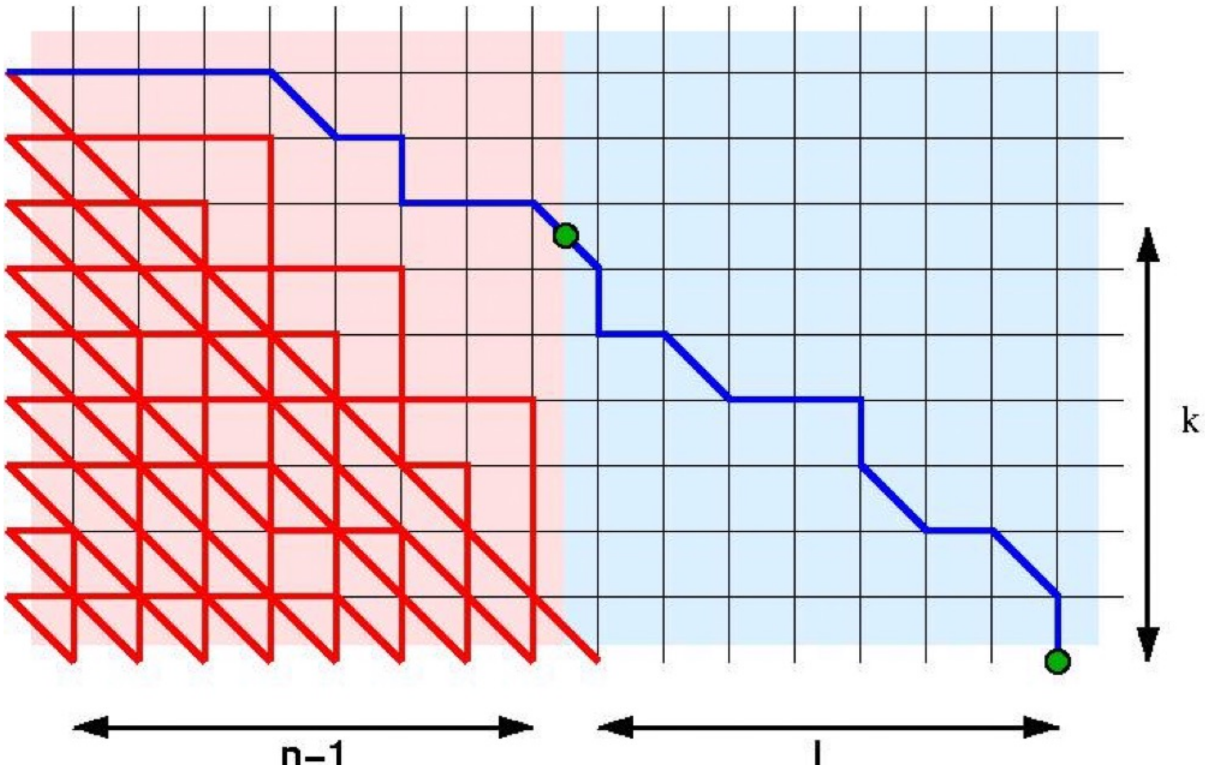




6V DWBC



20V DWBC1



Recipe

compute both the pink and blue partition functions!

