Operator entanglement

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Hubert's 60 birthday, Saclay, September 2021

Computing entanglement entropies:

-a business that was ‘suspiciously’ trendy in the late 2000s
(2007-2010: my PhD years with Hubert and Jesper)

-a topic I entered thanks to Jean-Marie Stéphan, my office mate in Saclay
(wonderful time with other PhD students/postdocs in the group at the time: Constantin Candu, Azat Gainutdinov, Roberto Bondesan, Balazs Pozsgay…)

-although at the time, Hubert was not working on entanglement entropies, his support and encouragements to publish our results with Jean-Marie were very important

-a topic Hubert also contributed to, together with Jesper, Paolo Zanardi, Romain Vasseur, Edouard Boulat, Romain Couvreur, and others…

So, more than a decade later, I’m still computing entanglement entropies…

okay, but this time it’s about operators! (as opposed to states)

What is 'operator entanglement'?

The definition

Take a quantum system corresponding to a Hilbert space

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

For simplicity, take $\mathcal{H}_A = \mathbb{C}^{d_A}$ and $\mathcal{H}_B = \mathbb{C}^{d_B}$.

Take an operator O acting on $\mathcal{H}$, namely a complex matrix of size $d_A d_B \times d_A d_B$.

We would like to know **'how far'** this operator is from a **product operator**

$$O \overset{?}{\simeq} O^A \otimes O^B$$

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$$O \overset{?}{\simeq} O_1^A \otimes O_1^B + O_2^A \otimes O_2^B$$

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One can write an **‘operator Schmidt decomposition’**

$$\frac{O}{\sqrt{\operatorname{tr} O^\dagger O}} = \sum_j \lambda_j O_j^A \otimes O_j^B$$

where $\operatorname{tr}[O_i^{A\dagger} O_j^A] = \operatorname{tr}[O_i^{B\dagger} O_j^B] = \delta_{i,j}$ and $\lambda_j \geq 0$

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Take an operator O acting on $\mathcal{H}$, namely a complex matrix of size $d_A d_B \times d_A d_B$.

Operator entanglement is then defined as

$$S_\alpha(O) \stackrel{\text{def}}{=} \frac{1}{1-\alpha} \log \left(\sum_j \lambda_j^{2\alpha} \right)$$

with Renyi index $\alpha > 0$. (The limit $\alpha \to 1$ works as usual.)

What is ‘operator entanglement’?

The definition

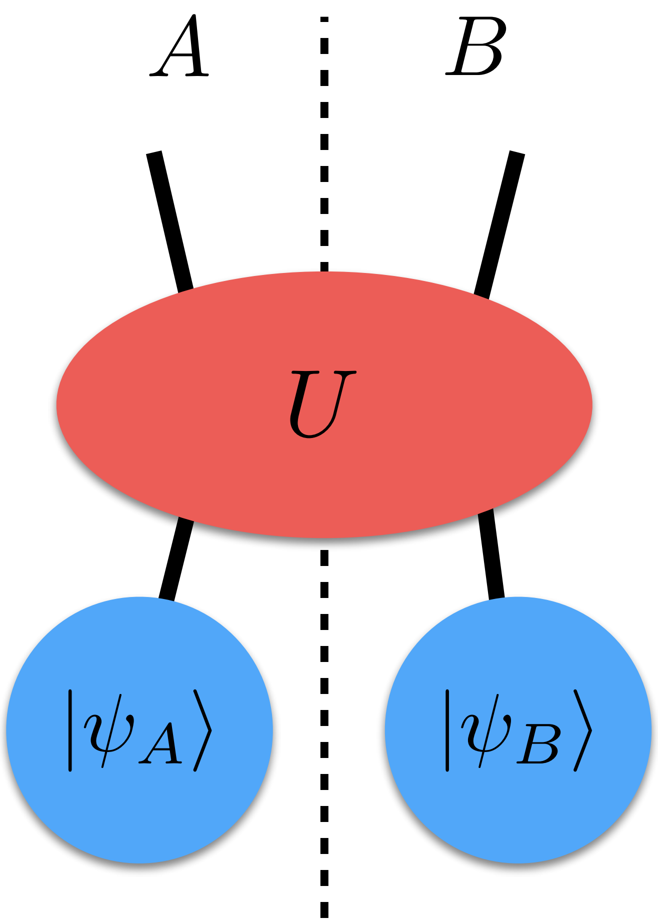
Of course, this is nothing but saying that the $d_A d_B \times d_A d_B$ matrix O acting on $\mathcal{H}$ can be viewed as a vector $\left|O\right\rangle$ in the the larger Hilbert space $\mathcal{H} \otimes \bar{\mathcal{H}}$ of dimension $d_A^2 \times d_B^2$. We are simply looking at the usual entanglement entropy of that state $\left|O\right\rangle$.

Why care about 'operator entanglement'?

One motivation: study of entanglement power

[Zanardi, Zalka, Faoro 2000], [Zanardi 2001], [Wang, Zanardi 2002]

Question: how much entanglement is produced by a unitary operator U, in average?



Zanardi et al. studied the **entanglement power:**

$$e(U) = \overline{E(U\,|\psi_A\rangle\,|\psi_B\rangle)}^{|\psi_A\rangle|\psi_B\rangle}$$

Why care about ‘operator entanglement’?

One motivation: study of entanglement power

[Zanardi, Zalka, Faoro 2000], [Zanardi 2001], [Wang, Zanardi 2002]

In particular, they investigated whether

$$\overline{S_\alpha(U\,|\psi_A\rangle\,|\psi_B\rangle)}^{|\psi_A\rangle|\psi_B\rangle}$$

had something to do with the operator entanglement.

Answer: in general, no direct relation.
However for the Renyi-2 entropy, there is a relation. It reads

$$\begin{aligned} &\overline{\exp[\,-S_2(U\,|\psi_A\rangle\,|\psi_B\rangle)\,]}^{|\psi_A\rangle|\psi_B\rangle} \\ \propto\quad &1-\tfrac{d^2}{(d+1)^2}+\tfrac{d^2}{(d+1)^2}\left(\exp[-S_2(U)]+\exp[-S_2(U\mathcal{S})]-\exp[-S_2(\mathcal{S})]\right) \end{aligned}$$

in the case $d_A = d_B = d$. $\mathcal{S}$ is the operator that swaps the subsystems A and B.

Why care about 'operator entanglement'?

Another motivation

Question: can an operator O acting on a 1d system (spin chain) be well approximated by a Matrix Product Operator (MPO) with small bond dimension?

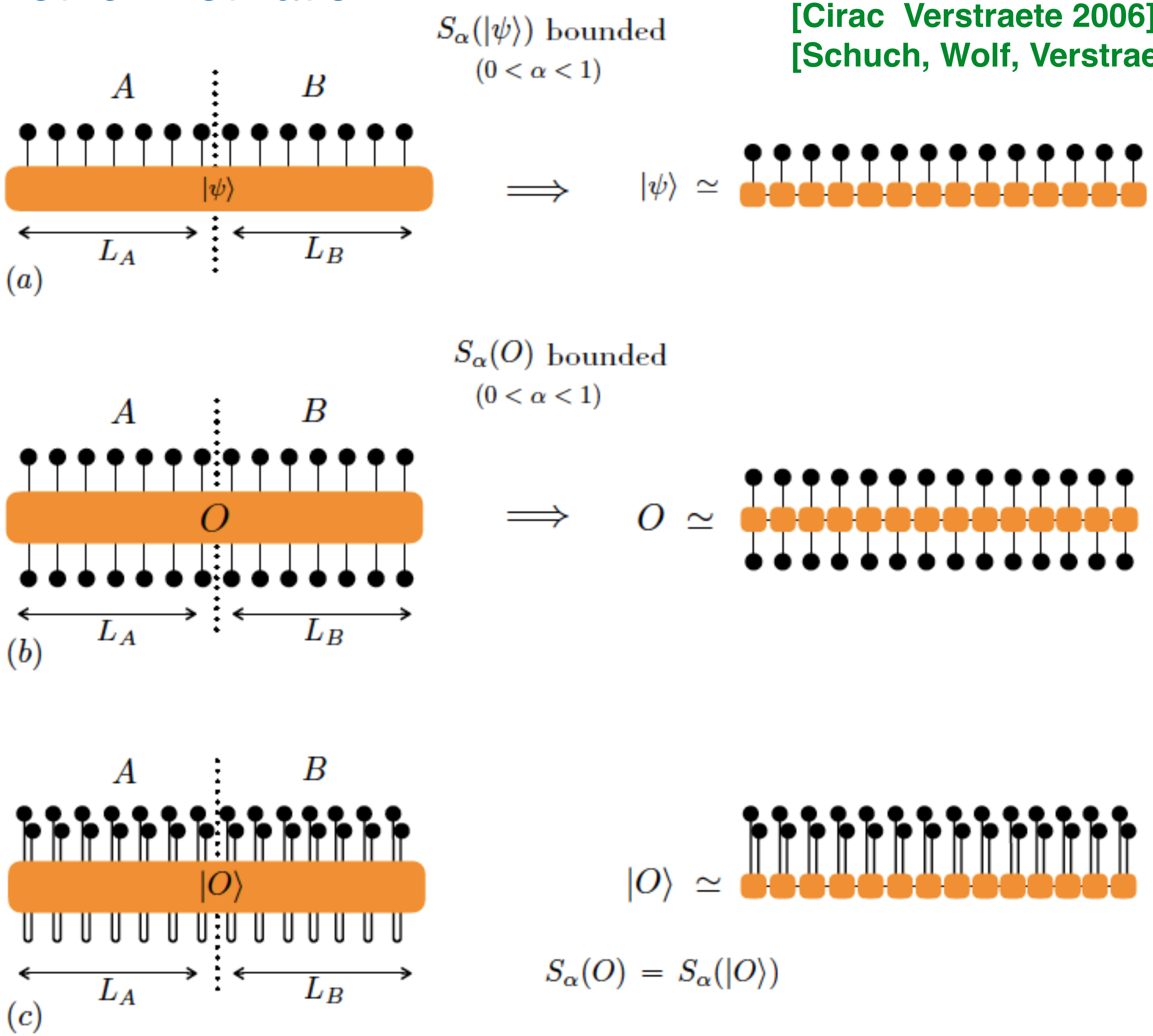
In that context, operator entanglement was first investigated by **[Prosen, Pizorn 2007], [Znidaric, Prosen, Pizorn, 2008]** (they were calling it 'Operator Space Entanglement Entropy' or OSEE).

