

Integral Means Spectrum, SLE and Riemann Zeta Function

Bertrand Duplantier[†], Eero Saksman[‡], Michel Zinsmeister

Dmitry Beliaev, B.D., M.Z.

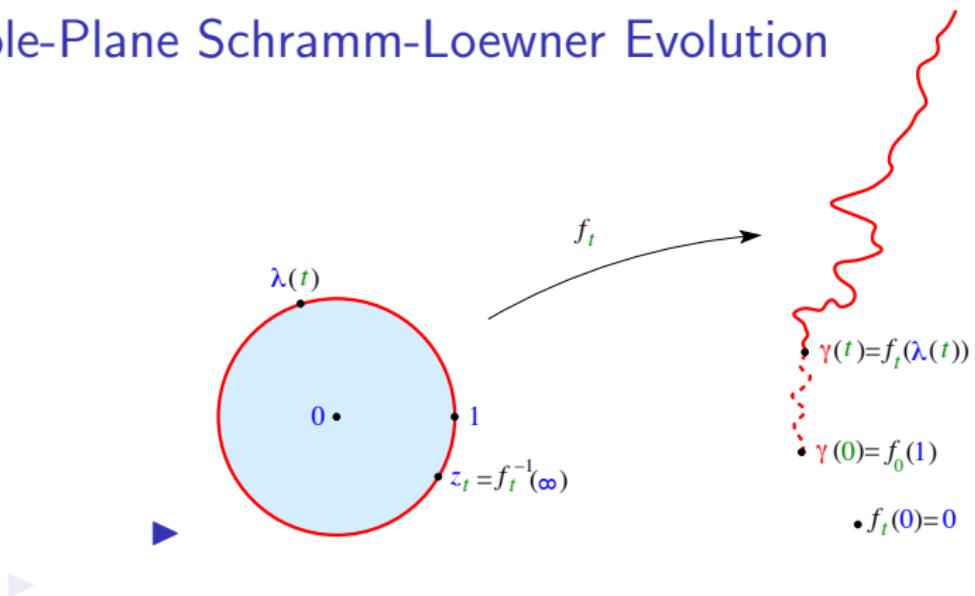
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THE ART OF MATHEMATICAL PHYSICS M \cap Φ
Hubert Saleur is 60
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Whole-Plane Schramm-Loewner Evolution

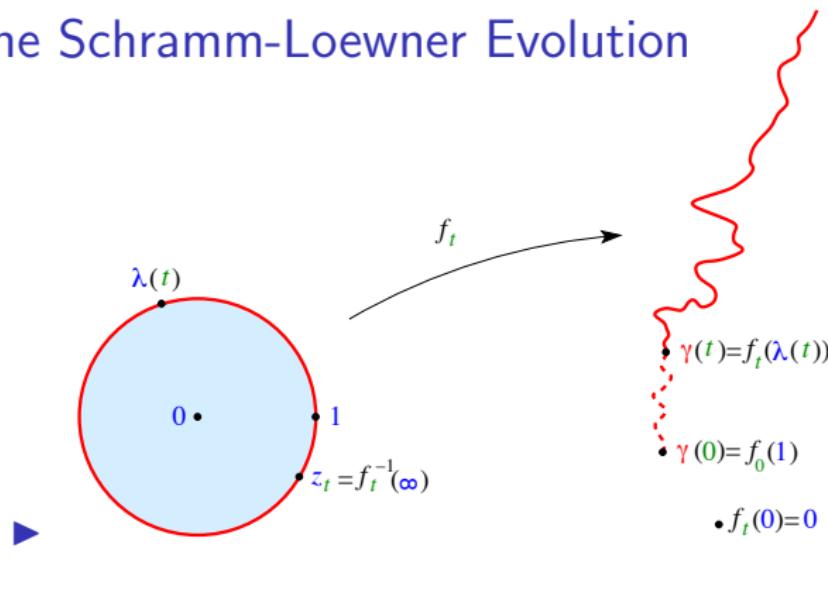


$$z \in \mathbb{D}, \quad \frac{\partial}{\partial t} f_t(z) = z \frac{\partial}{\partial z} f_t(z) \frac{\lambda(t) + z}{\lambda(t) - z}, \quad \lambda(t) = \exp(i\sqrt{\kappa} B_t)$$

$$f_t(e^{-t}z) \rightarrow z, \quad t \rightarrow +\infty; \quad \kappa = 0, \quad f_t(z) = \frac{e^t z}{(1-z)^2} \quad (\text{Koebe})$$

- ▶ $1/f(1/z)$ is the **bounded exterior version** from $\mathbb{C} \setminus \overline{\mathbb{D}}$ to the slit plane [Beliaev & Smirnov, Lawler].

