Correlation functions from hydrodynamics beyond the Boltzmann-Gibbs paradigm

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in part with Giuseppe Del Vecchio Del Vecchio

"The art of mathematical physics"

pour le 60ième anniversaire d'Hubert Saleur.

Comme on dit par ici, bonne fête Hubert!

(Paris)

One-dimensional many-body interacting systems with conservation laws

We are given some Hamiltonian H for an extensive (infinitely long) system in one dimension, which admits a certain number of conservation laws,

$$\partial_t q_i + \partial_x j_i = 0, \qquad \partial_t Q_i = 0, \qquad Q_i = \sum_{x\in\mathbb{Z}} q_i(x)$$

For illustration purposes, I concentrate on the XX model:

$$H = -\sum_{x\in\mathbb{Z}} \left[\sigma_x^1\sigma_{x+1}^1 + \sigma_x^2\sigma_{x+1}^2 + h\sigma_x^3\right]$$

but all ideas hold for more general systems, integrable or not.

I will consider throughout a thermal state

$$\langle\cdots\rangle = Z^{-1}\mathrm{Tr}\Big(e^{-\beta H}\cdots\Big)$$

but all ideas hold for more general states such as GGEs $e^{-\sum_i \beta^i Q_i}$.

Correlations at large scales of space and time

The problem under study: **dominant correlations at large scales of space and time**:

$$\langle a(x,t)b(0,0)\rangle^{\rm c} \sim \quad ??? \qquad (|x|,t\to\infty)$$

where $\langle a(x,t)b(0,0)\rangle^{\rm c} = \langle a(x,t)b(0,0)\rangle - \langle a(x,t)\rangle\langle b(0,0)\rangle$ is the covariance.

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- Even in integrable systems, with Bethe ansatz and inverse scattering, this is a monumental problem: dynamics is difficult
- Hydrodynamics offers a number of universal principles, valid with or without integrability, which give asymptotics in terms of much simpler quantities; much of these are fully accessible in integrable systems via TBA and GHD.

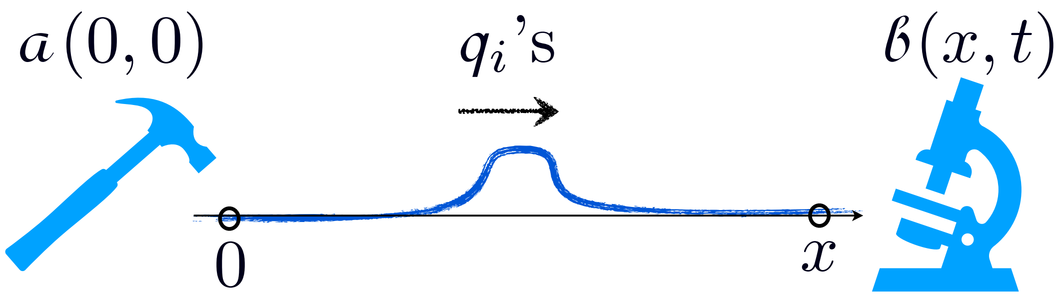
Boltzmann-Gibbs principle: physical picture

There is a **reduction of the number of degrees of freedom**, a projection onto slowly decaying hydrodynamic modes (or ballistic waves),

$$\lim_{x,t\to\infty} \langle a(0,0) b(x,t)\rangle^{\rm c} \sim \text{projection of } a \text{ onto modes } q_i$$

$$\times \quad \text{propagation } \langle q_i(0,0) q_j(x,t)\rangle^{\rm c}$$

$$\times \quad \text{projection of modes } q_j \text{ onto } b$$



Boltzmann-Gibbs principle: theorems

Define inner product ("susceptibilities" with wavenumber k) (**we look for $k \to 0$**)

$$\langle a, b \rangle_k = \sum_x e^{\mathrm{i}kx} \langle a^\dagger(x) b(0) \rangle^{\mathrm{c}}$$

Define matrix of susceptibilities (or **static correlation matrix**)

$$\mathsf{C}_{ij} = \langle q_i, q_j \rangle_0, \qquad \text{with inverse } \mathsf{C}_{ij}\mathsf{C}^{jm} = \delta_i^m$$