Recently, revived interest in this quantity:
[JD 2016], [Zhou, Luitz 2016], [Jonay, Huse, Nahum 2018], [Xu, Swingle 2018], [van Nieuwenburg, Zilberberg 2018], [Pal, Lakshminarayan, 2018], [Wang, Zhou, 2019], [Lezama, Luitz, 2019], [Alba, JD, Medenjak 2020], [Bertini, Kos, Prosen 2020] etc.

Why care about ‘operator entanglement’?

Another motivation

[Cirac Verstraete 2006],
[Schuch, Wolf, Verstraete Cirac 2008]



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A warning:

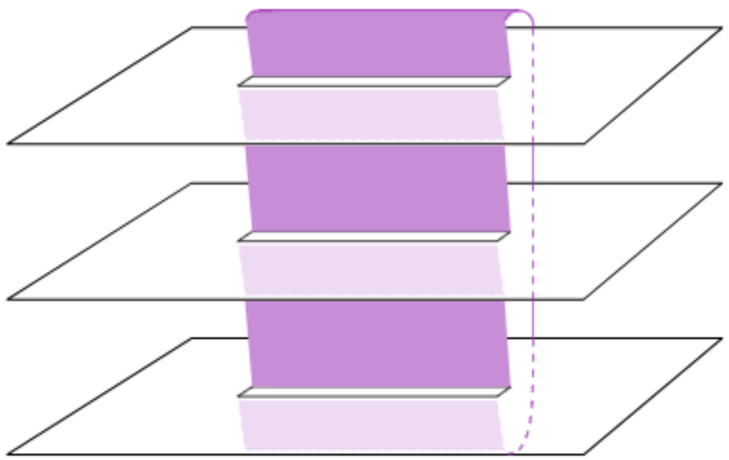
- by treating the operators as states, we tackle 'approximability' only with respect to the L^2-norm. Not the L^1-norm.

Plan of the rest of the talk

1. basic results from 2d CFT **[JD 2016]**
    1. OE of evolution operator
    2. OE of thermal state
    3. OE of reduced density matrix after global quench
2. OE of Heisenberg-picture operators: integrable vs. non-integrable
   [Alba, JD, Medenjak 2020, Bertini, Kos, Prosen 2020, etc.]
3. OE of the density matrix under dissipative evolution
   work in progress with V. Alba, J. Schachenmayer, G. Preisser, D. Wellnitz, 2021

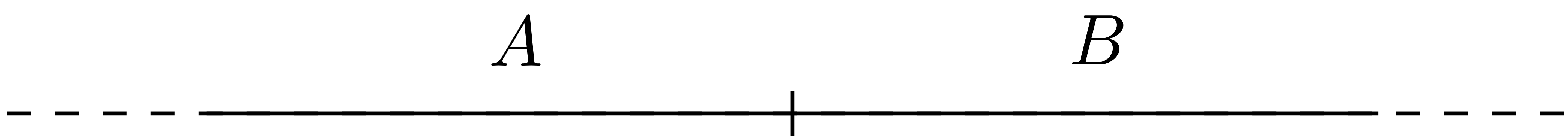
1.1 OE of the evolution operator

CFT prediction: linear growth



By adapting the α-sheeted surface/**twist field trick** of **[Calabrese Cardy 2004], [Calabrese Cardy 2005] [Cardy, Castro-Alvaredo, Doyon 2008]**, one can easily derive the following result.

The evolution operator on an infinite line has an OE that blows up linearly in time.



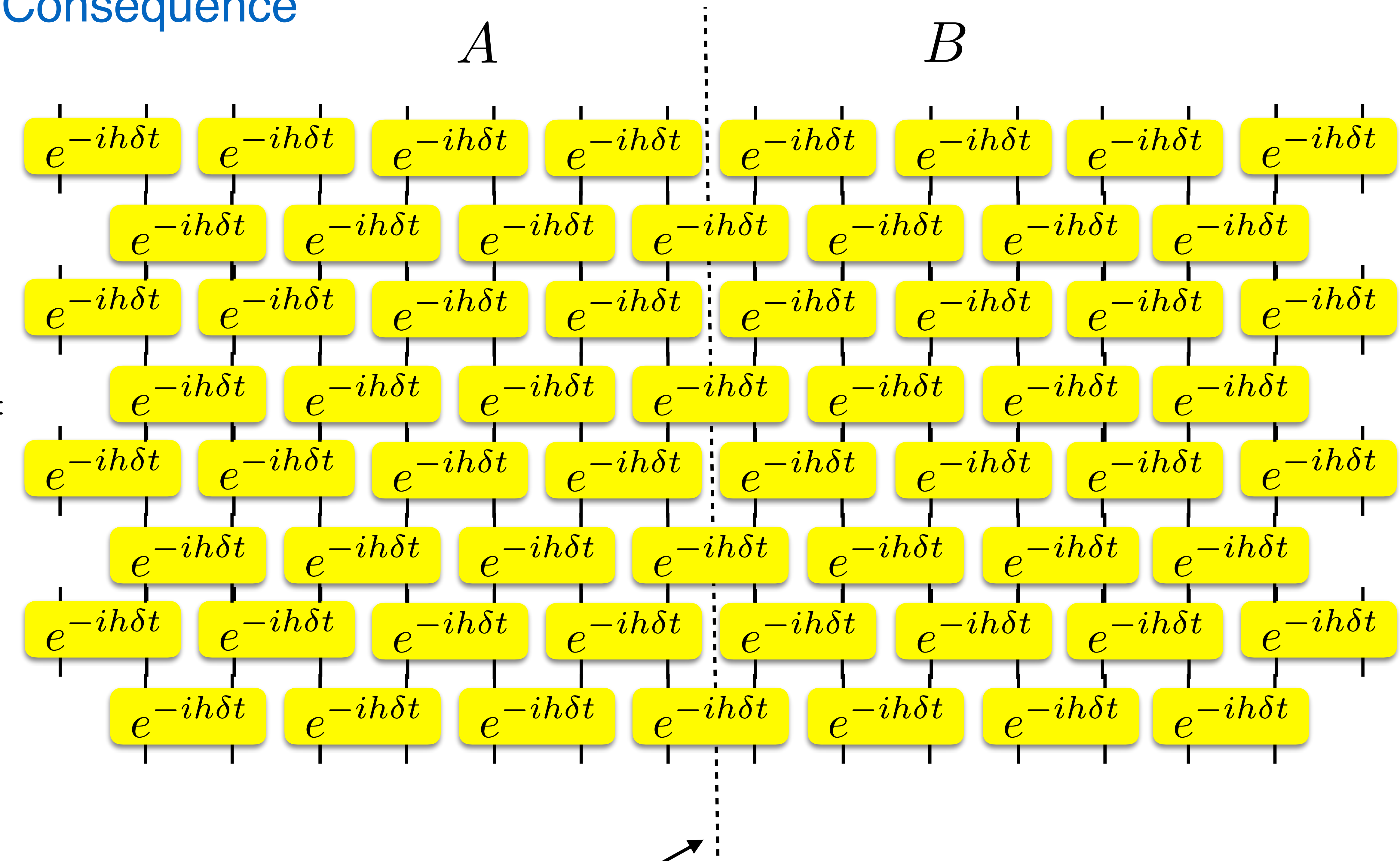
$$S_\alpha(e^{-iHt}) \propto t$$

Similarly to the linear growth of entanglement after a global quench **[Calabrese Cardy 2005]**, **this conclusion holds beyond CFT**.
Same conclusion reached by **[Zhou, Luitz 2016], [Jonay, Huse, Nahum 2018]**.