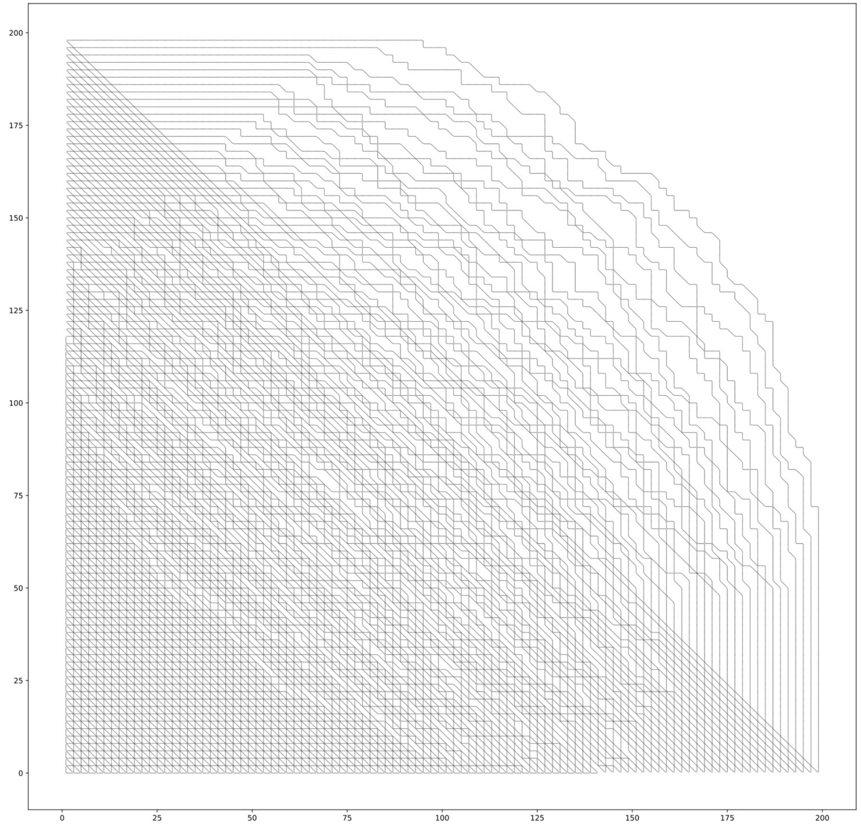
+ large n, ℓ, k estimates

+ saddle-point solution

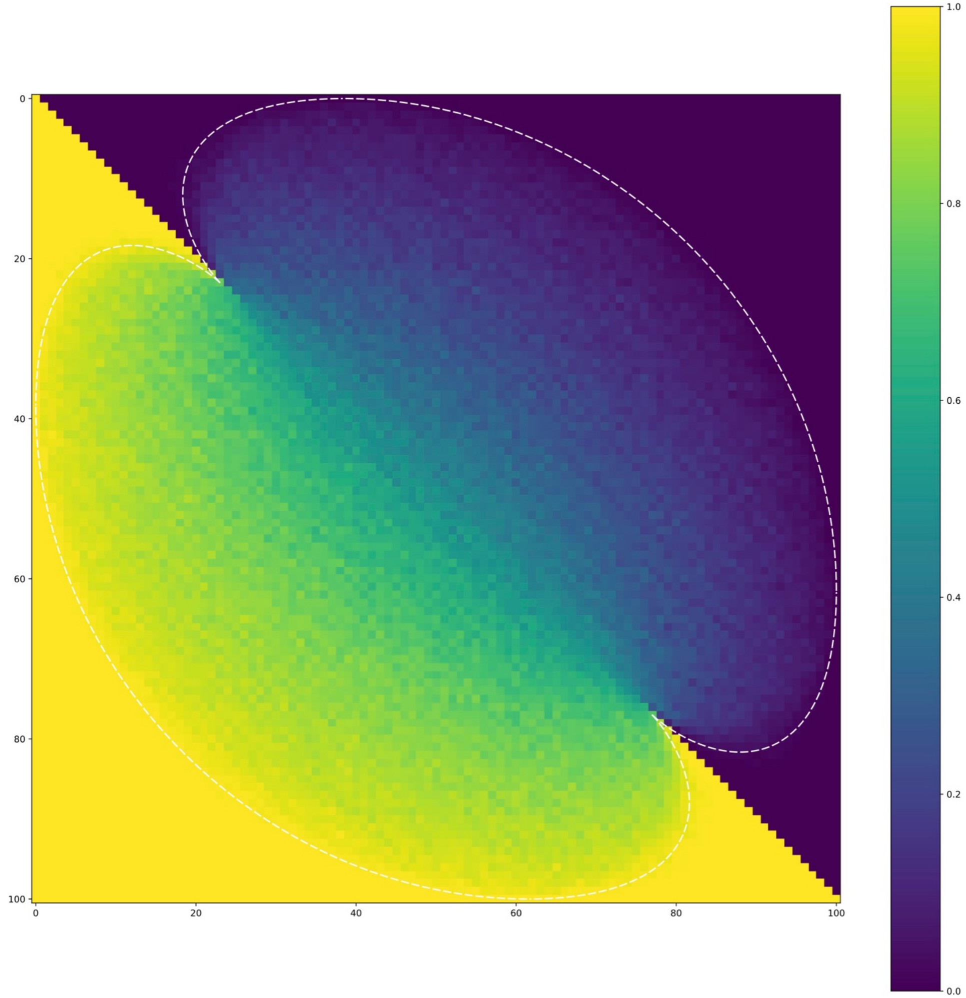
$\Rightarrow k(\ell)$

