Web models as generalisations of statistical loop models

Joint with Augustin LAFAY and Jesper JACOBSEN

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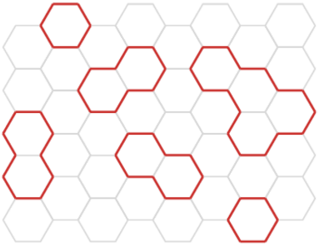
The art of Mathematical Physics
Hubert's 60th Birthday

IPHT Saclay, 23 September 2021

Loop models

- Self-avoiding simple curves on 2d lattice (nodes and links)
- Place bonds on some links so as to form set of loops
- Weight x per bond and N per loop
- For $|N| \leq 2$, dense and dilute critical points x_c
- Continuum limit of compactified free bosonic field (Coulomb gas) [Nienhuis, Di Francesco-Saleur-Zuber, Duplantier, Cardy]



Loop models and $U_q sl(2)$

We all love them!

- Applications (polymers, percolation, Ising spin clusters, ...)
- Links to invariant theory and knot theory (Jones polynomials)
- Connection to non-rational CFTs (Koo–Saleur formulas, 3pt and 4pt functions calculations)

Quantum group symmetry

- Local formulation of loop models = Theory of $U_q sl(2)$ invariants
- Local bits of the loop model are $U_q sl(2)$-invariant operators!
- Temperley–Lieb diagrams and (dilute) Temperley–Lieb algebra

Theory of diagrammatical $U_q sl(n)$ invariants

Math

- Well-known for mathematicians starting from the pioneering work of Kuperberg in 1992-1996
- He's developed a diagrammatic approach to classification of invariants of $U_q sl(3)$ – spider diagrams – they classify invariants in $\square \otimes \square \otimes \overline{\square} \otimes \ldots$
- Then many people have contributed to generalisations for higher ranks:
  - H. Murakami, T. Ohtsuki and S. Yamada in 1998 (MOY invariants)
  - M.-J. Jeong and D. Kim in 2005
  - H. Wu in 2009
  - S. Cautis, J. Kamnitzer and S. Morrison in 2012

Physics

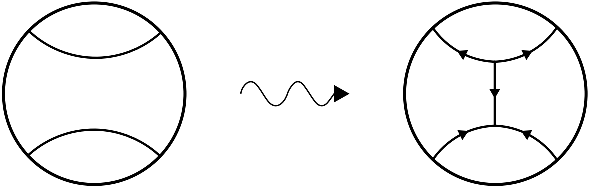
- The math works are all motivated by low-dimensional topology (knot theory, mfd invariants, etc)
- Theory of $U_q sl(n)$-type diagrams is largely unknown to physicists

From Loop to Web models

Quantum group symmetry

Elementary/local bits of the model are $U_q sl(n)$-invariant operators!

Example of $U_q sl(3)$-type generalisation:



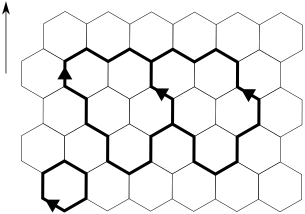
Web models

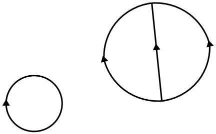
Why are we interested in them?

- Allow for branchings and bifurcations (with weights)
- Domain walls in spin systems
- CFT in the continuum limit with W-algebra chiral symmetry
- Koo-Saleur formula in higher rank?

$U_q sl(3)$ Web model: Lattice configurations

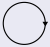
- Hexagonal lattice $\mathbb{H}$ with nodes and links
- Configuration c by drawing bonds on some links, with constraints:
  - Nodes have valence 0, 2 or 3: closed web with 3-valent vertices
  - Each bond is oriented. Orientations conserved at 2-valent nodes
  - Vertices are sources or sinks (all bonds point in or out)
- Each configuration as an abstract graph (keep only vertices and edges) is closed, planar, trivalent, bipartite.