1.1 OE of the evolution operator

Consequence

$$e^{-iHt} =$$



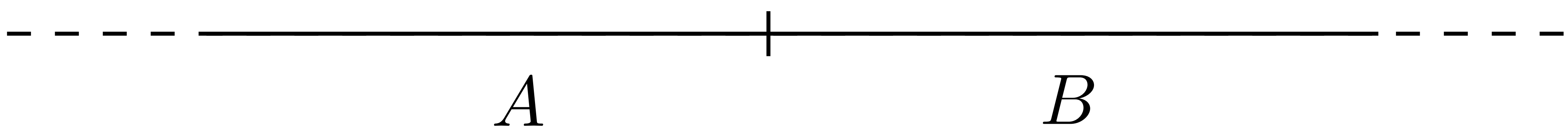
bond dimension will blow up exponentially with time

1.2 OE of thermal density matrix

CFT prediction: area law

The thermal density matrix $\rho = e^{-\beta H}$ on an infinite line has bounded OE:

$\beta < \infty$



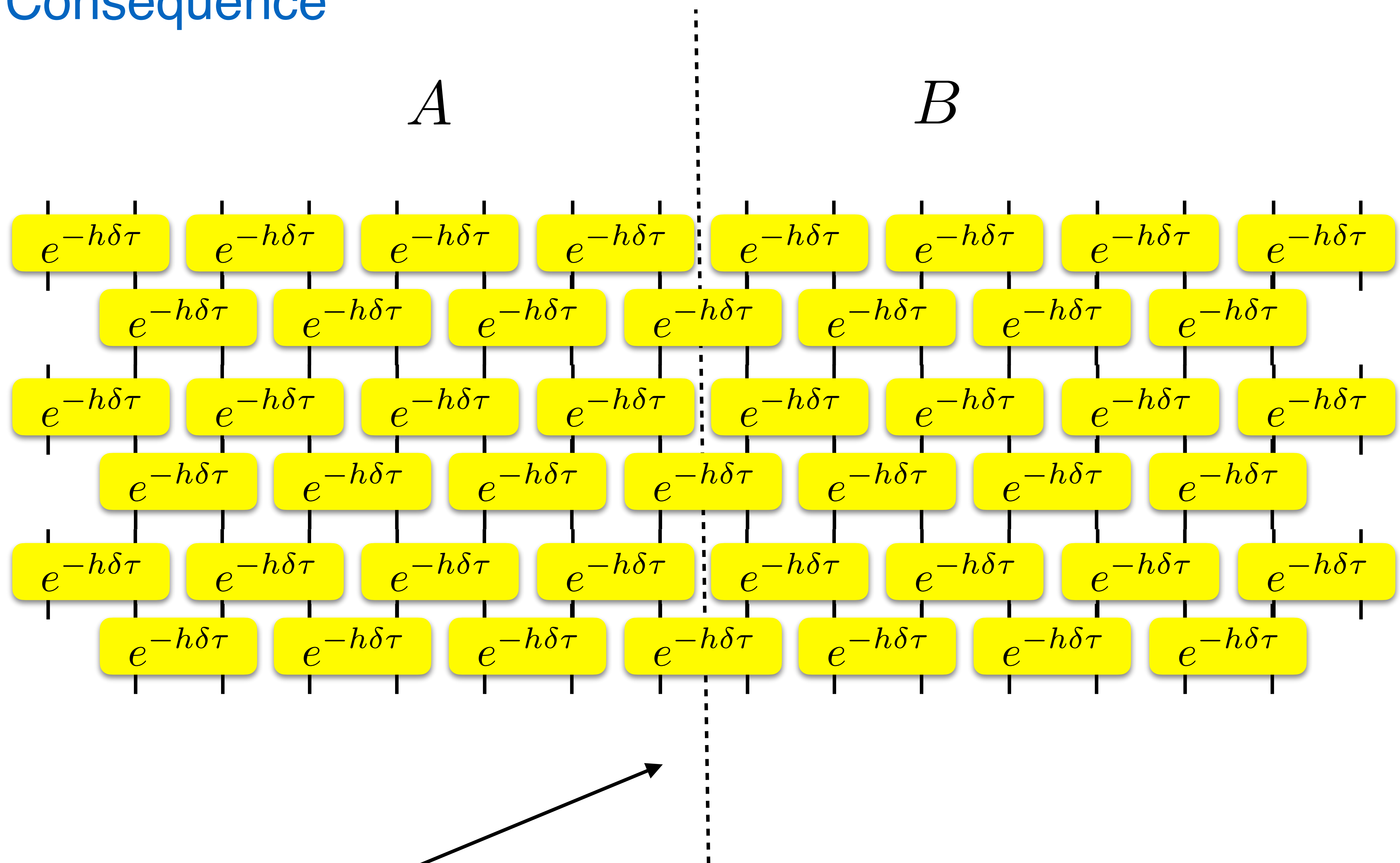
$$S_\alpha(e^{-\beta H}) = \frac{c}{6}\left(1 + \frac{1}{\alpha}\right)\log\beta$$

Note: it is formally the same calculation as for the entanglement entropy of the ThermoField Double state. The logarithmic growth of the OE was also conjectured in **[Znidaric, Prosen, Pizorn 2008]**.

1.2 OE of thermal density matrix

Consequence

$$e^{-\beta H} =$$



suggests that an efficient compression is possible, so MPO approximation should work well at finite temperature

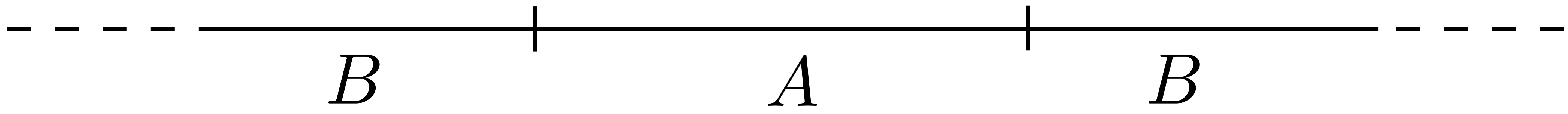
This is a well-known fact, and it holds also for PEPS in higher D

[Zwolak, Vidal 2004], [Hastings 2006], [Molnar, Schuch, Verstraete, Cirac 2015], etc.

1.3 OE after a global quench

Global quench **[Calabrese Cardy 2006]**: $|\psi_0\rangle \to e^{-iHt}|\psi_0\rangle$

At time t, take the reduced density matrix of a finite interval $\rho_A(t)$.



Question: can $\rho_A(t)$ be approximated by an MPO?

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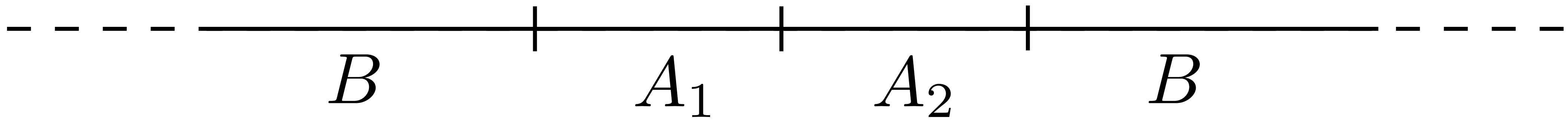
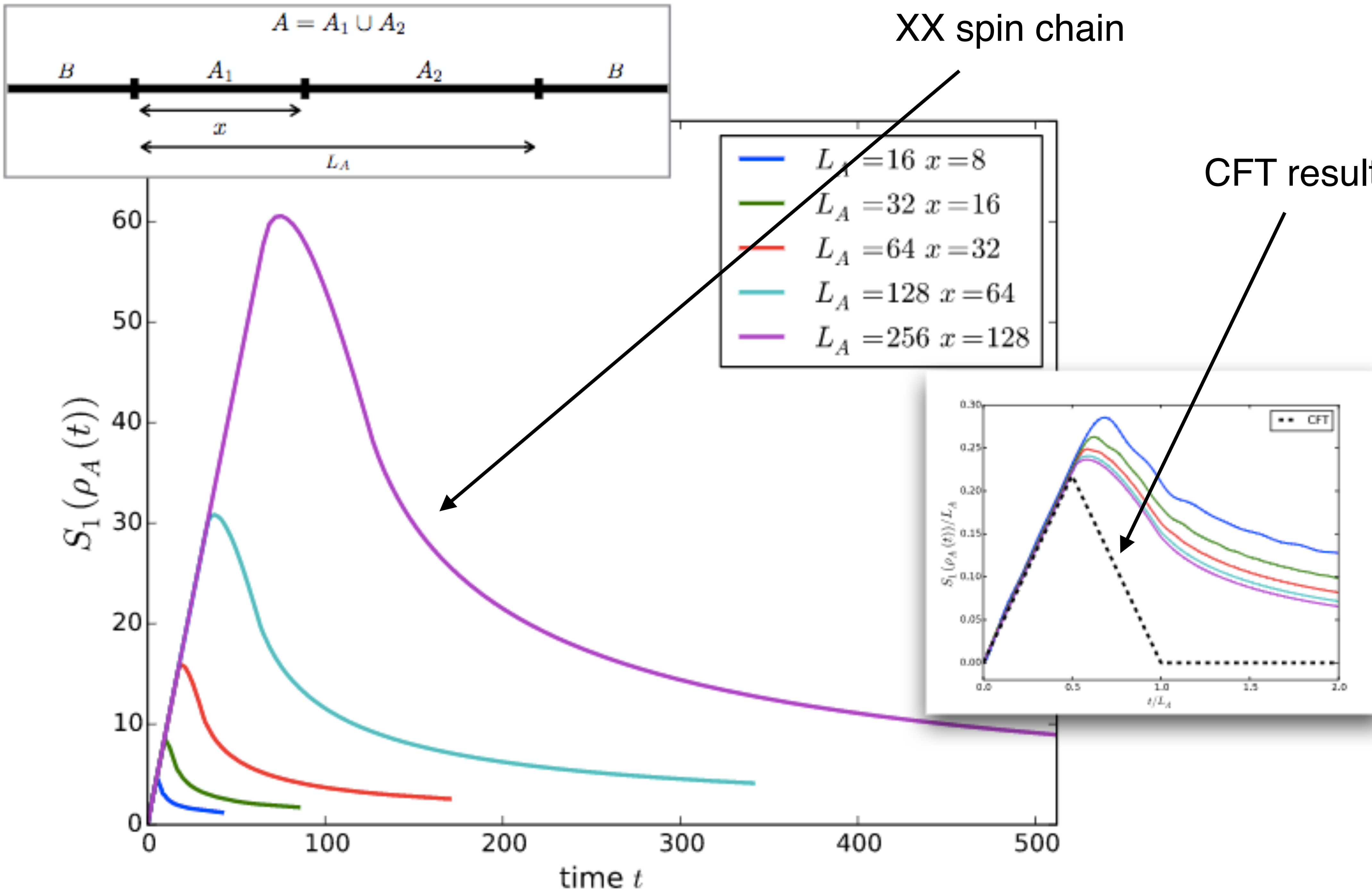


Question: can $\rho_A(t)$ be approximated by an MPO?

