Yang-Baxter Integrable Lindblad equations

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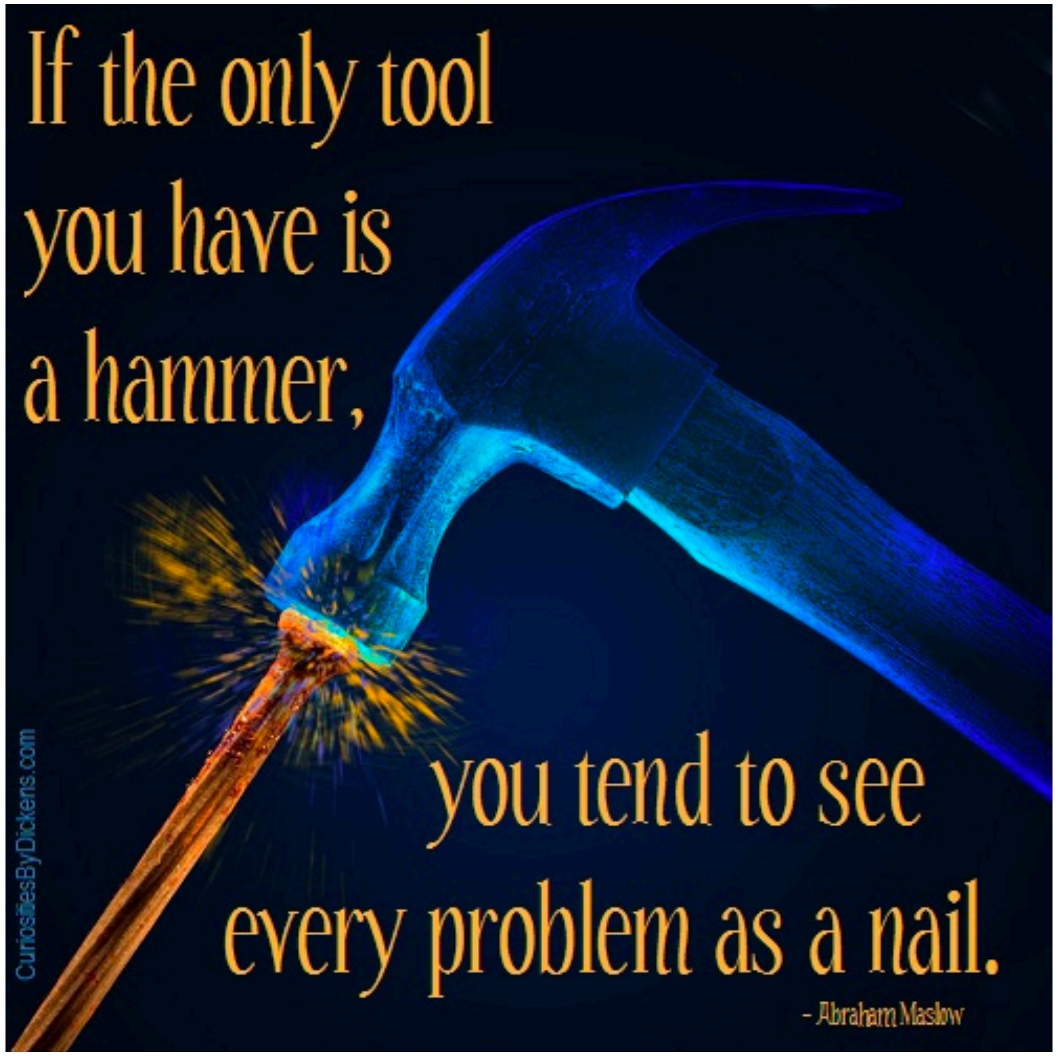
Integrability is a very useful tool:

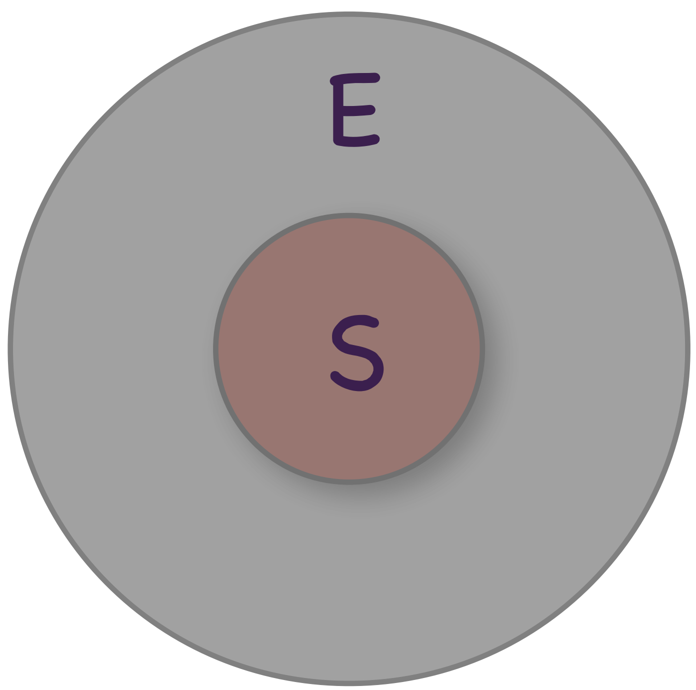
- quantum condensed matter
- quantum impurity problems
- classical stat. mech.
- stochastic processes
- random tilings
- string theory
- ultra-cold cold atoms
- combinatorics
- ...



A (fairly) new kind of nail: **Open Quantum Systems.**

Lindblad equation for open quantum systems

Important as even cold atom systems are not perfectly isolated



Hamiltonian $$H = H_S + H_E + H_{\text{int}}$$

density matrix $$\rho(t) = e^{-iHt}\rho(0)e^{iHt}$$

reduced DM $$\rho_S(t) = \mathrm{Tr}_E\left[\rho(t)\right]$$

initial state $$\boxed{\rho(0) = \rho_S(0) \otimes \rho_E(0)}$$

Goal: determine e.g. Tr[$\rho_S(t)$ $\mathcal{O}_S(x)$]

$\mathcal{O}_S(x)$ = local operator acting on system

Assumptions: separation of time scales

$$\tau_S \gg \delta t \gg \tau_E$$

(integrate out bath avoiding retardation)

Lindblad equation for the reduced density matrix

$$\frac{\partial \rho_S}{\partial t} = -i[H_S, \rho_S] + \underbrace{\sum_{k=1}^{M} L_k \rho_S(t) L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho_S(t)\}}_{D[\rho_S]}$$

L_k: jump operators, describe coupling to environment

Are there integrable Lindblad equations?

"Superoperator formalism"

$$\rho = \sum_{n,m=1}^{\dim\mathcal{H}} \langle n|\rho|m\rangle \; |m\rangle\langle n|$$

space of operators is a linear vector space itself

"vectorization"

$$|\boldsymbol{\rho}\rangle = \sum_{n,m} \langle n|\rho|m\rangle \; |m\rangle|n\rangle\!\rangle$$

Operators acting from the left/right:

$$\mathcal{O}\rho \to \mathcal{O}|\boldsymbol{\rho}\rangle \equiv \sum_{n,m} \langle n|\rho|m\rangle \; \big(\mathcal{O}|m\rangle\big)|n\rangle\!\rangle$$

$$\rho\mathcal{O} \to \widetilde{\mathcal{O}}|\boldsymbol{\rho}\rangle \equiv \sum_{n,m} \langle n|\rho|m\rangle \; |m\rangle\big(\widetilde{\mathcal{O}}|n\rangle\!\rangle\big)$$

$$\langle n'|\widetilde{\mathcal{O}}|n\rangle = \langle n|\mathcal{O}|n'\rangle$$

for bosonic ops

Lindblad eqn becomes

$$\frac{\partial}{\partial t}|\rho\rangle = \mathscr{L}\,|\rho\rangle$$

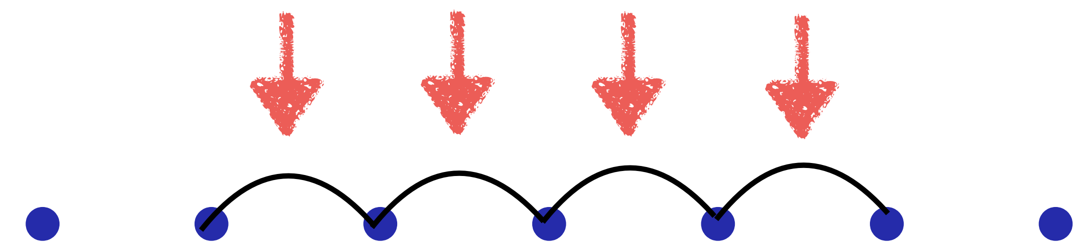
$$\mathscr{L} = -iH + i\widetilde{H} + \sum_a \gamma_a \left[L_a \widetilde{L}_a^\dagger - \frac{1}{2}\left(L_a^\dagger L_a + \widetilde{L}_a \widetilde{L}_a^\dagger \right) \right]$$

Search for Lindblad equations for which $\mathscr{L}$ is the (non-hermitian) Hamiltonian of a quantum integrable model.