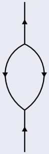


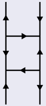
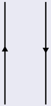
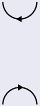


$U_q sl(3)$ Web model: Non-local weight of a configuration

Reduction rules: spider relations [Kuperberg'96]

 $= [3]_q = q^2 + 1 + q^{-2}$

 $= [2]_q$

 $=$  $+$ 

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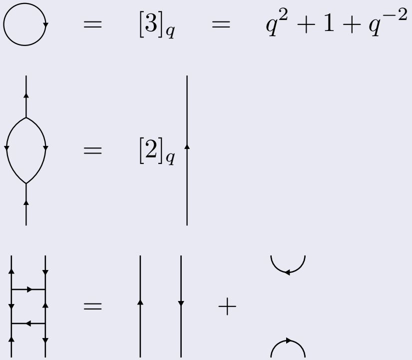


- Rotated and arrow-reversed diagrams
- A web component always has ≥ 1 polygon(s) of degree 0, 2 or 4
- The three rules evaluate any web configuration c to its weight $w_{\mathrm{K}}(c)$

$U_q sl(3)$ Web model: Partition function

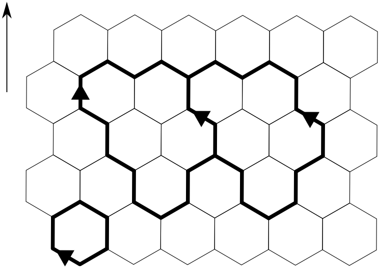
Statistical model

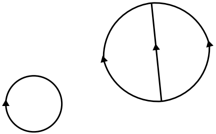
- Local fugacities for bonds and vertices:
  - x_1 (up bond) x_2 (down bond)
  - y (sink vertex) z (source vertex)
- Sum over all configurations $c \in \mathrm{K}$ on $\mathbb{H}$
- Partition function:

$$Z_{\mathrm{K}} = \sum_{c \in K} x_1^{N_1} x_2^{N_2} (yz)^{N_V} w_{\mathrm{K}}(c)$$

with N_1 up-bonds, N_2 down-bonds, and N_V vertex pairs

$U_q sl(3)$ Web model: Partition function





Example

- Non-local weight $w_{\mathrm{K}}(c) = [2]_q [3]_q^2$
- Total weight of the configuration on $\mathbb{H}$ is $x_1^{22} x_2^{13} yz [2]_q [3]_q^2$

$U_q sl(3)$ Web model: Relation to spin models

$\mathbb{Z}_3$-spin model

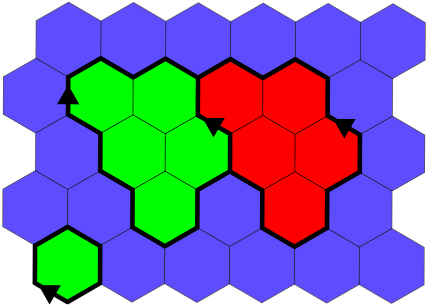
- Spins $\sigma_i \in \mathbb{Z}_3 = \{0, 1, 2\}$ on nodes i of a triangular lattice $\mathbb{T} = \mathbb{H}^*$
- Interaction along link (ij) of $\mathbb{T}$ defined as $x_{\sigma_j - \sigma_i}$ with j to the right of i

Mapping to the web model

- Set $x_0 = 1$, so this spin model associates a non-trivial weight, x_1 or x_2, to each piece of domain wall between unequal spins, i.e. a link of $\mathbb{H}$ that is dual to link (ij) of $\mathbb{T}$ satisfying $\sigma_i \neq \sigma_j$
- Orient accordingly: the link on $\mathbb{H}$ is up (down) if $\sigma_j - \sigma_i = 1$ $(= 2)$
- Vertices appear at a junction of three spin clusters, and they are either sources or sinks according to how the spins around them are arranged.