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Define an appropriate generalised limit $k \to 0, t \to \infty$ with kt fixed (**hyperbolic scaling**)

$$\lim_{kt=\kappa} f(t,k) = \mathrm{banach}\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{d}t\, f(t,\kappa/t)$$

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Theorem [BD 2020]. In every short-range homogeneous quantum spin chain, for any local operator $a\,$, b, we have

$$\lim_{kt=\kappa} \langle a\,(t), b\rangle_k = \sum_{i,j,m,n} \langle a\,, q_i\rangle_0\, \mathsf{C}^{ij}\, \mathsf{S}_{jm}(\kappa)\, \mathsf{C}^{mn}\, \langle q_n, b\rangle_0$$

where

$$\mathsf{S}_{jm}(\kappa) = \lim_{kt=\kappa} \langle q_j(t), q_m\rangle_k$$

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- $\langle \cdot, \cdot \rangle_0$ gives rise to an inner product and a Hilbert space, interpreted as the **Hilbert space of extensive homogeneous observables** $\mathcal{H}$.
- Time evolution $\tau_t = e^{\mathrm{i}t[H,\cdot]}$ gives rise to a **unitary operator on** $\mathcal{H}$.

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- Time evolution $\tau_t = e^{\mathrm{i}t[H,\cdot]}$ gives rise to a **unitary operator on** $\mathcal{H}$.
- The space of conserved charges is **rigorously defined** as $\mathcal{Q} = \cap_t \ker \tau_t \subset \mathcal{H}$.
- The conserved quantities q_i are any finite or countable basis, $\mathcal{Q} = \mathrm{span}\{q_i\}$.

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- The conserved quantities q_i are any finite or countable basis, $\mathcal{Q} = \mathrm{span}\{q_i\}$.
- $\mathcal{Q}$ is infinite-dimensional in integrable systems. It is conjectured to be finite-dimensional in chaotic systems.

Boltzmann-Gibbs principle: theorems

Theorem [BD 2020]. Under an appropriate extension of the space of local operators (as elements of the Gelfand-Naimark-Segal space), every local density q_i has an associated local current j_i satisfying

$$\partial_t q_i(x,t) + \partial_x j_i(x,t) = 0$$

and for every local $q_i,\ q_m$,

$$\frac{\mathrm{d}}{\mathrm{d}\kappa}\mathsf{S}_{im}(\kappa) = \mathrm{i}\sum_n \mathsf{A}_i{}^n \mathsf{S}_{nm}(\kappa) = \mathrm{i}\sum_n \langle j_i, q_l\rangle_0 \mathsf{C}^{ln}\mathsf{S}_{nm}(\kappa)$$

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The solution is $\mathsf{S}(\kappa) = \exp[\mathrm{i}\mathsf{A}\kappa]\mathsf{C}$. Physically, the equation comes from linear response, leading to wave-propagation equations

$$\partial_t \delta q_i(x,t) + \sum_j \mathsf{A}_i^{\ j}\partial_x \delta q_j(x,t) = 0, \qquad \mathsf{A}_i^{\ j} = \frac{\partial\langle j_i\rangle}{\partial\langle q_j\rangle}$$

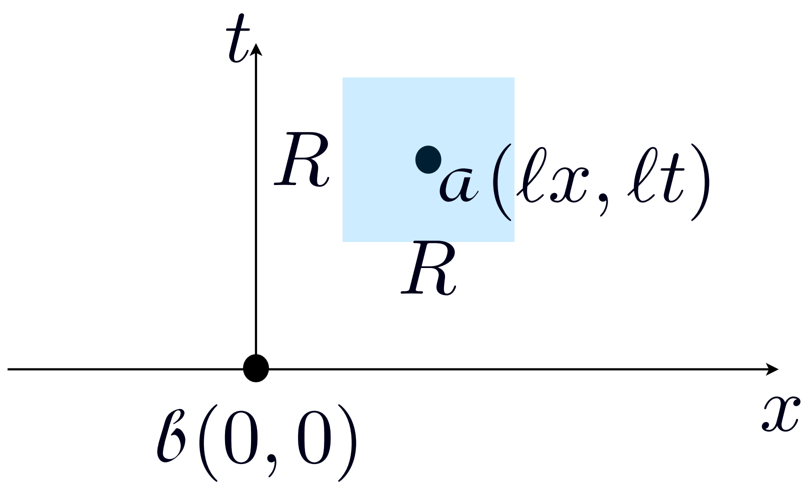
[physical argument can be extended to nonlinear response for higher-point functions: BD 2019 and especially Fava, Biswas, Gopalakrishnan, Vasseur, Parameswaran 2021]