+ envelope of lines thru $(\ell,0)$ & $(0,k)$

20V-DWBC1 uniform weights

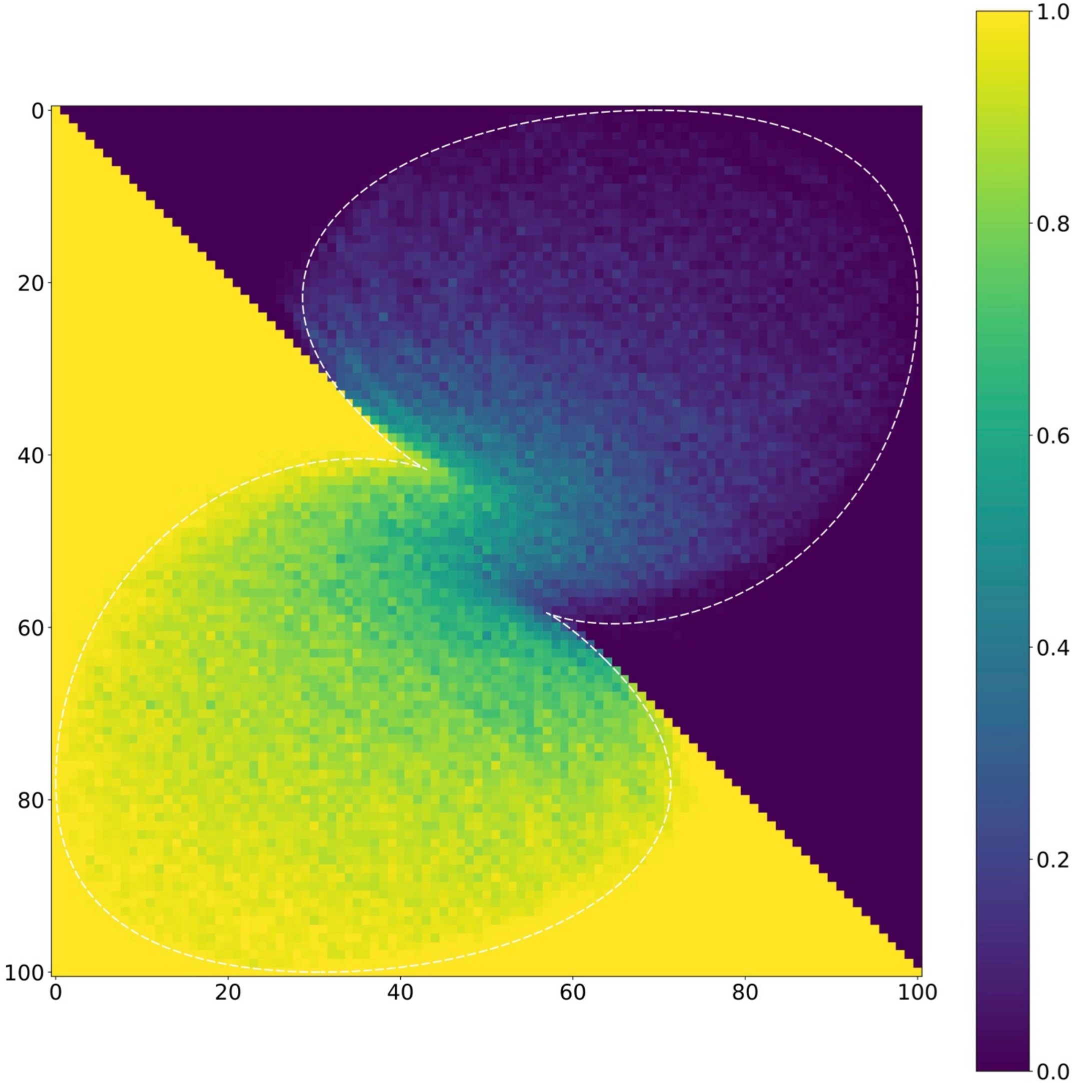


N=200

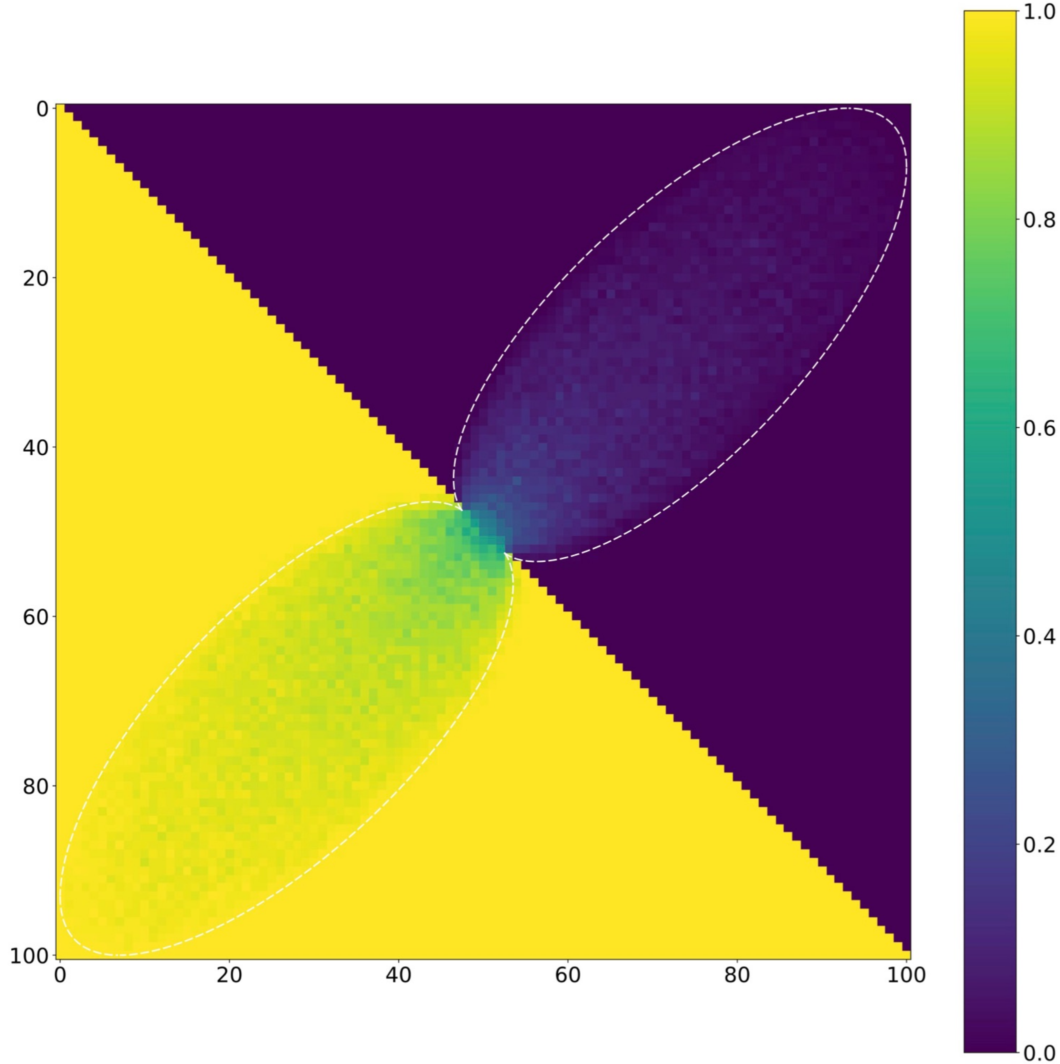


N=100