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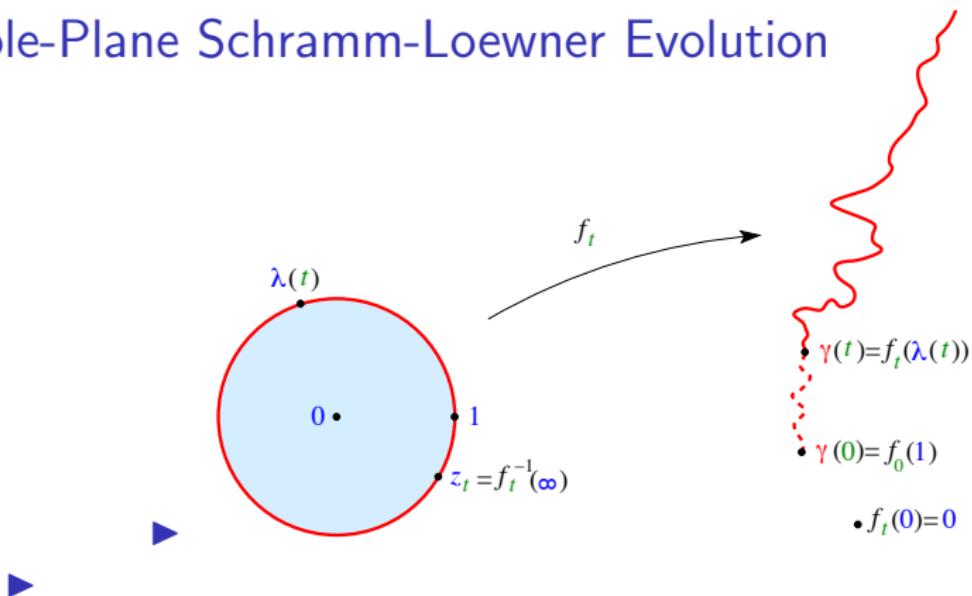


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Integral Means Spectrum

- ▶ Consider a Riemann map $\Phi : \mathbb{D} \rightarrow \mathbb{C}$
- ▶ The **integral means** of Φ are

$$\mathcal{I}(r, p, \Phi) := \int_0^{2\pi} |\Phi'(re^{i\theta})|^p d\theta, \quad 0 < r < 1, \quad p \in \mathbb{R};$$

- ▶ Φ random:
Expectation: $\mathbb{E} \mathcal{I}(r, p, \Phi) := \int_0^{2\pi} \mathbb{E} [|\Phi'(re^{i\theta})|^p] d\theta.$
- ▶ One then defines

$$\beta_\Phi(p) := \limsup_{r \rightarrow 1^-} \frac{\log(\mathcal{I}(r, p, \Phi))}{\log(\frac{1}{1-r})};$$

- ▶ If the limit exists,

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- ▶ The **integral means spectrum** is related to the **multifractal spectrum** of the **harmonic measure** ω on the boundary K of the image domain.
- ▶ Define, for $\alpha \geq 1/2$, \mathcal{E}_α as being the set of points z on the boundary K where

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as $r \rightarrow 0$.

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$$\beta_{\text{tip}}(p, \kappa) = -p - 1 + \frac{1}{4} \left(4 + \kappa - \sqrt{(4 + \kappa)^2 - 8\kappa p} \right),$$

$$\beta_0(p, \kappa) = -p + \frac{(4 + \kappa)^2}{4\kappa} - \frac{(4 + \kappa)}{4\kappa} \sqrt{(4 + \kappa)^2 - 8\kappa p},$$

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- a.s. β_{tip} [Johansson Viklund & Lawler '12]
- a.s. β_0 [Gwynne, Miller & Sun '18]