Boltzmann-Gibbs principle: fluid-cell averaging

A more physical way of re-writing the final result is using fluid-cell averaging

$$\langle \overline{\mathfrak{a}}(\ell x, \ell t)\mathfrak{b}(0,0)\rangle^{\mathrm{c}} \sim \ell^{-1} \sum_{i,j,n} \langle \mathfrak{a}\,, q_i\rangle_0\, \mathsf{C}^{ij}\, \delta(x - \mathsf{A}t)_j^{\ n}\, \langle q_n, \mathfrak{b}\rangle_0$$

$$\overline{\mathfrak{a}}(x,t) = \frac{1}{R^2} \sum_{y=-R}^{R} \int_{-R}^{R} \mathrm{d}s\, \mathfrak{a}(x+y, t+s)$$



$$\ell \gg R \to \infty$$

Boltzmann-Gibbs principle: application to integrable systems

The result can be evaluated in integrable systems

$$\langle \overline{a}(\ell x, \ell t) b(0,0)\rangle^{\rm c} \sim \ell^{-1} \sum_{i,j,n} \langle a\,,q_i\rangle_0\, \mathsf{C}^{ij}\, \delta(x-\mathsf{A}t)_j^{\;n}\, \langle q_n\,,b\rangle_0$$

- C: known from TBA
- A: known from GHD
- $\langle a\,,q_i\rangle_0 = -\partial\langle a\rangle/\partial\beta^i$: known if the GGE average $\langle a\rangle$ is known

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For conserved densities

$$\langle \overline{q}_i(\ell x, \ell t) q_j(0,0)\rangle^{\rm c} \sim \ell^{-1} \int {\rm d}k\, \rho_{\rm p}(k)(1-n(k))\delta(x - v^{\rm eff}(k)t) h_i^{\rm dr}(k) h_j^{\rm dr}(k)$$

Boltzmann-Gibbs principle: application to XX model

XX model:

$$H = -\sum_{x\in\mathbb{Z}} \left[\sigma_x^1\sigma_{x+1}^1 + \sigma_x^2\sigma_{x+1}^2 + h\sigma_x^3\right]$$

By **Jordan-Wigner transformation:** $v^{\rm eff}(k) = 4\sin k$, and $\sigma^3 = 2a_x^\dagger a_x - 1$: conserved fermion density with $h(k) = 2$.

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$|x|, t \to \infty,\ x/t = \xi$

Boltzmann-Gibbs principle gives

$$\begin{aligned}\langle \overline{\sigma}_x^3(t)\sigma_0^3(0)\rangle^{\rm c} &\sim \frac{2}{\pi}\int_{-\pi}^{\pi} {\rm d}k\,\delta(x - v(k)t)n(k)(1-n(k)) \\ &= \frac{2}{\pi t\sqrt{16-\xi^2}}\sum_{a=\pm} n_a(1-n_a) \qquad (|\xi|<4)\end{aligned}$$

where

$$n_\pm = \frac{1}{1+\exp\left[-\beta\Big(2h \mp \sqrt{16-\xi^2}\Big)\right]}$$

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By direct Wick's theorem and saddle point analysis

$$\langle \sigma_x^3(t)\sigma_0^3(0)\rangle^{\rm c} \sim \frac{2}{\pi|t|\,\sqrt{16-\xi^2}}\sum_{a=\pm} \times$$

$$\times\ n_a\Big(1-n_a + a{\rm i}\,(1-n_{-a})(-1)^x e^{-2a{\rm i}(x\arcsin(\xi/4)+t\sqrt{16-\xi^2})}\Big)$$

Boltzmann-Gibbs principle: application to XX model

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$|x|, t\to\infty,\ x/t = \xi$

By direct Wick's theorem and saddle point analysis; **fluid-cell averaging cancels oscillatory term**

$$\langle \overline{\sigma}^3_x(t)\sigma^3_0(0)\rangle^{\rm c} \sim \frac{2}{\pi|t|\sqrt{16-\xi^2}} \sum_{a=\pm} \times$$

$$\times\ n_a\left(1-n_a + \cancel{a{\rm i}\,(1-n_{-a})(-1)^x e^{-2a{\rm i}(x\arcsin(\xi/4)+t\sqrt{16-\xi^2})}}\right)$$