Bijection between web configurations and spin configurations modulo the global $\mathbb{Z}_3$ action

$U_q sl(3)$ Web model: Relation to spin models



$U_q sl(3)$ Web model: Relation to spin models

Partition function

$$Z_{\text{spin}} = 3 \sum_{c \in \mathrm{K}} x_1^{N_1} x_2^{N_2}$$

Equivalence condition

- Neither vertex weights nor non-local weights in the spin model
- Equivalent to the web model ($Z_{\text{spin}} = 3 Z_{\mathrm{K}}$) if the following relation holds for all configurations:

$$(yz)^{N_V} w_{\mathrm{K}}(c) = 1$$

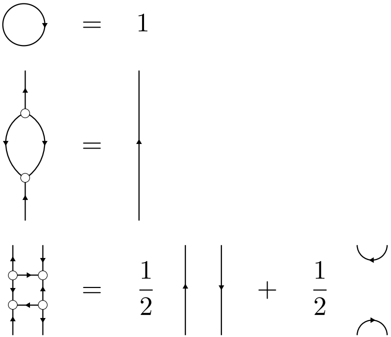
$U_q sl(3)$ Web model: Relation to spin models

Equivalence at a special point

$$q = e^{i\frac{\pi}{4}}$$

$$yz = 2^{-\frac{1}{2}}$$

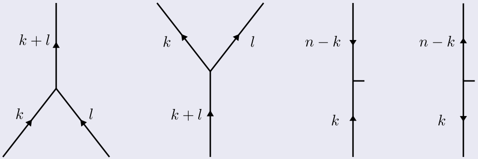
Idea of proof: Absorb y and z into the vertices and use $[3]_q = 1$ and $[2]_q = \sqrt{2}$



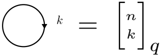
$U_q sl(n)$ Web models

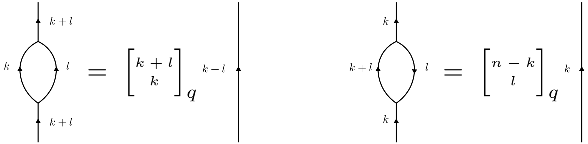
Use spiders of Cautis-Kamnitzer-Morrison'12

- Webs are still closed, oriented, planar, trivalent graphs
  But not always bipartite as before
- Edges carry an integer flow $i \in [\![1, n-1]\!]$.
  - Flow labels fundamental representations of $U_q sl(n)$
  - Orientation distinguishes between two duals
- Generators conserve flow, or change by n due to tags:

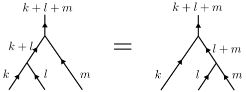


Spider relations:

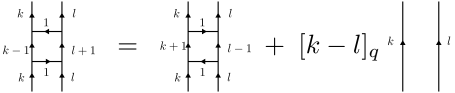




and an associativity rule:



and a square rule:



and the tag rules...

Short summary of results

- Case $n = 3$ gives back Web model based on Kuperberg's spiders
- Case $n = 2$ gives the well-known Nienhuis loop model
- Special point $q = e^{i\pi/(n+1)}$ equivalent to $\mathbb{Z}_n$ spin model

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$U_q sl(n)$ Web models: Criticality?

- $\mathbb{Z}_n$ spin models known to be critical
  $\rightarrow U_q sl(n)$ Web models have the special critical point for any $n \geq 2$
- Web models likely have larger critical manifold varying q and x, y, z

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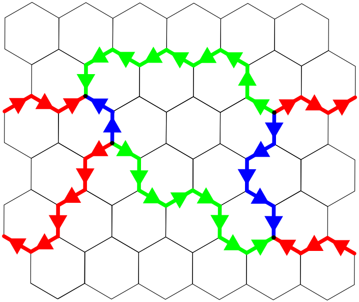
To investigate criticality we need a local formulation

- Analogous to vertex models and O(N) models
- The locality enables us to define a transfer matrix
- Good for numerical study and critical exponents calculation

$U_q sl(3)$ Web models: Local reformulation

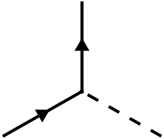
Basic idea

- Decorate bonds by extra degrees of freedom ($n = 3$ colours RGB)
- They allow to redistribute the web weight locally
- Summing over colours gives back the undecorated model
- Each link can now be in 7 different states of $\mathscr{H} = \mathbb{C} \oplus \mathbb{C}^3 \oplus \overline{\mathbb{C}}^3$