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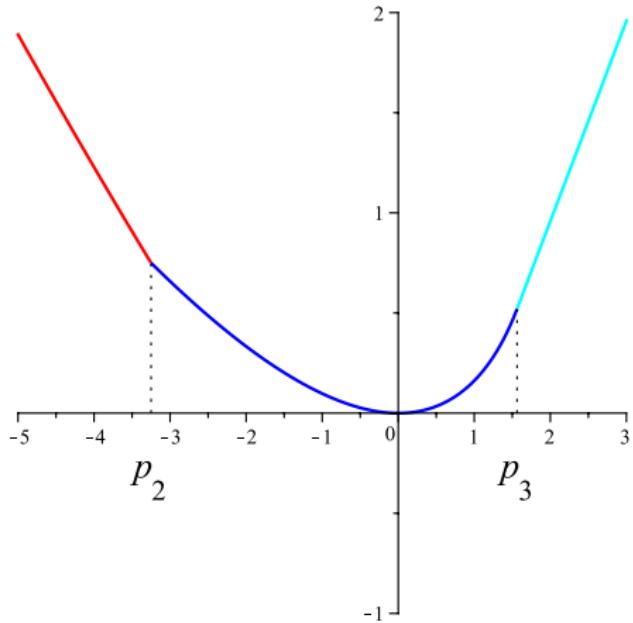
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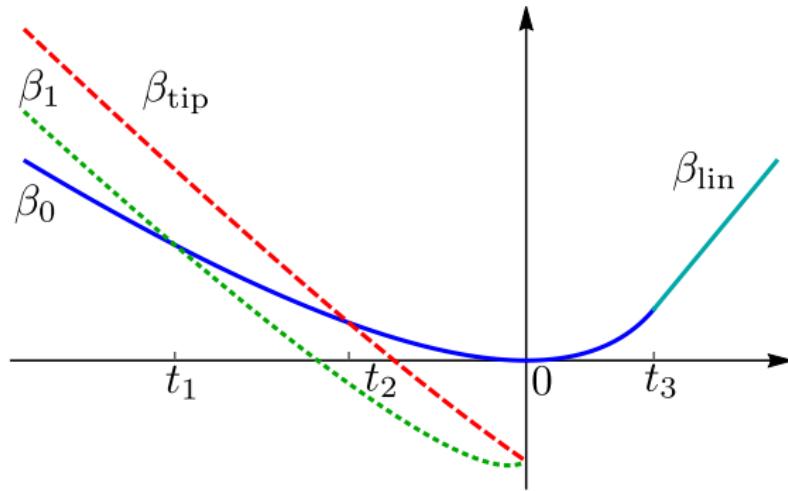
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$$p_2 = -1 - \frac{3\kappa}{8}, \quad p_3 = \frac{3(4 + \kappa)^2}{32\kappa}$$

Average integral means spectrum for **bounded** whole-plane SLE.



$$\beta_1(p, \kappa) := -p - \frac{1}{2} - \frac{1}{2}\sqrt{1 - 2\kappa p}.$$

'Second tip' spectrum

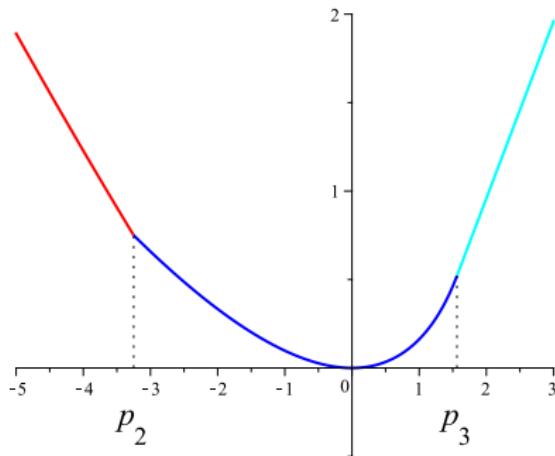
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Unbounded Whole-plane SLE

- In this case, [Loutsenko & Yermolayeva '13] and [D., Nguyen, Nguyen & Zinsmeister '14] have shown the existence of a phase transition at $p_0 := \frac{(4+\kappa)^2 - 4 - 2\sqrt{2(4+\kappa)^2 + 4}}{16\kappa}$ to

$$\hat{\beta}_1(p, \kappa) := 3p - \frac{1}{2} - \frac{1}{2}\sqrt{1 + 2\kappa p}.$$