$=$ Boltzmann-Gibbs result

Boltzmann-Gibbs principle: application to XX model

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Boltzmann-Gibbs principle: application to XX model

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Boltzmann-Gibbs principle gives

$$\langle\sigma_x^+(t)\sigma_0^-(0)\rangle^{\mathrm{c}} \sim 0$$

Boltzmann-Gibbs principle: application to XX model

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$|x|, t \to \infty,\ x/t = \xi$

Fredholm determinant calculation gives **exponential decay, subleading to Euler scale**

[Its, Izergin, Korepin, Slavnov 1992; Jie (PhD thesis) 1998]

$$\begin{aligned}\langle\overline{\sigma}_x^+(t)\sigma_0^-(0)\rangle^{\mathrm{c}} &\asymp \exp[-\zeta(\xi)t] \\ &\sim 0 \\ &= \text{Boltzmann-Gibbs result}\end{aligned}$$

Boltzmann-Gibbs principle: application to XX model

Three different types of behaviours are seen:

- Monotonic algebraic decay: BG predicts
- Oscillating algebraic decay: BG does not predict
- Exponential decay: BG does not predict

Can we explain / evaluate oscillatory algebraic decay, and exponential decay, using **general hydrodynamic principles**?

Finite-frequency hydrodynamic projections

The theorems on hydrodynamic projections stay valid if we replace time evolution by

$$\tau_t = e^{\mathrm{i}t\,\mathrm{ad}H} \quad \to \quad \tilde\tau_t = e^{-\mathrm{i}\omega t} e^{\mathrm{i}t\,\mathrm{ad}H}$$

and if we replace space translation by

$$\iota_x : a(y) \mapsto a(y+x) \quad \to \quad \tilde\iota_x = e^{\mathrm{i}kx}\iota_x$$

That is, Hilbert space of “homogeneous” extensive observables $\tilde{\mathscr{H}}$ is based on $\tilde\iota_x$, and $\tilde\tau_t$ is valid time-evolution unitary operators it. [BD 2020]

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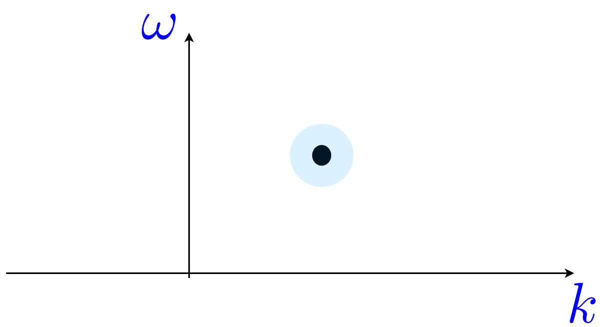
Redo the same hydrodynamic projection construction, but based on oscillating time evolution and space translation: hydrodynamics near (ω, k) instead of $(0, 0)$.



Finite-frequency hydrodynamic projections

New fluid-cell average that extracts the oscillating part:

$$\overline{a}(x,t) = \frac{1}{R^2}\sum_{y=-R}^{R}\int_{-R}^{R}\mathrm{d}s\, e^{\mathrm{i}ky-i\omega s}a\,(x+y,t+s)$$

New "susceptibility"

$$\langle a\,,b\rangle_0 = \sum_x e^{\mathrm{i}kx}\langle a^\dagger(x)b(0)\rangle^{\mathrm{c}}$$