Medvedyeva, Essler & Prosen '16
Rowlands & Lamacraft '18, Shibata & Katsura '19
Essler& Ziolkowska '20, Essler& Piroli '21, Robertson & Essler '21
Buca et al '20, Nakagawa, Kawakami & Ueda `20
de Leeuw, Paletta & Pozsgay '21 ...

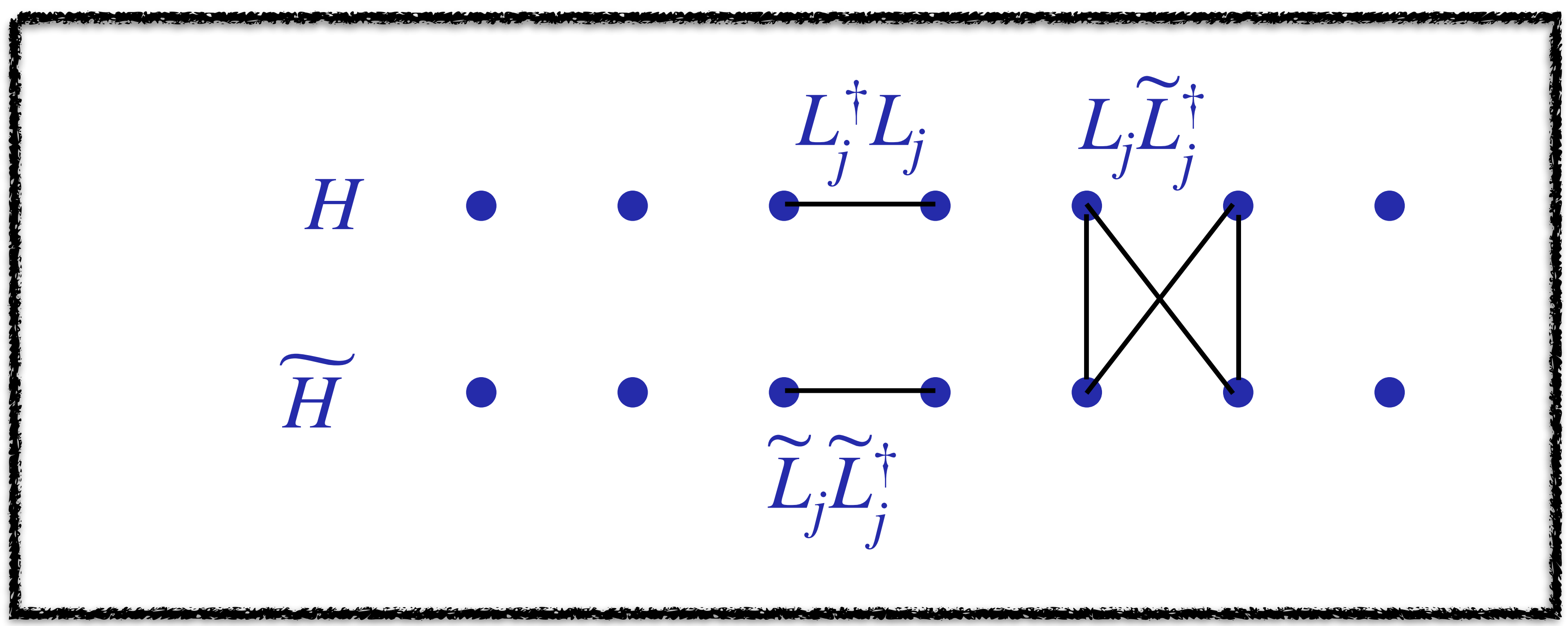
Setup considered here

jump ops act on all n.n. bonds



Lindbladian: $$\mathscr{L} = -iH + i\widetilde{H} + \gamma \sum_j \left[L_j \widetilde{L}_j^\dagger - \frac{1}{2}\left(L_j^\dagger L_j + \widetilde{L}_j \widetilde{L}_j^\dagger\right)\right]$$

structure of a 2-leg ladder



Lindblad structure is **very restrictive** and most integrable ladders cannot be accommodated.