20V-DWBC1 – Non-uniform integrable weights



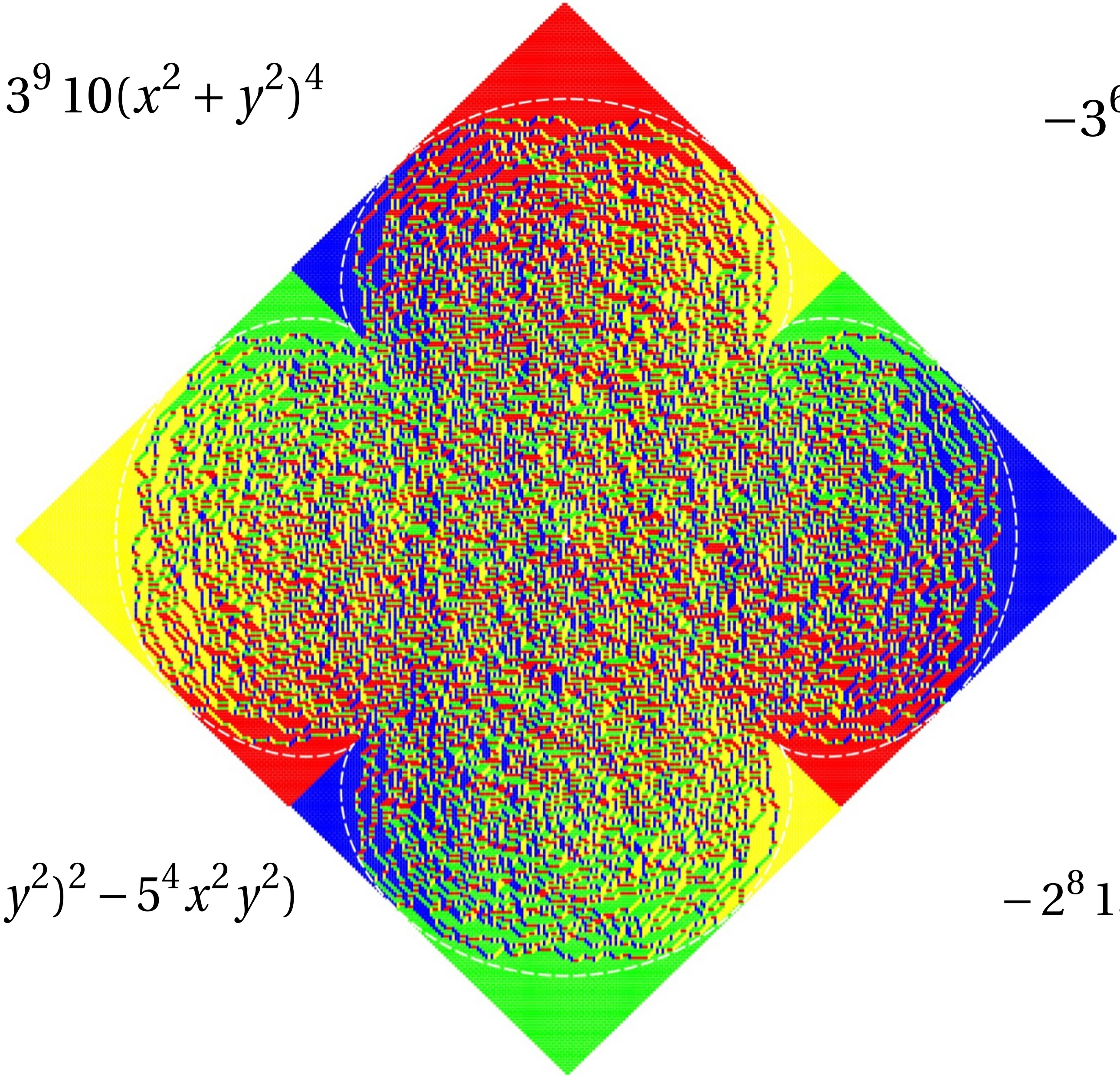
N=100



N=100

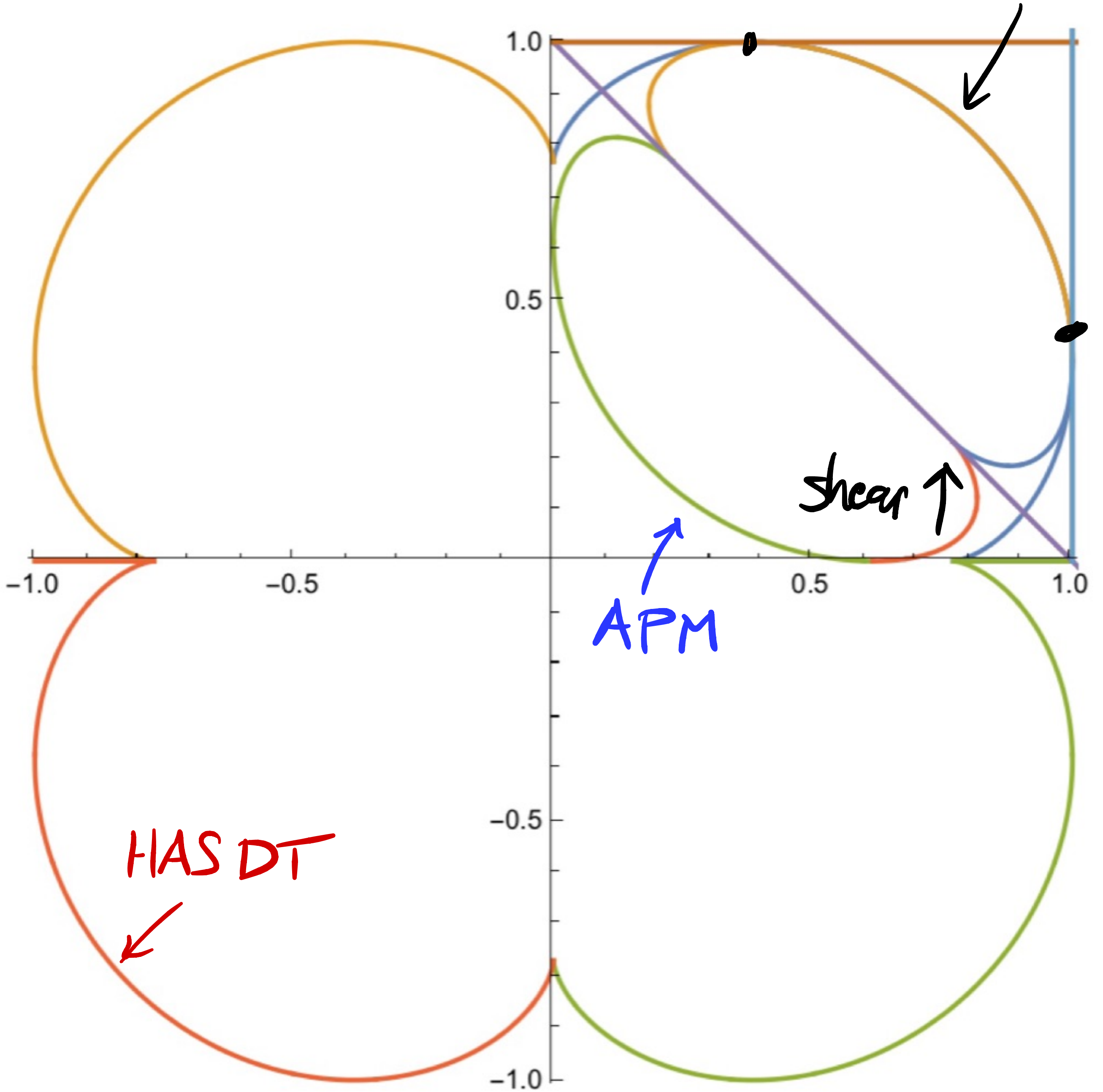
Holey Aztec square domino tilings (uniform weights)