The answer will depend on the behavior of $S_\alpha(\rho_A(t))$.

1.3 OE after a global quench

CFT prediction: ‘entanglement barrier’ at intermediate times



(The CFT calculation parallels the one of **[Coser, Tonni, Calabrese 2014]** for the negativity, which has a very similar behavior.) See also more recent studies by **[Alba Calabrese 2018], [Wang Zhou 2019], [Bertini Klobas]**

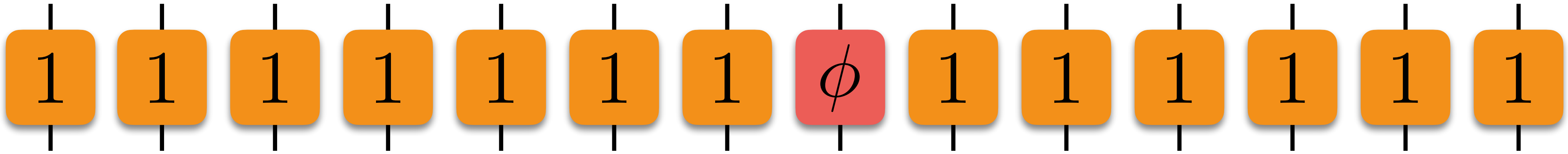
2. OE of Heisenberg picture operators

Motivation: Heisenberg-picture DMRG/TEBD

Goal: calculate $\langle\psi_0|\,\phi(t)\,|\psi_0\rangle$ after a global quench.

Idea of DMRG/TEBD in Heisenberg picture: instead of approximating $e^{-iHt}\,|\psi_0\rangle$ by an MPS, one could approximate $\phi(t) = e^{iHt}\phi e^{-iHt}$ by an MPO.

[Hartmann, Prior, Clark, Plenio, 2009]



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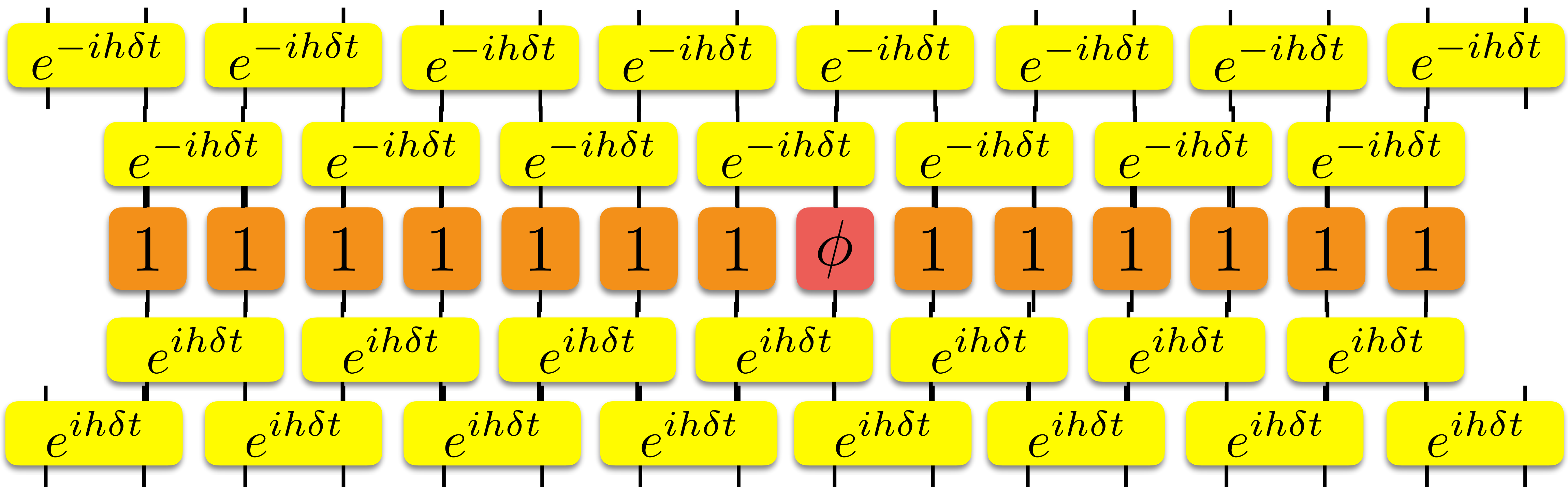

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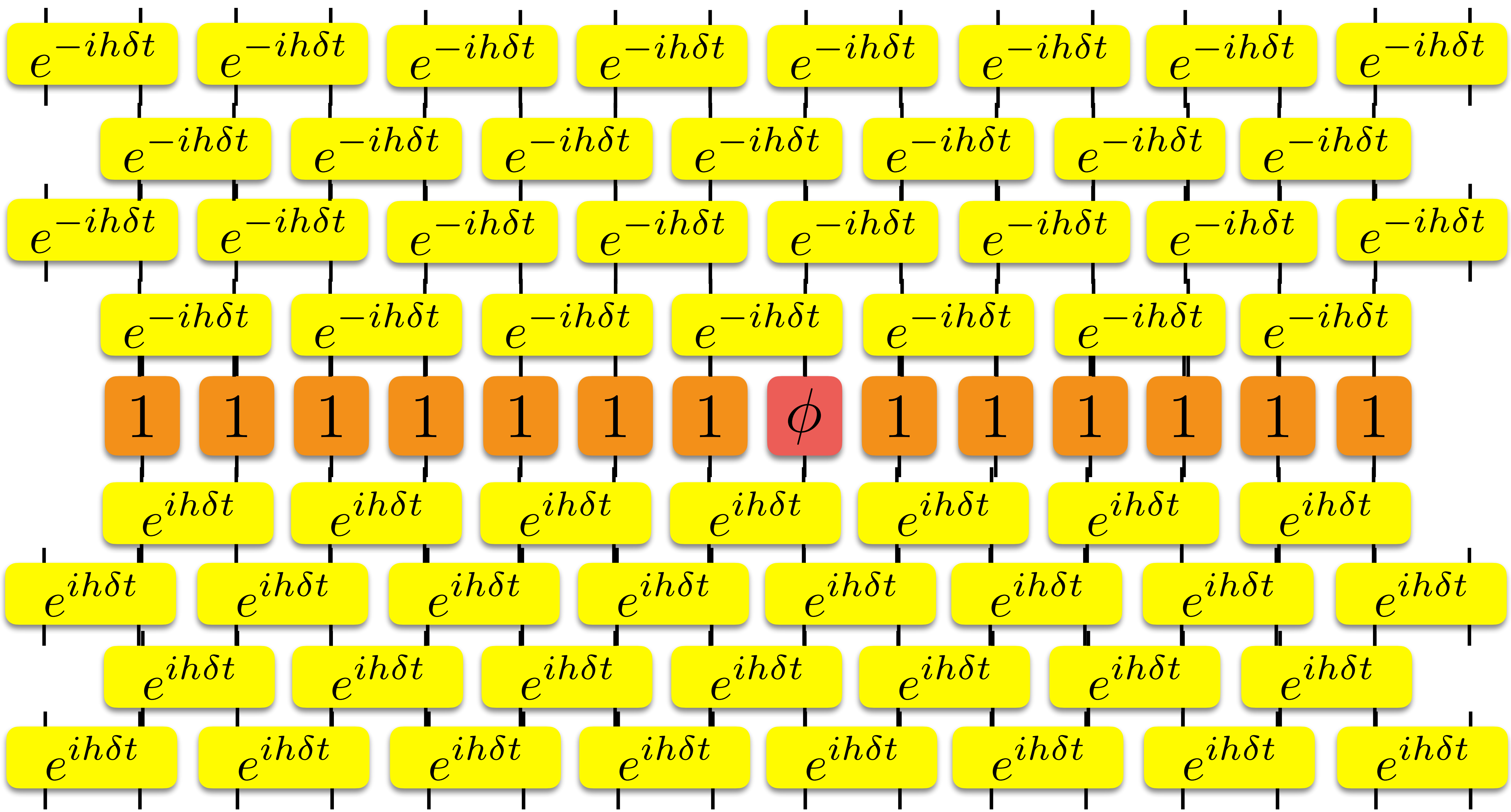