Hubbard-like models

Integrability structure of Hubbard is unusual

Shastry '88

"Glueing together" two XX models

$$R_{[12],[34]}(\lambda,\mu) = r_{13}(\lambda-\mu)\ r_{24}(\lambda-\mu) + h(\lambda,\mu)\ r_{13}(\lambda+\mu)\ C_1\ r_{24}(\lambda+\mu)\ C_2$$

YBE: $$r_{12}(\lambda_{12})r_{13}(\lambda_{13})r_{23}(\lambda_{23}) = r_{23}(\lambda_{23})r_{13}(\lambda_{13})r_{12}(\lambda_{12})$$

"Decorated YBE":

$$r_{12}(\lambda_1+\lambda_2)C_1 r_{13}(\lambda_1-\lambda_3)r_{23}(\lambda_2+\lambda_3) = r_{23}(\lambda_2+\lambda_3)r_{13}(\lambda_1-\lambda_3)C_1 r_{12}(\lambda_1+\lambda_2)$$

Conjugation matrix: $C^2 = \mathbf{1}$

This structure (essentially) ensures a Lindblad interpretation.

Tight-binding model with dephasing noise

Medvedyeva et al '16

$$H = \sum_j a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j \qquad L_j = \sqrt{2u}\, a_j^\dagger a_j$$

or, via Jordan-Wigner

$$H = \sum_j \sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ \qquad L_j = \sqrt{\frac{u}{2}}\, \sigma_j^z$$

$$\mathscr{L} = -i \sum_{j,\sigma} c_{j,\sigma}^\dagger c_{j+1,\sigma} + c_{j+1,\sigma}^\dagger c_{j,\sigma} + 4u \sum_j \left(n_{j,\uparrow} - \frac{1}{2}\right)\left(n_{j,\downarrow} - \frac{1}{2}\right)$$

imaginary t Hubbard model

Maassarani models and GL(N,M) generalisations

Maassarani '98

Drummond et al '07

Maassarani: N bosonic states per site

$$H = \sum_j P^{(3)}_{j,j+1}\,, \qquad L_j = C_j$$

$$P^{(3)}_{j,j+1} = \sum_{a\in A}\sum_{b\in B} e^{i\varphi_{ab}} E_j^{ba} E_{j+1}^{ab} + e^{-i\varphi_{ab}} E_j^{ab} E_{j+1}^{ba} \qquad C_j = \sum_{b\in B} E_j^{bb} - \sum_{a\in A} E_j^{aa}$$

$$\mathscr{L} = -i \sum_j P^{(3)}_{j,j+1} - \widetilde{P}^{(3)}_{j,j+1} + U \sum_j C_j \widetilde{C}_j,$$

Integrable Liouvillian

Example N=3:

$$H = -\mathscr{P} \sum_{j,\sigma} \left[c^\dagger_{j+1,\sigma} c_{j,\sigma} + \mathrm{h.c.} \right] \mathscr{P}, \qquad \mathscr{P} = \prod_{j=1}^{L} (1 - n_{j,\uparrow} n_{j,\downarrow})$$

$$L_j = 2 - 2(1 - n_{j,\uparrow})(1 - n_{j,\downarrow})$$

Infinite-U Hubbard model with dephasing noise

Quantum ASEP

Jin, Krajenbrink & Bernard '20
Bauer, Bernard & Jin '19, '20

spin-1/2 chain coupled to quantum noise

$$H(t) = \sum_{j=1}^{L} \kappa_j(t)\sigma_j^+\sigma_{j+1}^- + \bar{\kappa}_j(t)\sigma_j^-\sigma_{j+1}^+$$

$$\mathrm{Tr}_E\left[\rho_E \kappa_j(t)\bar{\kappa}_k(t')\right] = J_1 \delta_{j,k}\delta(t-t') \qquad \mathrm{Tr}_E\left[\rho_E \bar{\kappa}_j(t)\kappa_k(t')\right] = J_2 \delta_{j,k}\delta(t-t')$$

Average over quantum noise → **Lindblad equation**

(i) $L_j^{(1)} = \left(L_j^{(2)}\right)^\dagger = \sigma_j^+\sigma_{j+1}^-$ 2 jump operators/link