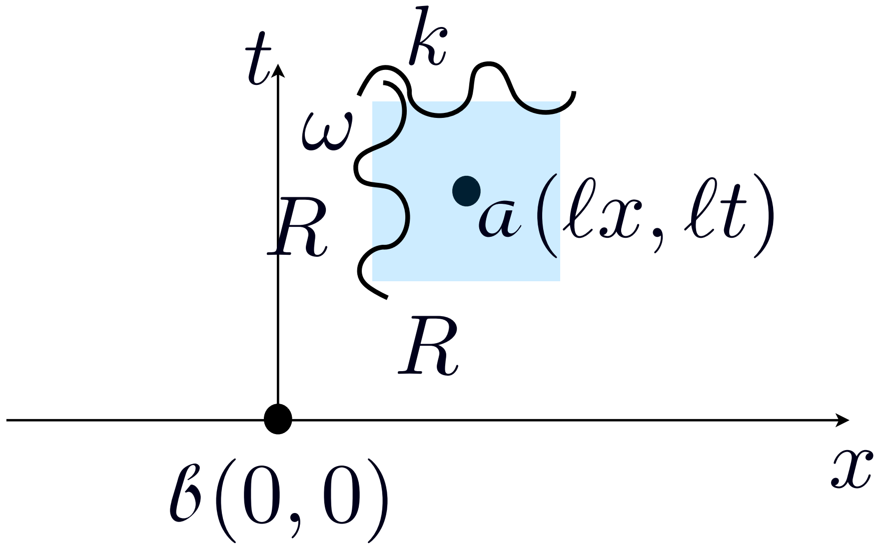
New condition of conservation

$$\mathrm{i}[H,Q_i] = \omega Q_i, \qquad Q_i = \sum_x e^{\mathrm{i}kx}q_i(x)$$

Finite-frequency hydrodynamic projections

Same formula

$$\langle \overline{a}(\ell x, \ell t) b(0,0)\rangle^{\rm c} \sim \ell^{-1} \sum_{i,j,n} \langle a\,, q_i\rangle_0\, \mathsf{C}^{ij}\, \delta(x - \mathsf{A}t)_j^{\;n}\, \langle q_n\,, b\rangle_0$$



Application to XX model

For every $\xi \in (-4,4)$ there are two modes of velocity ξ

$$k_\pm : v(k_\pm) = 4\sin(k_\pm) = \xi$$

Then, there is an extensive (ω,k)-conserved quantity with $\omega = E(k_+) - E(k_-)$ and $k = k_+ - k_-$:

$$Q = c^\dagger(k_+)c(k_-)$$

Using this, finite-frequency hydro projection correctly predicts

$$\langle \overline{\sigma}_x^3(t)\sigma_0^3(0)\rangle^{\rm c} \overset{\omega,k}{\sim} \frac{2{\rm i}}{\pi t\sqrt{16-\xi^2}} \sum_{a=\pm} a n_a(1-n_{-a})(-1)^x e^{-2a{\rm i}(x\arcsin(\xi/4)+t\sqrt{16-\xi^2})}$$

Finite-frequency hydrodynamic projections give the correct oscillating algebraic decay! It is due to the presence of two modes of different energies but with the same velocity.

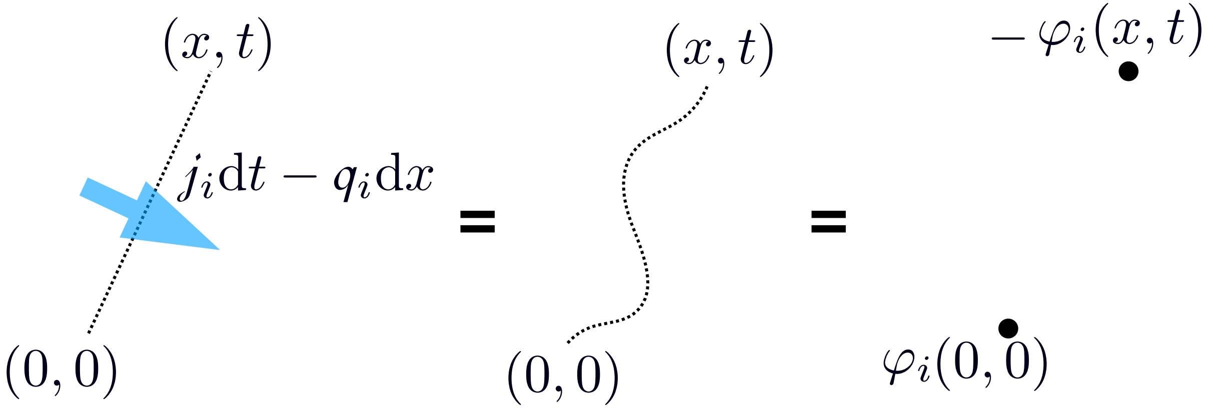
Correlation functions of twist fields and ballistic fluctuation theory