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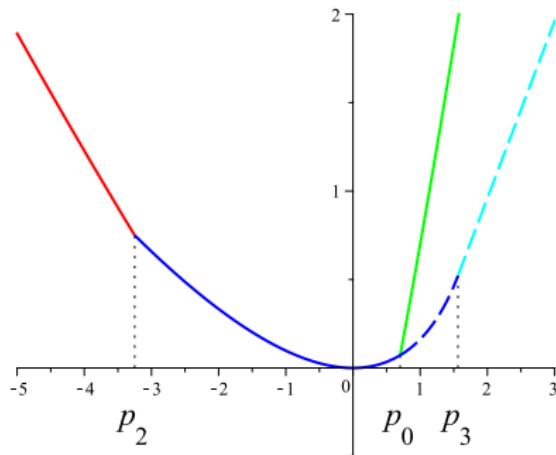


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Universal Integral Means Spectrum

- $B_{\text{bd}}(p) = \sup\{\beta_\Phi(p), \Phi \in \mathcal{S}, \Phi \text{ injective \& bounded}\}.$

Conjecture: $B_{\text{bd}}(p) = \frac{|p|}{2}$, $|p| \leq 2$; $B_{\text{bd}}(p) = |p| - 1$, $|p| \geq 2$.

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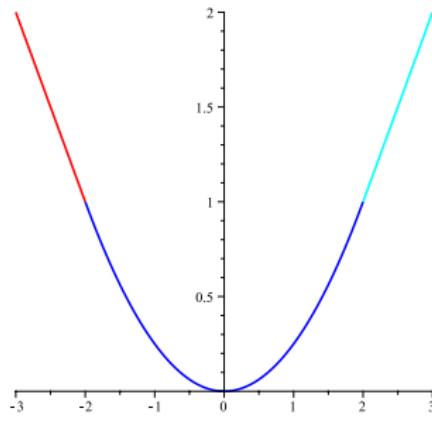
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- $B(p) = \sup\{\beta_\Phi(p), \Phi \in \mathcal{S}\}.$

where \mathcal{S} is the set of all Φ such that Φ bounded}.

- Theorem (Makarov '99):

$$B(p) = \max\{B_{\text{ad}}(p), 3p - 1\}.$$

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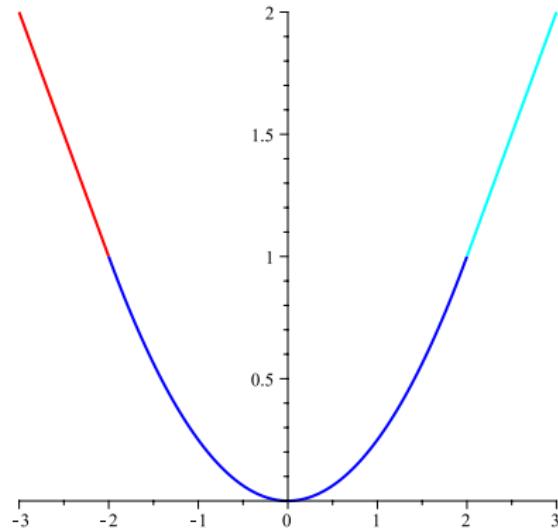
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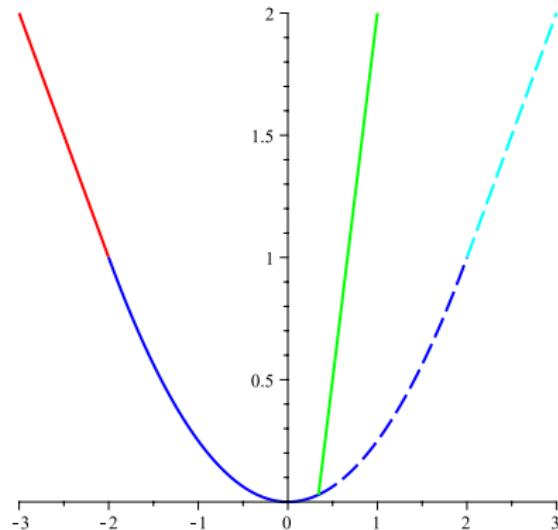
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Riemann Zeta Function

- $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}, \quad \Re s > 1$
+ analytic continuation

► Riemann conjecture: zeroes on the critical line $s = \frac{1}{2} + i\mathbb{R}$.

► Randomized Riemann zeta function:

$$\zeta_{\text{rand}}(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s} e^{i\theta_p}}, \quad \Re s > 1/2$$

The θ_p 's are i.i.d. uniform random variables on $[0, 1]$.

For $\Re s = 1/2$, $\zeta_{\text{rand}}(s)$ is a generalized function.