(ii) no Hamiltonian part in the LE

Superoperator formalism:

$$|\uparrow\rangle_j\ {}_j\langle\uparrow| \Rightarrow |1\rangle_j$$
$$|\downarrow\rangle_j\ {}_j\langle\uparrow| \Rightarrow |2\rangle_j$$
$$|\uparrow\rangle_j\ {}_j\langle\downarrow| \Rightarrow |3\rangle_j$$
$$|\downarrow\rangle_j\ {}_j\langle\downarrow| \Rightarrow |4\rangle_j$$

Basis of superoperators: $E_j^{ab} \equiv |a\rangle_j\ {}_j\langle b|\ , \quad a,b \in \{1,2,3,4\}$

Lindblad equation:

$$\frac{d|\rho_S(t)\rangle}{dt} = \mathscr{L}|\rho_S(t)\rangle$$

$$\mathscr{L} = \sum_j J_1 E_j^{14} E_{j+1}^{41} + J_2 E_j^{41} E_{j+1}^{14} - J_1 E_j^{44} E_{j+1}^{11} - J_2 E_j^{11} E_{j+1}^{44}$$
$$-\frac{1}{2}\sum_j (E_j^{22} + E_j^{33})(J_1 E_{j+1}^{11} + J_2 E_{j+1}^{44}) + (E_{j+1}^{22} + E_{j+1}^{33})(J_2 E_j^{11} + J_1 E_j^{44})$$
$$+\frac{J_1 + J_2}{4}\sum_j (E_j^{22} E_{j+1}^{33} + E_j^{33} E_{j+1}^{22})\,.$$