A "twist field" is a field $e^{\lambda\varphi_i(x,t)}$ where the "potential" $\varphi_i(x,t)$ is formally defined by solving the continuity equations:

$$q_i(x,t) = \partial_x\varphi_i(x,t), \qquad j_i(x,t) = -\partial_t\varphi_i(x,t)$$

The two-point function is an exponential of a path-independent line integral in space-time:

$$\langle e^{-\lambda(\varphi_i(x,t)-\varphi_i(0,0))}\rangle = \langle \exp\left[\lambda\int_{(0,0)}^{(x,t)}(j_i\mathrm{d}t - q_i\mathrm{d}x)\right]\rangle$$



Correlation functions of twist fields and ballistic fluctuation theory

Recall that

$$\sigma_x^+ = a_x^\dagger \exp\left(\mathrm{i}\pi \sum_{y=0}^{x-1} a_y^\dagger a_y \right)$$

This involves the fermion density $q_0(x,t) = a_x^\dagger(t) a_x(t)$.

Doing properly the JW transformation, one gets a "**space-time Jordan-Wigner string**"

$$\langle \sigma_x^+(t)\sigma_0^-(0)\rangle \asymp \langle a_x^\dagger(t) e^{\mathrm{i}\pi(\varphi_0(x,t)-\varphi_0(0,0))} a_0(0)\rangle$$

The presence of a "string" leads to a field that is semi-local, and this is one reason why $\sigma_x^\pm$ do not project onto extensive conserved charges.

Correlation functions of twist fields and ballistic fluctuation theory

To the quantity

$$\left\langle \exp\left[\lambda\int_{(0,0)}^{(x,t)}(j_i\mathrm{d}t - q_i\mathrm{d}x)\right]\right\rangle$$

we apply **large-deviation theory**:

$$\left\langle \exp\left[\lambda\int_{(0,0)}^{(\ell x,\ell t)}(j_i\mathrm{d}t - q_i\mathrm{d}x)\right]\right\rangle \asymp \exp\left[\ell F_i(\lambda; x, t)\right]$$

$F_i(\lambda; x, t)$ may be evaluated using **Euler-scale Macroscopic Fluctuation Theory** [BD, Sasamoto, Yoshimura to appear], or using the **ballistic fluctuation theory** [Myers, Bhaseen, Harris, Doyon 2019; Doyon, Myers 2019], in terms solely of objects from Euler hydrodynamics, as explained in Takato's talk.

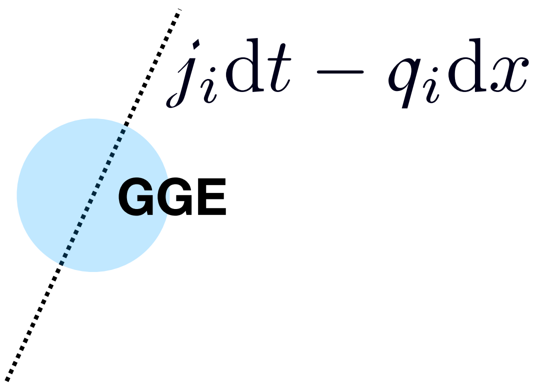
Correlation functions of twist fields and ballistic fluctuation theory

Its basis is the concept of "measure bias"

$$\lim_{\ell\to\infty} \frac{\langle \exp\left[\lambda \int_{(-\ell x,-\ell t)}^{(\ell x,\ell t)} (j_i \mathrm{d}t - q_i \mathrm{d}x)\right] \cdots \rangle}{\langle \exp\left[\lambda \int_{(-\ell x,-\ell t)}^{(\ell x,\ell t)} (j_i \mathrm{d}t - q_i \mathrm{d}x)\right] \rangle} = \langle \cdots \rangle_\lambda$$

By using path-invariance and hydrodynamic projections, one can show, order by order in λ, that $\langle \cdots \rangle_\lambda$ must be a (G)GE, and that the λ-dependent GGE satisfies a **flow equation**

$$\partial_\lambda \beta^j(\lambda;\xi) = \mathrm{sgn}\big(x\,\mathbf{1} - t\,\mathsf{A}(\lambda;\xi)\big)_i^{\;j}, \qquad \beta^j(0;\xi) = \beta^j, \quad \xi = x/t.$$