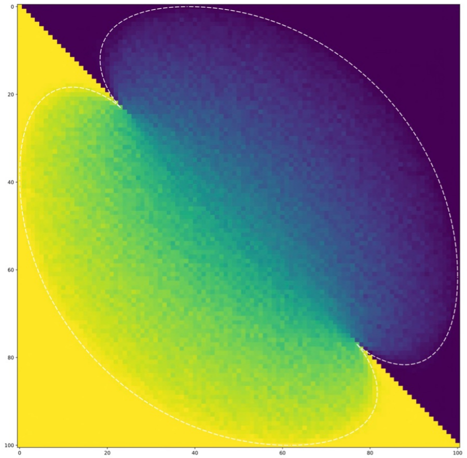
$$3^{11}(x^2+y^2)^5 + 3^9\,10(x^2+y^2)^4 - 3^6\,5(x^2+y^2)^3$$



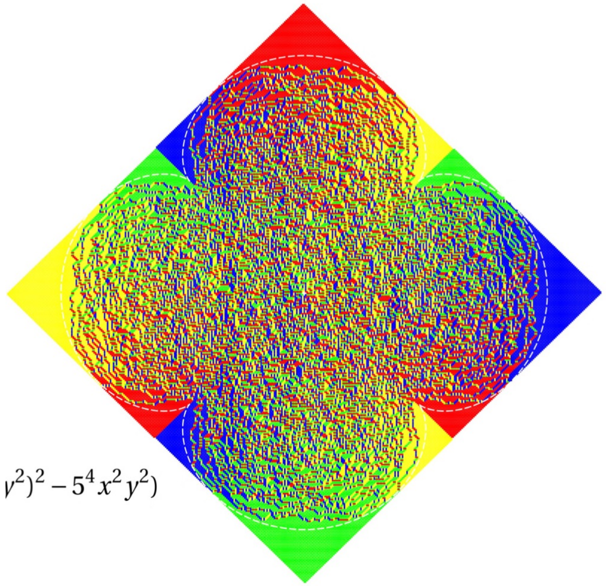
$$+\,6^2\,20(73(x^2+y^2)^2 - 5^4x^2y^2) - 2^8\,15(x^2+y^2) - 2^{12} = 0\,.$$

APM - holey Aztec Domino Tiling





Happy 60th Hubert!



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