It is a distribution.

It is a tempered distribution.

It is a tempered generalized function.

It is a tempered generalized function of exponential type.

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Riemann Zeta Function

- $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}, \quad \Re s > 1$
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↳ Riemann hypothesis

$$\zeta_{\text{rand}}(s) = \prod_{p \text{ prime}} \frac{1}{1 - e^{2\pi i \theta_p} p^{-s}}, \quad \Re s > 1/2$$

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For $\Re s = 1/2$, $\zeta_{\text{rand}}(s)$ is a generalized function.

► Stronger mode for $\log(\zeta(1/2 + it + h))$,

+ uniform $\in [T, 2T]$, $T \rightarrow \infty$, $h \in [0, 1]$

$$X_t^h = \sum_{p \leq T} \frac{\Re(e^{2\pi i \theta_p} p^{-it-h})}{p^{1/2}}$$

The X_t^h 's are highly correlated to each other.

↳ Correlation function

↳ Correlation function of the Riemann zeta function

Riemann Zeta Function

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For $\Re s = 1/2$, $\zeta_{\text{rand}}(s)$ is a generalized function.

- Harper's model for $\log |\zeta(1/2 + i(\tau + h))|$,
 τ uniform $\in [T, 2T]$, $T \rightarrow \infty$, $h \in [0, 1]$:

$$X_h^T := \sum_{p \leq T} \frac{\Re(e^{2\pi i \theta_p} p^{-ih})}{p^{1/2}}$$

The X_h^T 's are log-correlated (Harper '13).

- Identification in the Euler product:

$$(p^{-s}, p \text{ prime}) \iff (e^{2\pi i \theta_p}, p \text{ prime})$$

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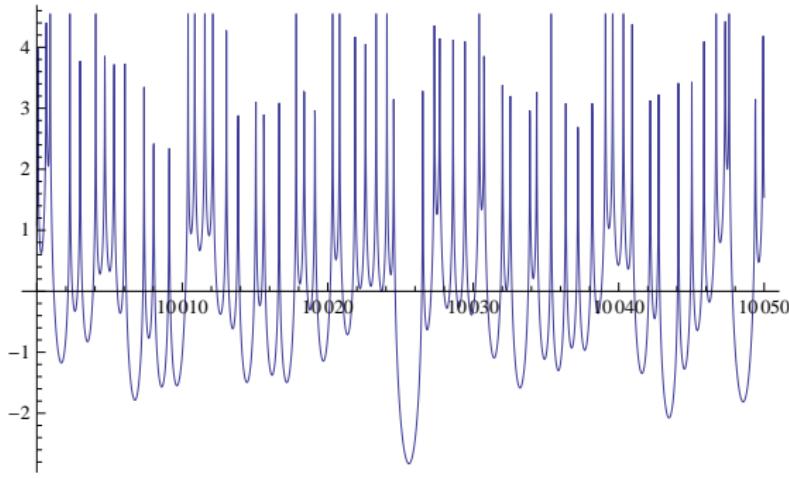
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$-\log |\zeta(1/2 + i(\textcolor{blue}{T} + h))|$ for $\textcolor{red}{T} = 10^4$ and $h \in [0, 50]$
(Courtesy of L.-P. Arguin)

Moments of the (Randomized) Riemann Zeta Function

- ▶ Many recent results for ζ , or its randomized version, **on the critical line**, for the **moments**, **maxima**, **relation to Gaussian multiplicative chaos**. (Fyodorov-Hiary-Keating; Najnudel; Arguin; Belius; Bourgade; Harper; Radziwill; Soundararajan; Saksman-Webb; Kistler; Hartung …)

→ Moments on a finite interval of the critical line, for large T :

$$M_r(\gamma) = \int_0^1 |\zeta(1/2 + i(\gamma + h))|^r dh, \quad \gamma > 0$$

→ Moments of the random version X_h^T of $\log |\zeta|$:

$$M_r(\gamma) = \int_0^1 \exp(rX_h^T) dh$$

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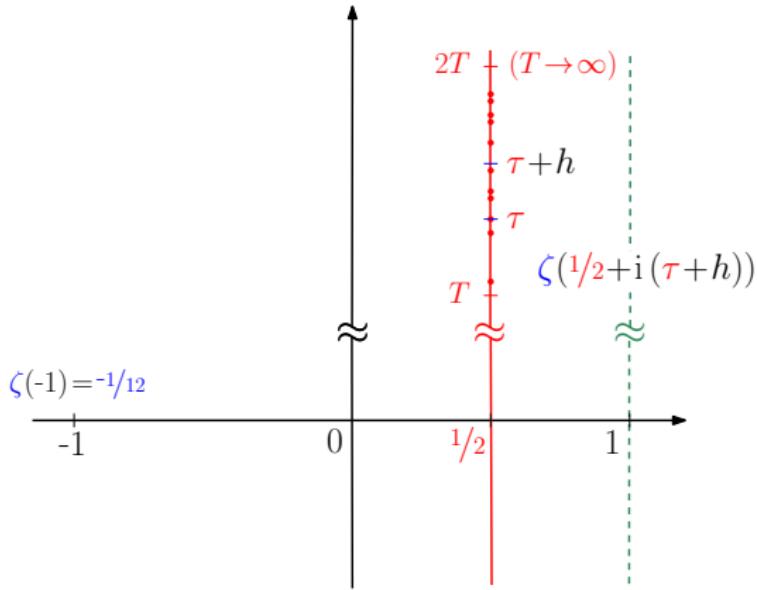
$$M_\tau(\gamma) := \int_0^1 |\zeta(1/2 + i(\tau + h))|^\gamma dh, \quad \gamma > 0$$

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Moments of the (Randomized) Riemann Zeta Function

$$M_\tau(\gamma) := \int_0^1 |\zeta(1/2 + i(\tau + h))|^\gamma dh,$$



$$\text{Random } \mathcal{M}_\tau(\gamma) := \int_0^1 \exp\left(\gamma X_h^T\right) dh$$

Moments and Free Energy

- Quenched free energy of the random version on the critical line

$$\mathcal{F}_T(\gamma) := \mathbb{E} [\log \mathcal{M}_T(\gamma)] = \mathbb{E} \left[\log \int_0^1 \exp \left(\gamma X_h^T \right) dh \right]$$

- Then (Assumption 1.7 and 2.2)

$$\lim_{T \rightarrow \infty} \frac{1}{\log \log T} \mathcal{F}_T(\gamma) = f(\gamma)$$

where $f(\gamma) = \gamma^2/4, \gamma < 2$ and $f(\gamma) = \gamma - 1, \gamma \geq 2$

- In particular, $\mathcal{M}_T(\gamma)$ has the same form as for $\gamma \in [2, 27]$, $T \rightarrow \infty$

$$\mathcal{M}_T(\gamma) = (\log T)^{f(\gamma) + o(1)},$$

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- Then (Arguin & Tai '17)

$$\lim_{T \rightarrow \infty} \frac{1}{\log \log T} \mathcal{F}_T(\gamma) = f(\gamma)$$

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- Arguin, Ouimet, Radziwill '19: a.s. for $\tau \in [T, 2T]$, $T \rightarrow \infty$:

$$M_\tau(\gamma) = (\log T)^{f(\gamma)+o(1)}.$$

Off-Critical Line Moments

- Move off the critical line: $s = \sigma + i\mathbb{R}$, $\sigma > 1/2$

Then define the moments of the Riemann zeta function for $\sigma > 1/2$:

$$ME(\gamma) := \int_0^\infty |C(\sigma + i(\gamma + h))|^2 dh, \quad \gamma > 0$$

Off-Critical Line Moments

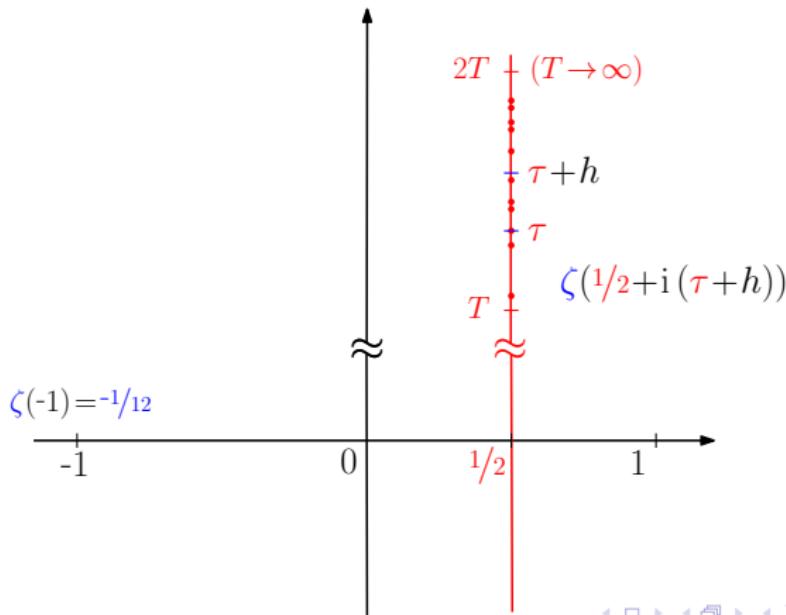
- ▶ Move off the critical line: $s = \sigma + i\mathbb{R}$, $\sigma > 1/2$
- ▶ Then define moments on a finite interval off axis, for $\tau \rightarrow \infty$:

$$M_{\tau}^{\sigma}(\gamma) := \int_0^1 |\zeta(\sigma + i(\tau + h))|^{\gamma} dh, \quad \gamma > 0$$

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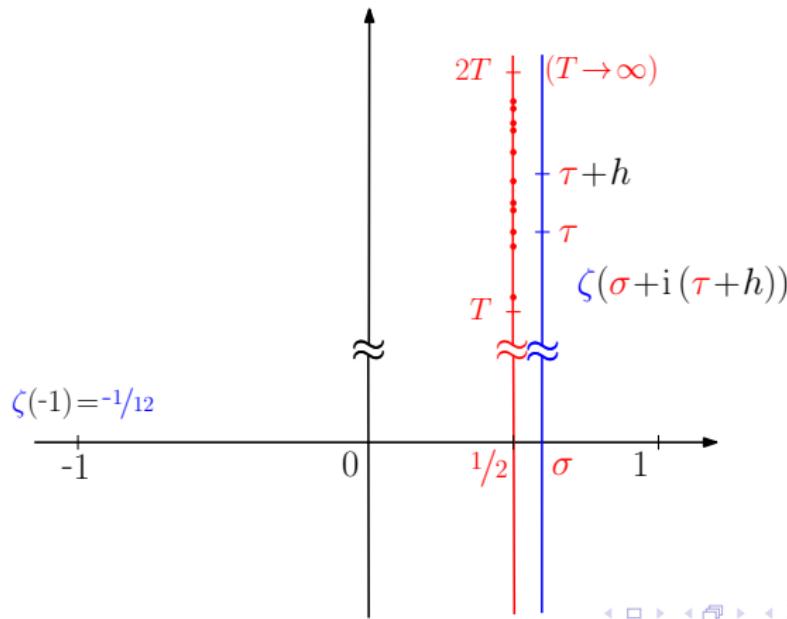
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Off-Critical Line Moments

- Move off the critical line: $s = \sigma + i\mathbb{R}$, $\sigma > 1/2$

Motivation: Off the critical line, $\zeta(s)$ is analytic

$$MC(\gamma) := \int_0^\infty (\zeta(\sigma + i(\gamma + h)))^2 dh, \quad \gamma > 0$$

- Approximate $\log|\zeta_{\text{odd}}(\sigma + ih)|$ by the first term in its Euler product expansion

$$\chi_p = \sum_{n \geq 1} \frac{\Re(e^{2\pi i n h} p^{-nh})}{p^n}$$

θ_p 's I.I.D. uniform random variables on $[0, 1]$

Then $\log|\zeta_{\text{odd}}(\sigma + ih)| \approx \sum_p \chi_p \theta_p$

Then $MC(\gamma) \approx \int_0^\infty \left(\sum_p \chi_p \theta_p \right)^2 dh$

Then $MC(\gamma) \approx \sum_p \chi_p^2 \int_0^\infty \theta_p^2 dh$

Then $MC(\gamma) \approx \sum_p \chi_p^2 \cdot 1/3$

Off-Critical Line Moments

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- Moments on a finite interval off axis, for $\tau \rightarrow \infty$:

$$M_\tau^\sigma(\gamma) := \int_0^1 |\zeta(\sigma + i(\tau + h))|^\gamma dh, \quad \gamma > 0$$

- Approximate $\log |\zeta(\sigma + ih)|$ by the first term in its Euler product expansion

$$\chi_i = \sum_{p \leq i} \frac{\Re(e^{2\pi i \theta_p} p^{-it})}{p^s}$$

θ_p 's I.I.D. uniform random variables on $[0, 1]$

- Moments of the random version X_i^γ of $\log |\zeta|$

$$\int_0^1 |\zeta(\sigma + ih)|^\gamma dh \sim M^\sigma(\gamma) = \int_0^1 \exp(\gamma X_i) dh$$