Correlation functions of twist fields and ballistic fluctuation theory

The flow determines the large-deviation, with associated "specific free energy" – scaled cumulant generating function – given by

$$F_i(\lambda; x, t) = \int_0^\lambda \mathrm{d}\lambda' \big(t\, \mathsf{j}_i(\lambda'; \xi) - x\, \mathsf{q}_i(\lambda'; \xi)\big)$$

In the XX model for the fermion number ($i = 0$), the GGE along the flow is described by the function

$$w(\lambda; \xi; k) = \beta E(k) + \lambda\, \mathrm{sgn}\big(x - t\, v(k)\big)$$

and the scaled cumulant generating function is

$$F_0(\lambda; x, t) = \int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi}\, |x - t\, v(k)| \log\Big(\frac{1 + e^{-w(\lambda;\xi;k)}}{1 + e^{-w(k)}}\Big).$$

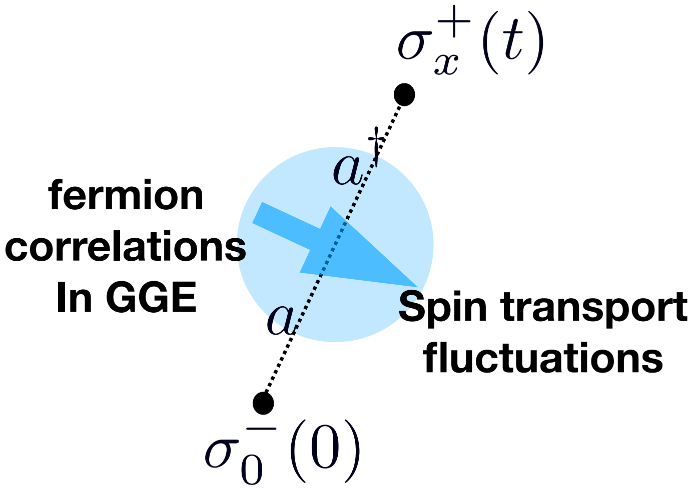
Correlation functions of twist fields and ballistic fluctuation theory

Using the flow on states and the SCGF, the required correlation function factorises into:

[BD, Del Vecchio Del Vecchio 2021]

- an exponential decay due to the interaction between the "boundary fermions" that occurs well within the region between them where λ-GGE is established,
- and a contribution from the large-deviation for fluctuations of total spin, or total spin transport:

$$\langle a^\dagger_{\ell x}(\ell t) e^{\lambda(\varphi_0(\ell x,\ell t)-\varphi_0(0,0))} a_0(0)\rangle \asymp \langle a^\dagger_{\ell x}(\ell t) a_0(0)\rangle_\lambda \exp\left[\ell F_0(\lambda;x,t\right]$$



Results in the XX model

An analysis of both factors (saddle point, and ballistic fluctuation theory) gives the correct results ($|x|, t \to \infty,\ x/t = \xi \in \mathbb{R}$)

$\langle \sigma_x^+(t)\sigma_0^-(0)\rangle$ $\qquad (E(k) = 4h - 2\cos k,\ v(k) = 4\sin k)$

$$\asymp \begin{cases} \exp\left[|t|\int_{-\pi}^{\pi}\frac{\mathrm{d}k}{2\pi}\,|\xi - v(k)|\log\left|\tanh\frac{\beta E(k)}{2}\right|\right] & (|\xi| \le 4) \\ \exp\left[|x|\int_{-\pi}^{\pi}\frac{\mathrm{d}k}{2\pi}\,\log\left|\tanh\frac{\beta E(k)}{2}\right|\right] & (|\xi| > 4,\ |h| \le 2) \end{cases}$$