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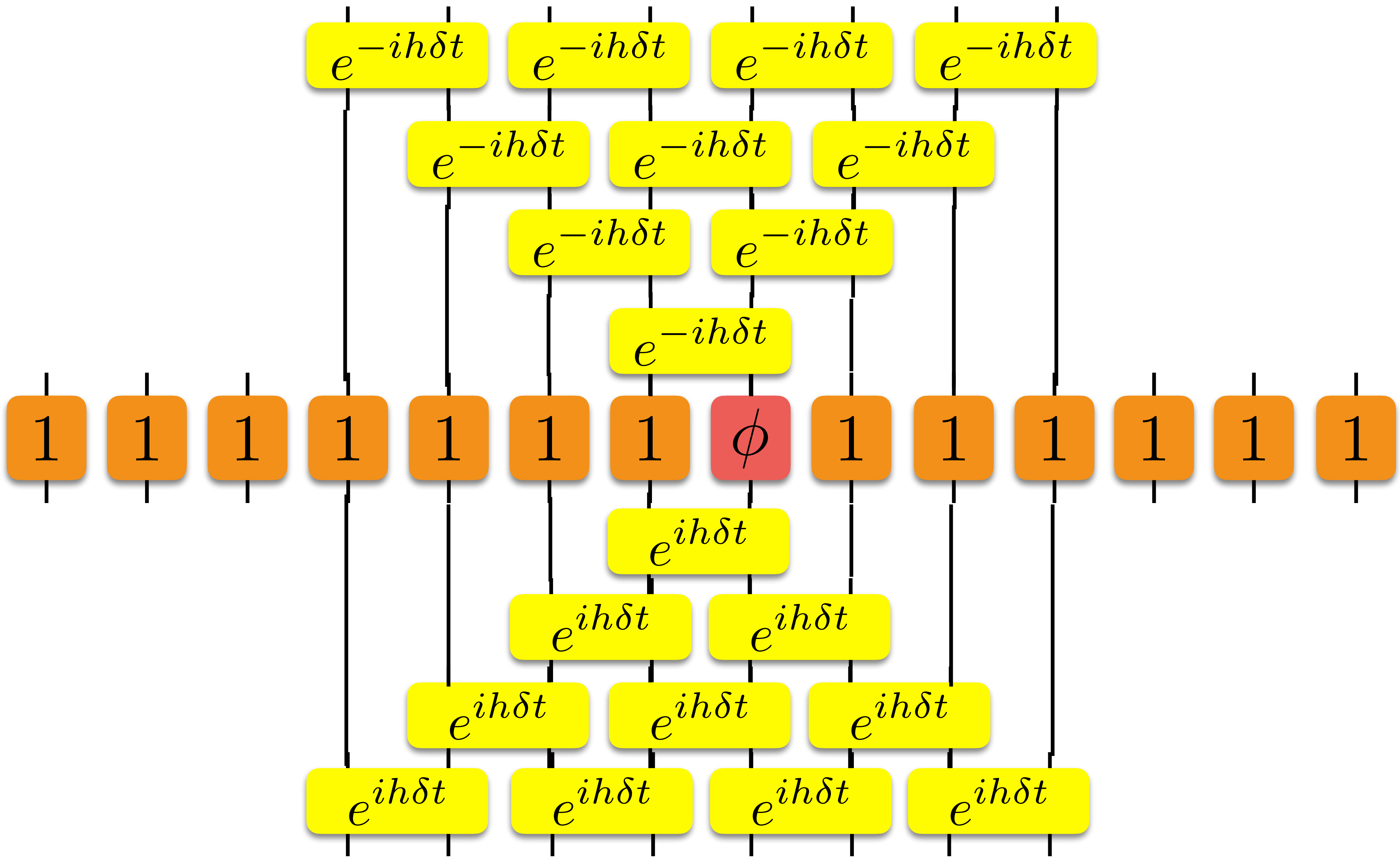
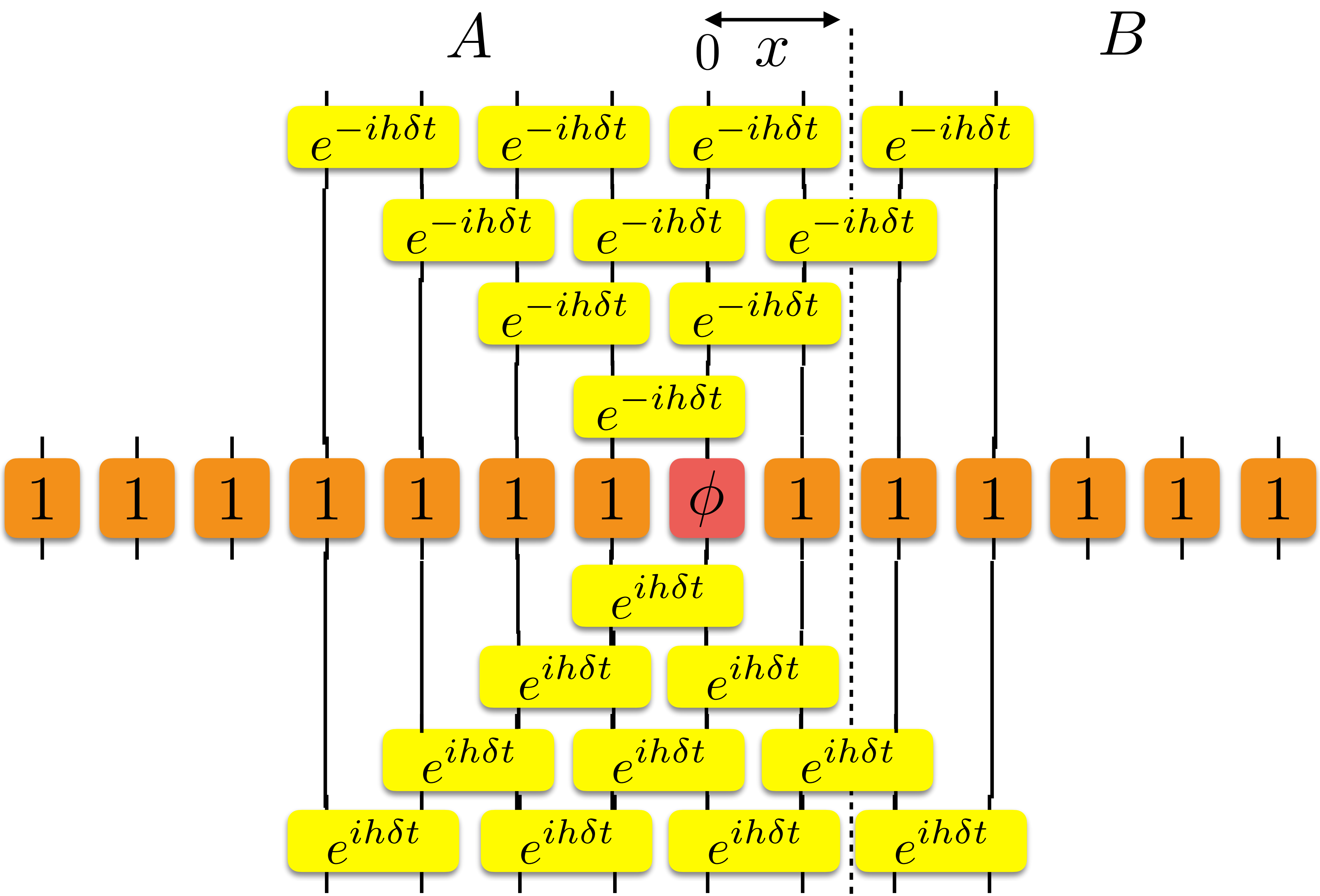


2. OE of Heisenberg picture operators

Motivation: Heisenberg-picture DMRG/TEBD

Question: how does $S_\alpha(x, \phi(t))$ grow with time?



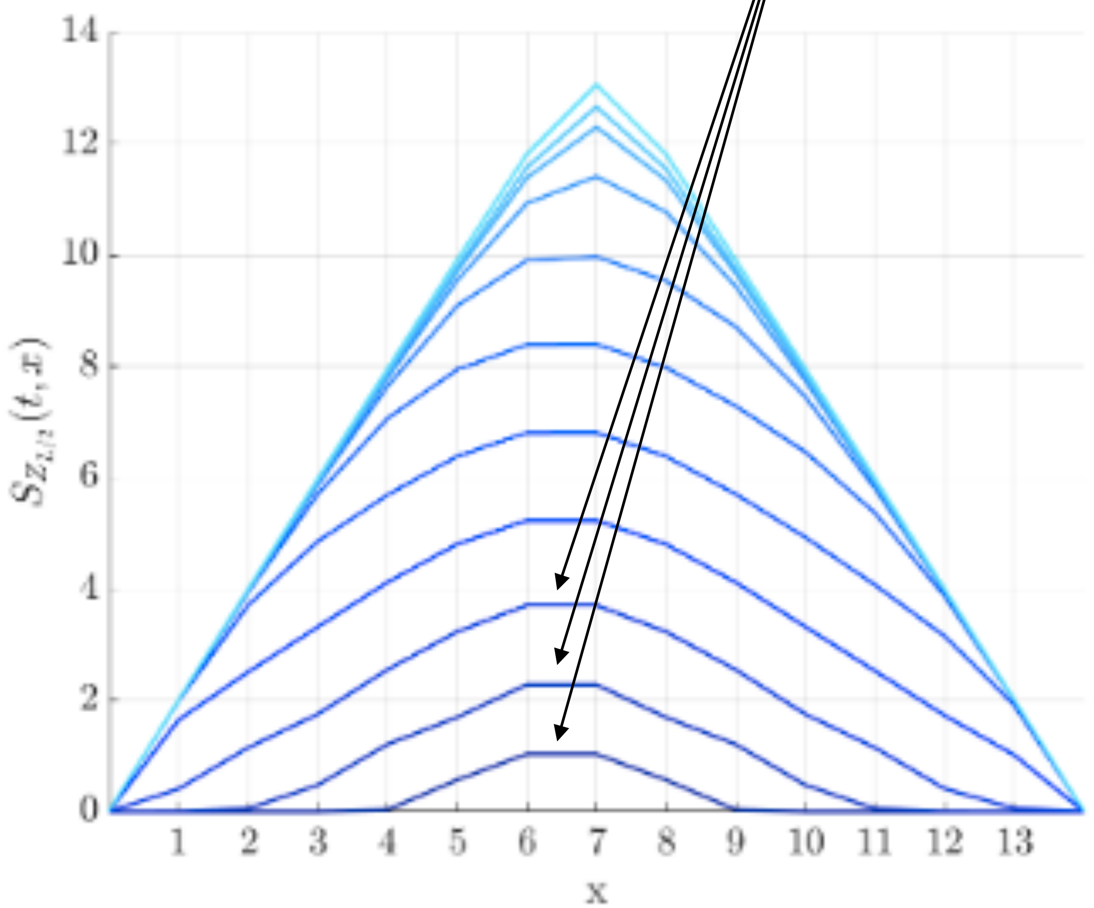
2. OE of Heisenberg picture operators

Known results (before 2020)