Operator space fragmentation

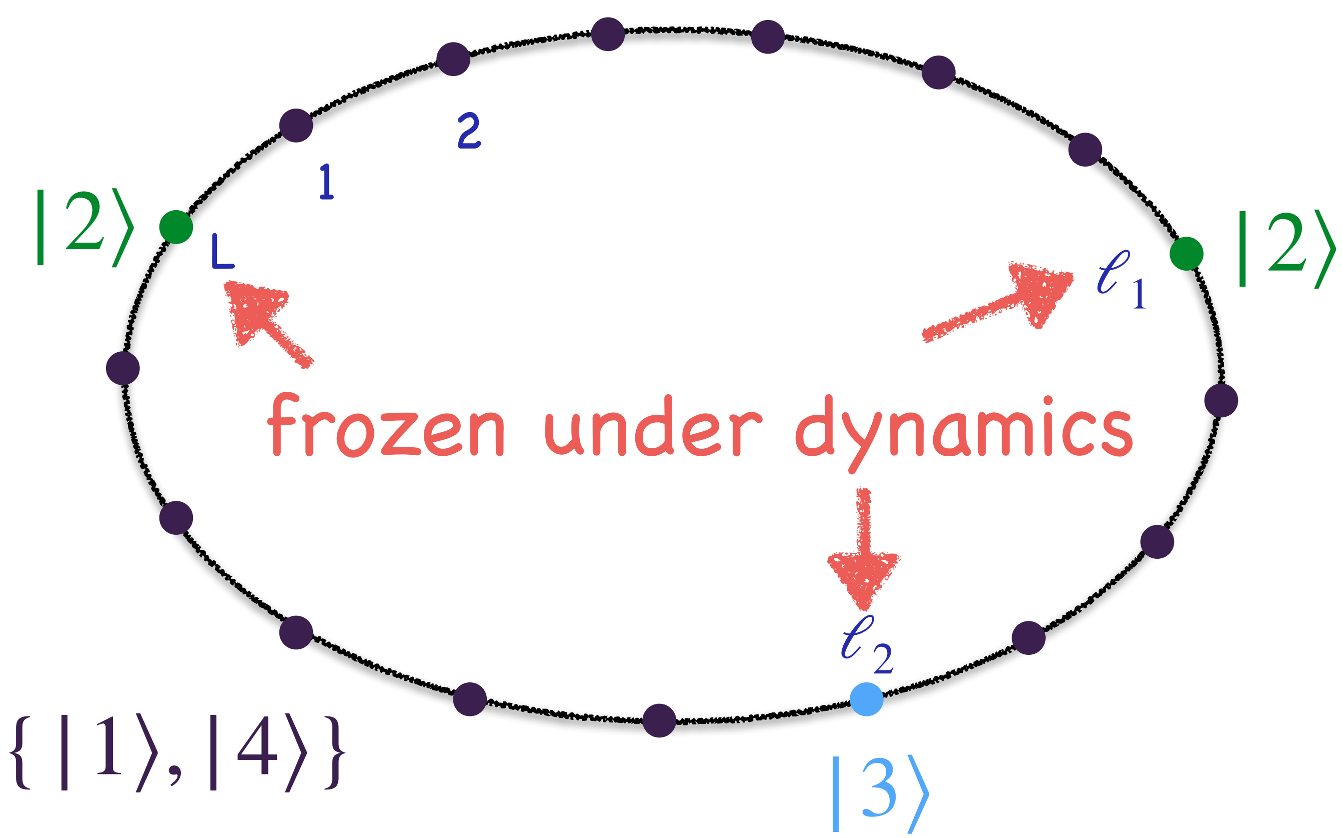
Essler&Piroli '20

$\mathscr{L}$ has an extensive number of strictly local conservation laws

$$[\mathscr{L}, E_j^{22}] = 0 = [\mathscr{L}, E_j^{33}]$$

j=1,..,L

→ $\mathscr{L}$ is **block-diagonal**



$|2\rangle, |3\rangle$= "defects"

$$\mathscr{L} \to \mathscr{L}_{[1,\ell_1-1]} + \mathscr{L}_{[\ell_1+1,\ell_2-1]} + \mathscr{L}_{[\ell_2+1,L-1]}$$

Each block of the Lindbladian is integrable!

$$S^{-1}\mathscr{L}_{[m,n]}S = -\sqrt{\frac{J_1 J_2}{2}}\left(2\Delta + \sum_{j=m}^{n-1}\left[\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta\left(\sigma_j^z\sigma_{j+1}^z - 1\right)\right]\right)$$

$$2\Delta = \sqrt{\frac{J_1}{J_2}} + \sqrt{\frac{J_2}{J_1}}$$

Defect-free sector: Asymmetric exclusion process (ASEP)

$$\mathscr{L}_{\mathrm{ASEP}} = \sum_{j=1}^{L}\left[J_1\sigma_j^+\sigma_{j+1}^- + J_2\sigma_j^-\sigma_{j+1}^+ + \frac{J_1+J_2}{4}\left(\sigma_j^z\sigma_{j+1}^z - 1\right)\right].$$

Spitzer '70
Gwa&Spohn '92
Derrida, Pasquier,...

$\rightarrow$ can use integrability methods to determine spectrum of $\mathscr{L}$

What about correlation functions?

$$\mathrm{Tr}\left[\rho(t)\sigma_1^+\sigma_\ell^-\right] = \langle \boldsymbol{\phi} \,|\, E_1^{22} e^{\mathscr{L}_{[2,\ell-1]}t} E_\ell^{33} e^{\mathscr{L}_{[\ell+1,L]}t} \,|\, \rho(0)\rangle$$

where $$\langle \boldsymbol{\phi} \,| = \otimes_{j=1}^{L} \left[{}_j\langle 1 \,| + {}_j\langle 4 \,| \right]$$

Not known how to calculate this for ASEP/XXZ.

Look at simpler problem:

$$L_j^{(1)} = \left(L_j^{(2)}\right)^\dagger = \sigma_j^+ \sigma_{j+1}^- \qquad L_j^{(3)} = \left(L_j^{(4)}\right)^\dagger = \sigma_j^+ \sigma_{j+1}^+$$

Here the corresponding Lindbladian has a **free fermion point** **&** exhibits operator-space fragmentation

Here we can calculate

$$\mathrm{Tr}\left[\rho(t)\sigma_1^+\sigma_\ell^-\right] = \langle \boldsymbol{\phi} \,|\, E_1^{22} e^{\mathscr{L}_{[2,\ell-1]}t} E_\ell^{33} e^{\mathscr{L}_{[\ell+1,L]}t} \,|\, \rho(0)\rangle$$

for **simple** $|\rho(0)\rangle$ by non-standard free-fermion methods.

Summary

1. Certain quantum master equations can be related to Yang-Baxter integrable models in interesting ways.
2. Spectral properties can be analysed using integrability.
3. Calculation of observables requires new methods.

Happy Birthday Hubert!