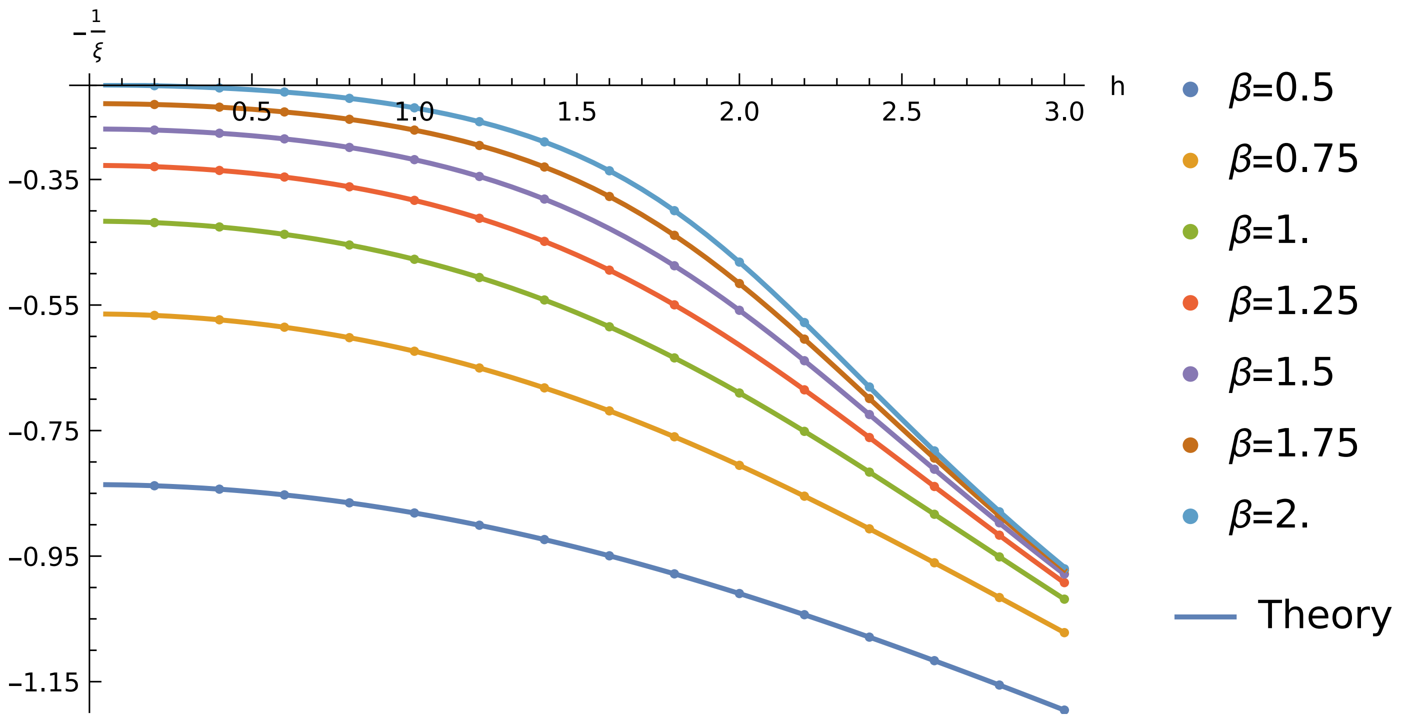
[Its, Izergin, Korepin, Slavnov 1992; Jie (PhD thesis) 1998][– asymptotics of Fredholm determinants]

$$e^{\mathrm{i}\Phi(x,t)}\exp\Big[-|x|\min\Big(\operatorname{arccosh}(|h|/2),\ \operatorname{arccosh}(|\xi|/4) - \sqrt{1-\frac{16}{\xi^2}}\Big)\Big]\times$$
$$\times\exp\left[|x|\int_{-\pi}^{\pi}\frac{\mathrm{d}k}{2\pi}\,\log\left|\tanh\frac{\beta E(k)}{2}\right|\right] \qquad (|\xi| > 4,\ |h| > 2)$$

[BD, Del Vecchio Del Vecchio 2021 – hydrodynamics]

Results in the XX model

Comparison with numerics, e.g. in space-like region, $e^{-|x|/\xi}$



Conclusions

Finite-frequency hydro projection

- Related works in interacting models where "dynamical symmetries" are used to bound finite-frequency Drude weights [Buça, Tindall, Jaksch 2019; Medenjak, Prosen Zadnik 2020].
- Probably the principles used here can be extended to generic integrable models using the finite-density form factors (see reviews [De Nardis, BD, Medenjak, Panfil 2021; Cortés Cubero, Yoshimura, Spohn 2021]).
- Finite-frequency hydrodynamic equations? Higher-point functions?

Twist fields

- Immediately applicable to other fields of interest such as $e^{\mathrm{i}\alpha\phi}$ in the sine-Gordon model [in progress with del Vecchio del Vecchio, Kormos], potentially for Ψ field in Lieb-Liniger.
- Can be used to study non-equilibrium dynamics of entanglement entropy....