Off-Critical Line Moments

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$$X_h^\sigma := \sum_{p \leq \infty} \frac{\Re(e^{2\pi i \theta_p} p^{-ih})}{p^\sigma},$$

θ_p 's I.I.D. uniform random variables on $[0, 1]$

- Moments of the random version X_τ^σ of $\log |\zeta|$

$$\int_0^1 |\zeta_{\text{rand}}(\sigma + ih)|^\gamma dh \sim M(\gamma) = \int_0^1 \exp(\langle X_h^\sigma \rangle) dh$$

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$$\int_0^1 |\zeta_{\text{rand}}(\sigma + ih)|^\gamma dh \sim \mathcal{M}^\sigma(\gamma) := \int_0^1 \exp(\gamma X_h^\sigma) dh$$

Off-Critical Moments and Free Energy

- For the randomized version **off the critical line**, define the 'quenched free energy':

$$\mathcal{F}^\sigma(\gamma) := \mathbb{E} [\log \mathcal{M}^\sigma(\gamma)] = \mathbb{E} \left[\log \int_0^1 \exp(\gamma X_h^\sigma) dh \right]$$

- Theorem (D. Belius, J. Bourgade, Y. Gu):

$$\lim_{\sigma \rightarrow 1/2} \frac{1}{\log(\sigma - \frac{1}{2})} \mathcal{F}^\sigma(\gamma) = f(\gamma)$$

where $f(\gamma) = \gamma^2/4, \gamma \leq 2$ and $f(\gamma) = \gamma - 1, \gamma \geq 2$

- Thus $f(\gamma)$ is the (Kraetzer) integral means spectrum of the primitive $\Phi(z) = \int_z^\infty \zeta_{\text{rand}}(s) ds$.

• Theorem (D. Belius, J. Bourgade, Y. Gu):
The function $f(\gamma)$ is the spectral measure of the random Schrödinger operator $H_\gamma = -\frac{d^2}{dx^2} + V(x)$ with potential $V(x) = \sum_{n=1}^\infty \frac{\gamma_n}{x+n}$ where γ_n are iid standard normal variables.

Off-Critical Moments and Free Energy

- For the randomized version **off the critical line**, define the 'quenched free energy':

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- Theorem (D., Saksman, Zinsmeister '21⁺)

$$\lim_{\sigma \rightarrow 1/2} \frac{1}{\log(\sigma - \frac{1}{2})^{-1}} \mathcal{F}^\sigma(\gamma) = f(\gamma)$$

where $f(\gamma) = \gamma^2/4, \gamma \leq 2$ and $f(\gamma) = \gamma - 1, \gamma \geq 2$

for some function $\mathcal{G}_{\text{crit}}$ and the function \mathcal{G}_{off} of the primitive $\Phi(x) = \int_0^x \mathcal{G}_{\text{off}}(s) ds$.

- Proofs of the above: via the prime number theorem and second moment estimates, or via the Saksman-Webb relation to Gaussian Multiplicative Chaos on the critical line.

Off-Critical Moments and Free Energy

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For the proof, one can either use the number theory and second moment estimates, or via the Saksman-Webb relation to Gaussian Multiplicative Chaos on the critical line.

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Univalence

- ▶ Kraetzer's conjecture concerns *univalent (injective) bounded functions.*

• ζ_{rand}

• *Conjecture (Kraetzer 1997)*: Let Φ stand for the primitive $\Phi(z) = \int_0^z \zeta_{\text{rand}}(t) dt$. Let $h > 0$ and $Q := (1/2, 1/2 + h) \times (0, h)$ be a small open square with its left side on the critical line. Then, almost surely, Φ is not injective on Q .

• ζ_{rand}

- ▶ According to the Koebe-Nehari necessary condition for univalence, it is enough to prove that

$$\sup_{\sigma > 1/2} (\sigma - 1/2) |\Phi''(\sigma + ih)/\Phi'(\sigma + ih)| = \infty \quad \text{almost surely.}$$

for $\Phi''/\Phi' = \zeta_{\text{rand}}/\zeta_{\text{rand}}$.

Univalence

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Theorem

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HAPPY BIRTHDAY, HUBERT!