$$S_1(x=0,t) \propto t$$

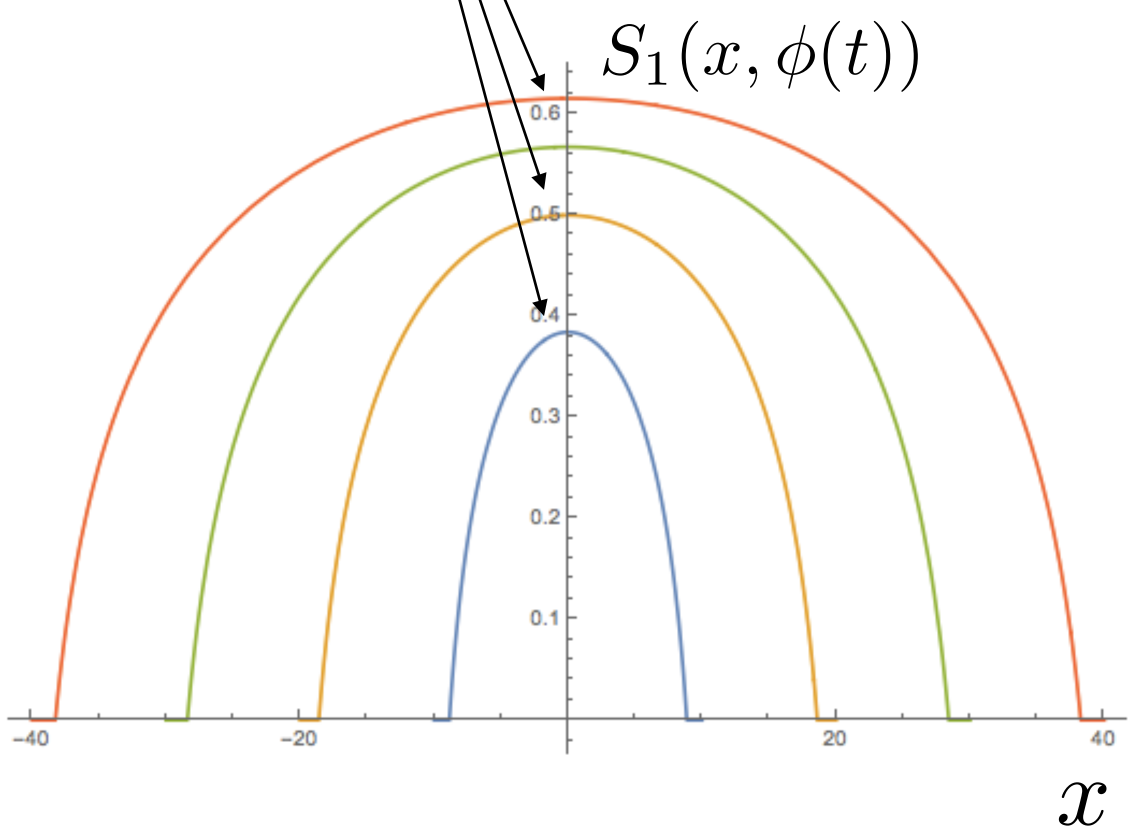
$$S_1(x=0,t) \sim \frac{1}{6}\log t$$

OE in quantum chaotic Ising chain



[Jonay, Huse, Nahum, 2018]

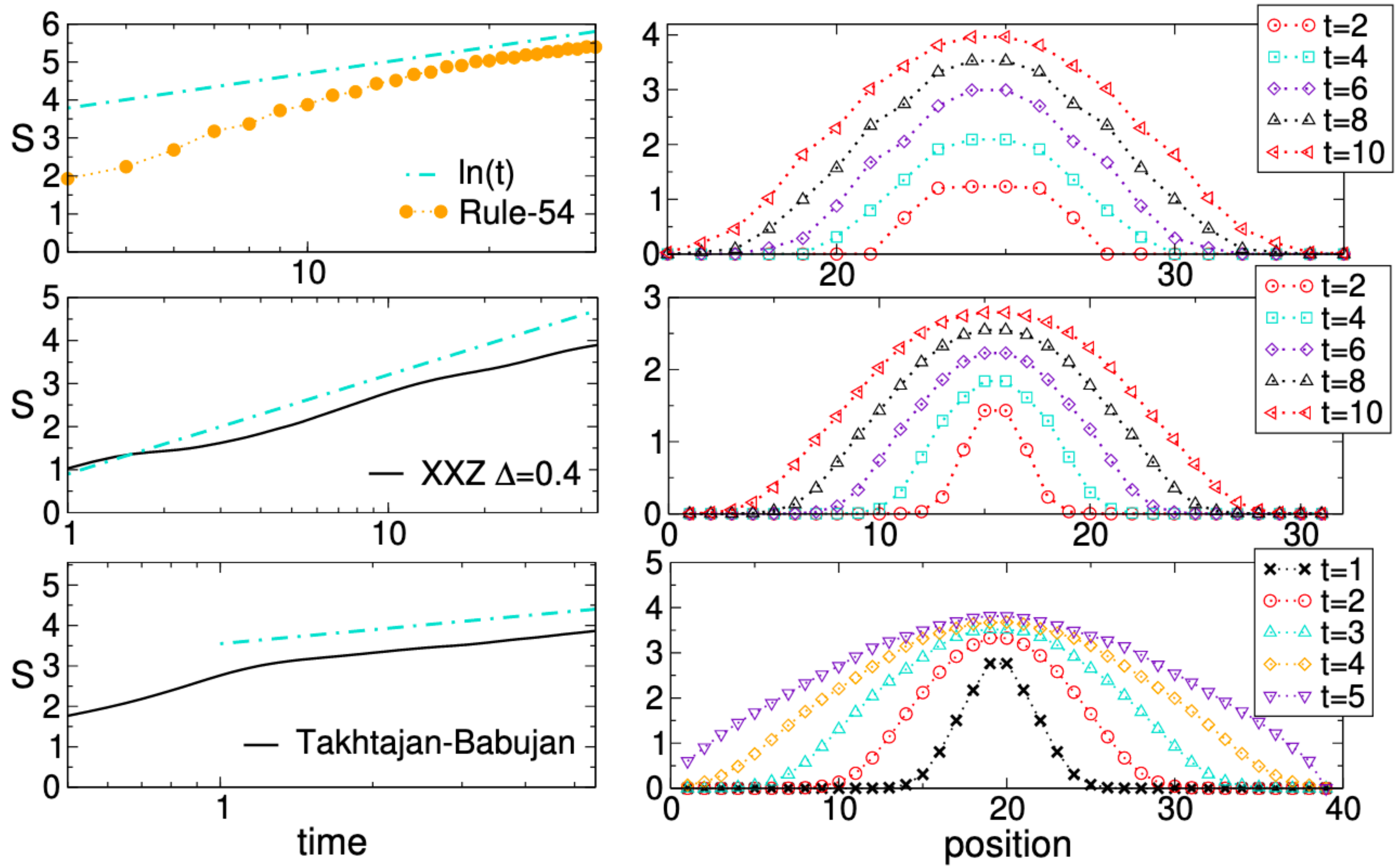
OE in Ising chain (free fermions)



[JD 2016]
(logarithmic growth found numerically by **[Prosen and Pizorn 2007]**)

2. OE of Heisenberg picture operators

Numerics in interacting integrable chains (2020)



[Alba, JD, Medenjak 2020]

2. OE of Heisenberg picture operators

A conjecture (2020)

Conjecture formulated in **[Alba, JD, Medenjak 2020]**
(closely related conjecture by **[Prosen and Znidaric 2007]**):

The operator entanglement of local operators in interacting integrable spin chains grows at most logarithmically with time (i.e. similarly to the free fermion case).

Consequently, operator entanglement distinguishes integrable dynamics from chaotic dynamics.

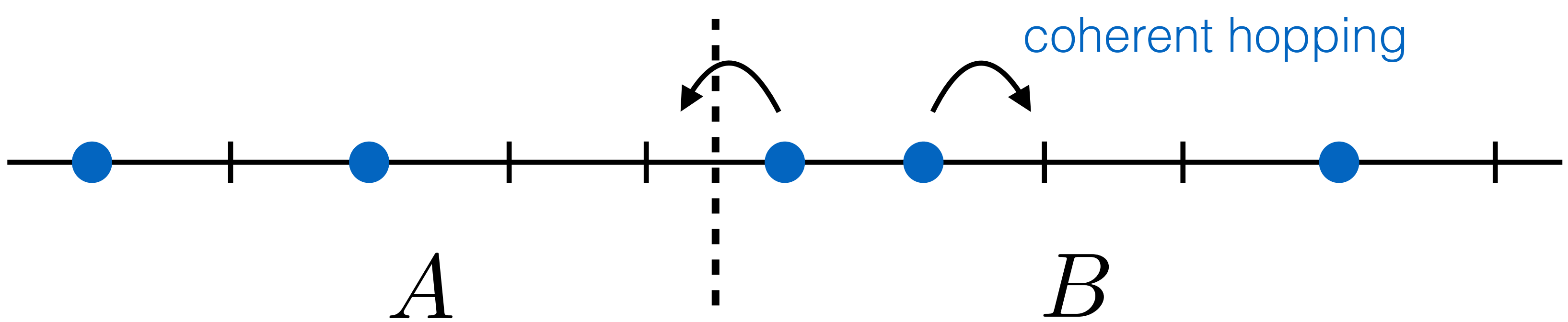
Confirmed by analytic results in various systems: rule 54 chain **[Alba, JD, Medenjak 2020]**, dual unitary circuits **[Bertini, Kos, Prosen 2020]**, holographic CFT **[Caputa, Simon, Stikonas, Takayanagi, Watanabe, 2015]**, etc.

3. OE of density matrix under Lindblad evolution

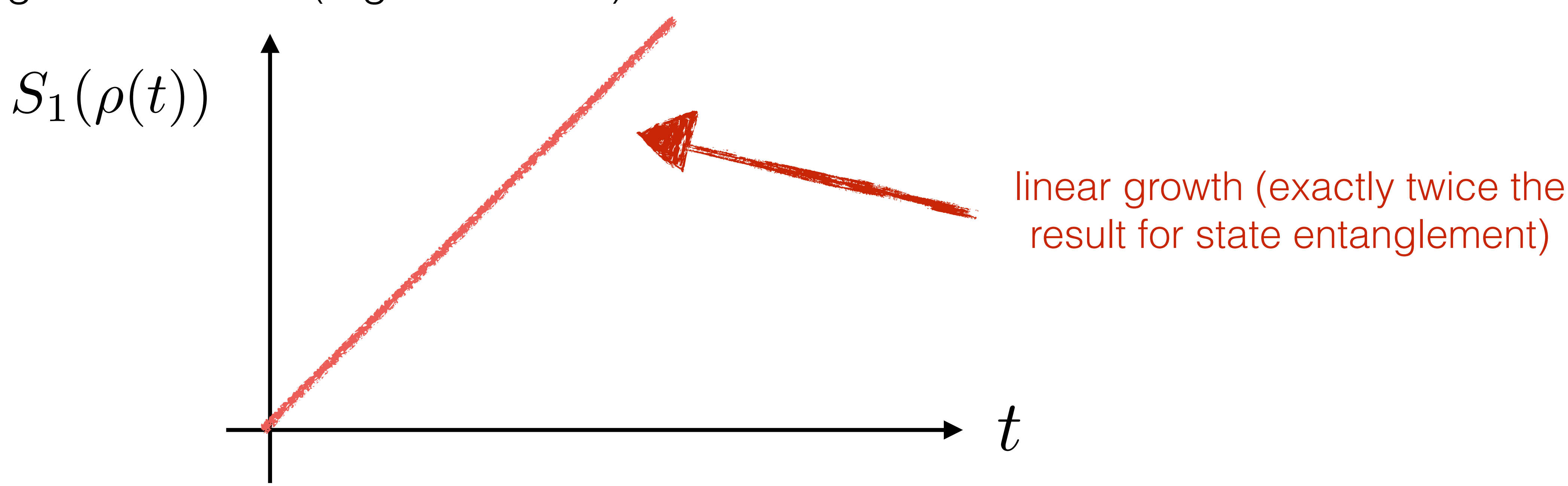
‘Standard’ scenario

Take a Hamiltonian for 1d system, for instance hard core bosons with hopping:

$$H = \sum_j a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j \qquad\qquad \frac{d}{dt}\rho = -i[H,\rho]$$



Question: how does the OE of the density matrix evolve after a quench from an initial state with short-range correlations (e.g. Néel state)?

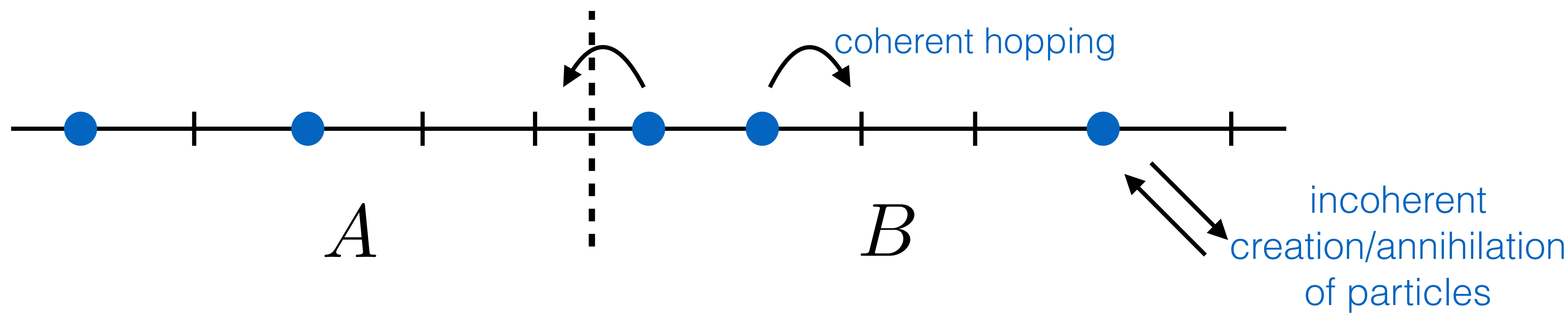


3. OE of density matrix under Lindblad evolution

'Standard' scenario

Now add a dissipative part, for instance incoherent creation/annihilation of particles:

$$\frac{d}{dt}\rho = -i[H,\rho] + \gamma \sum_j \left(a_j^\dagger \rho a_j - \frac{1}{2}\{a_j a_j^\dagger, \rho\} \right) + \gamma \sum_j \left(a_j \rho a_j^\dagger - \frac{1}{2}\{a_j^\dagger a_j, \rho\} \right)$$



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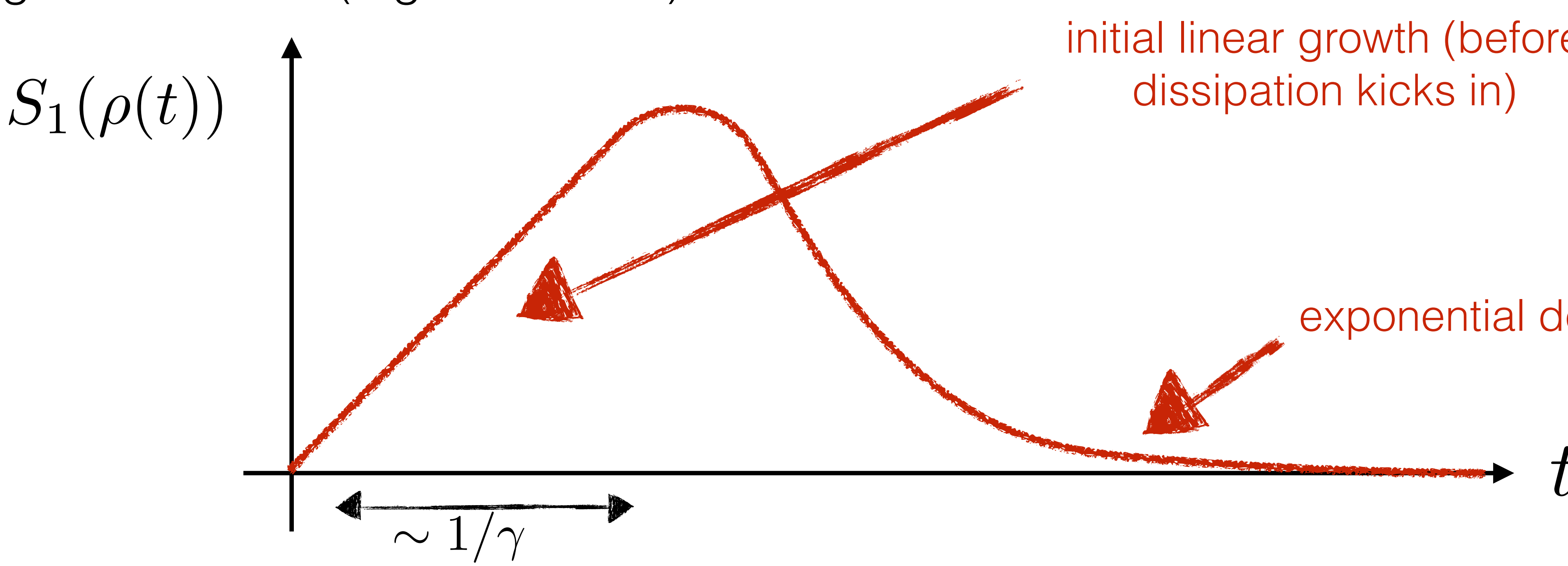


3. OE of density matrix under Lindblad evolution

'Standard' scenario

Remark: the same observation has been used recently to argue against the 'quantum supremacy' claim by Google **[Noh, Jiang, Fefferman 2020]**

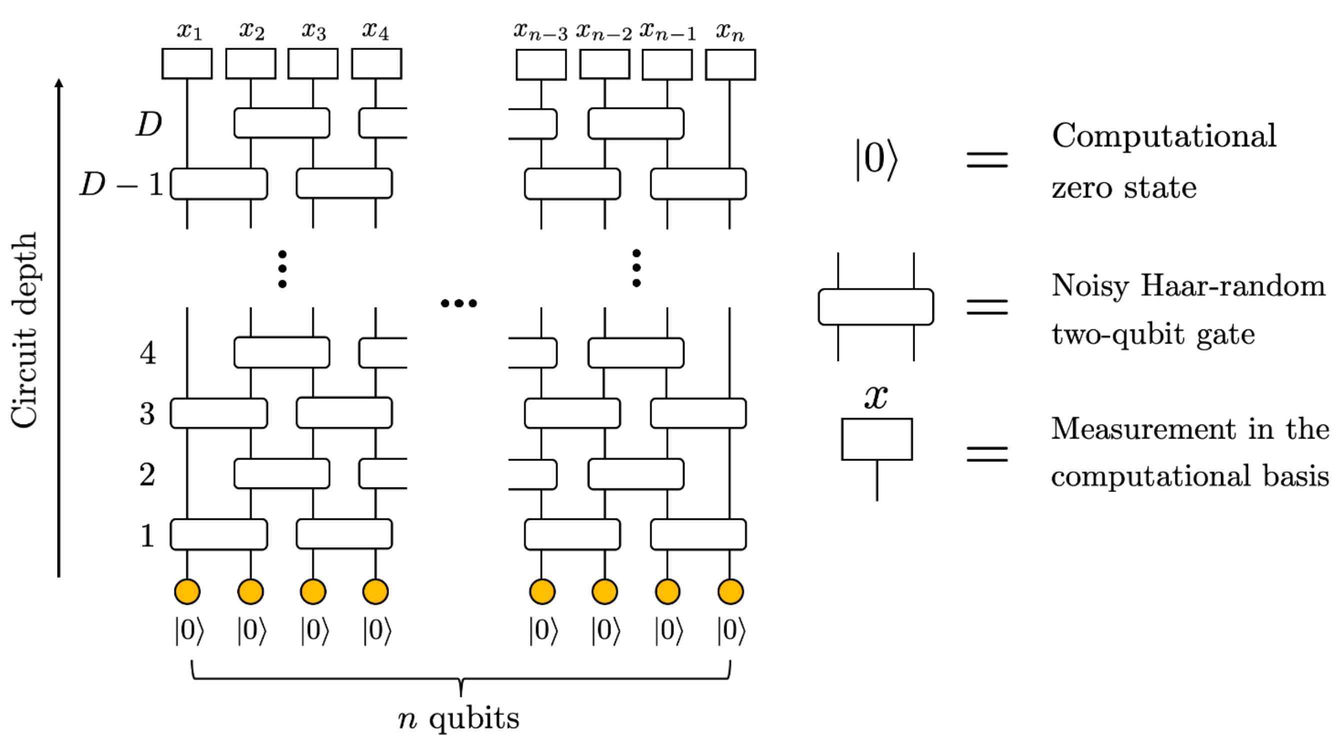


Figure 2: Noisy random circuit sampling in one dimension. Each noisy two-qubit gate is given by a 4×4 Haar-random unitary operation followed by a two-qubit depolarization channel $\mathcal{N}_2[p]$ with an error rate p. At the end of the circuit, all the qubits are measured in the computational basis. For simplicity, we only consider even number of qubits. Although the maximum circuit depth D is chosen to be even in the schematic illustration, we allow D to be odd as well.

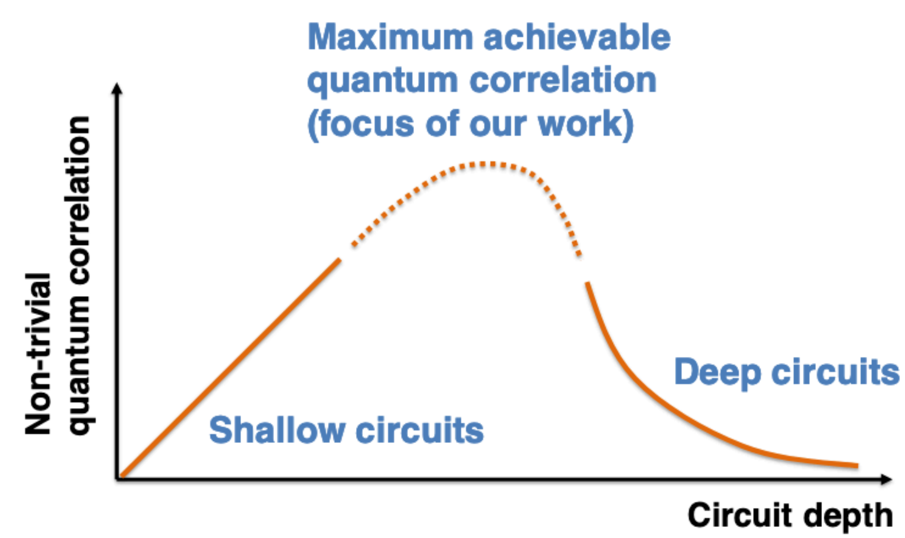


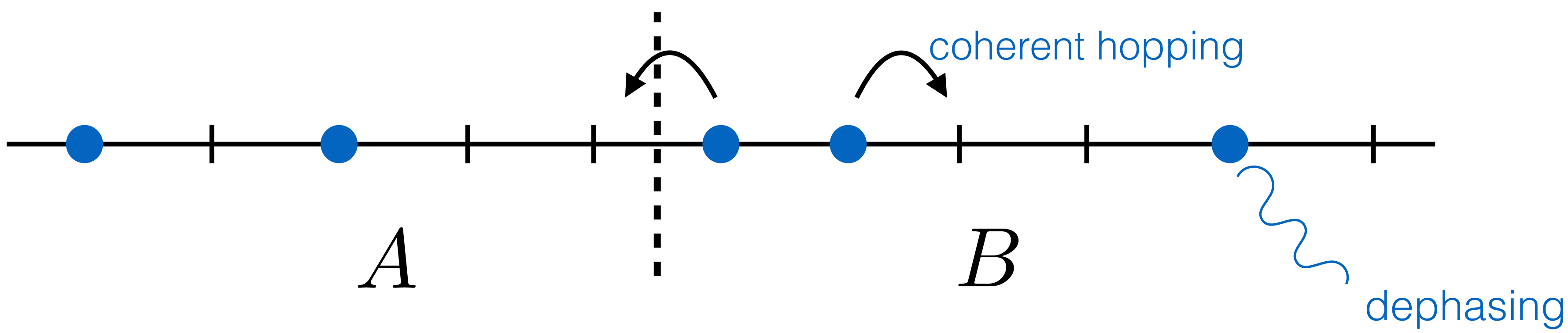
Figure 1: Schematic plot of the degree of non-trivial quantum correlation as a function of the circuit depth. When the circuit depth is small, quantum correlation grows linearly in the circuit depth. On the other hand, when the circuit depth is large, the system converges to a depolarized state and thus the non-trivial quantum correlations are washed away. The focus of our work is to understand the optimal regime where the maximum non-trivial quantum correlation is achieved. See also Figs. 5 and 9.

3. OE of density matrix under Lindblad evolution

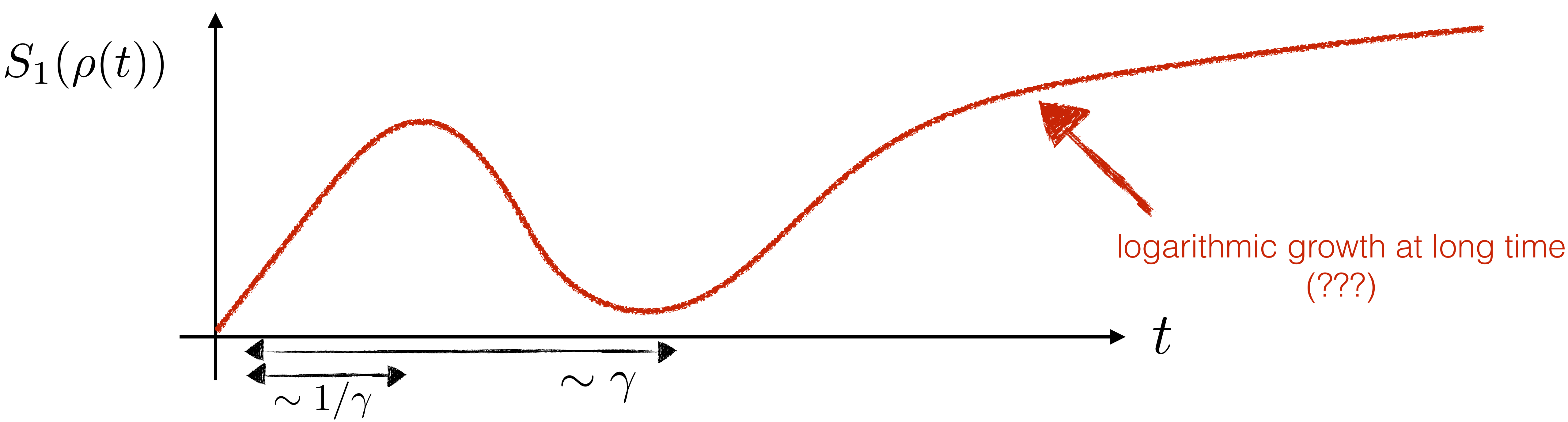
An exception to the standard scenario

Something odd happens for dephasing:

$$\frac{d}{dt}\rho = -i[H,\rho] + \gamma \sum_j \left((-1)^{a_j^\dagger a_j} \rho (-1)^{a_j^\dagger a_j} - \rho \right)$$



Observation (Schachenmayer+Wellnitz+Preisser, work in progress): OE of the density matrix after a quench from an initial state with short-range correlations



Conclusions

- Operator Entanglement is interesting, it is useful to determine whether an operator can be approximated by a Matrix Product Operator with small bond dimension

- nice analytic calculations to do within 1+1d CFT, or with toy models (integrable models, cellular automata, random circuits, dual unitary circuits, etc.)

- some funny surprises:
    - the OE of operators in Heisenberg picture distinguishes chaotic from integrable dynamics
    - logarithmic growth of the OE of the density matrix under Lindblad evolution

Happy birthday Hubert!