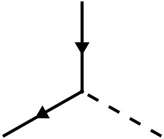


Reminder for loop models or $n = 2$ case

- Write the loop fugacity as $q + q^{-1} = [2]_q$
- Orient each loop in two ways (clockwise, anticlockwise)
- a piece carries weight $q^{-\frac{\vartheta}{2\pi}}$ when it bends an angle ϑ



$= xq^{-\frac{1}{6}}$



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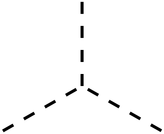

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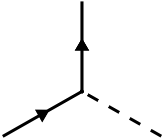

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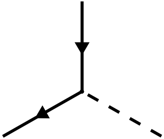

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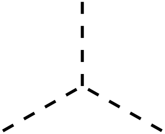


$= 1$

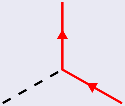
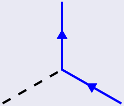
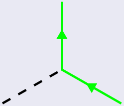
- Better to think of these two “orientation” as colourings
- The analogue for $n = 3$ is the three colours RGB
- The orientations distinguish (for $n = 3$) fundamental $\mathbb{C}^3$ and its dual $\overline{\mathbb{C}}^3$ but for $n = 2$ the two coincide!

$U_q sl(3)$ Web models: Local reformulation

Basic idea for $n = 3$

3 colours RBG and each coloured web c is assigned a weight $w_{\text{col}}(c)$:

- Loop weight $[3]_q = q^2 + 1 + q^{-2}$ for sum over clockwise loop


$= xq^{\frac{1}{3}}$,

$= x$,

$= xq^{-\frac{1}{3}}$

  Here, $x_1 = x_2 = x$ for convenience

- Opposite phases for anticlockwise loops

$U_q sl(3)$ Web models: Local reformulation

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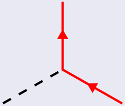
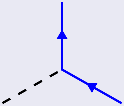
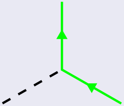
 $= xq^{\frac{1}{3}}$,  $= x$,  $= xq^{-\frac{1}{3}}$

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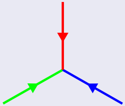
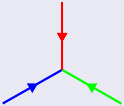
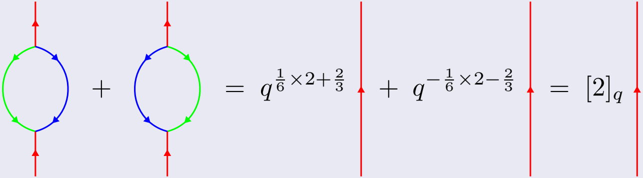
- Opposite phases for anticlockwise loops
- and more tricky for vertices (sinks/sources):

 $= zx^{\frac{3}{2}}q^{-\frac{1}{6}}$  $= zx^{\frac{3}{2}}q^{\frac{1}{6}}$

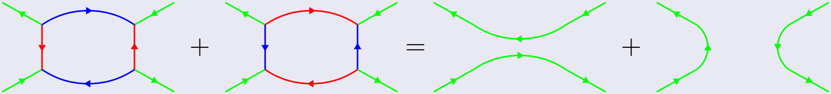
- Plus opposite phases for opposite directions

Idea of proof

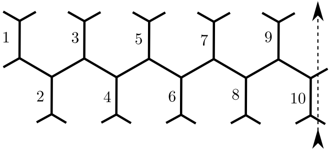
Digon rule



Square rule



Double row transfer matrix T

$T \quad =$ 

- Define in terms of elementary blocks – the local transfer matrices $t_{(1)} =$  and $t_{(2)} =$ 

$$t_{(1)} : \quad \mathscr{H} \otimes \mathscr{H} \to \mathscr{H}$$
$$t_{(2)} : \quad \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$$

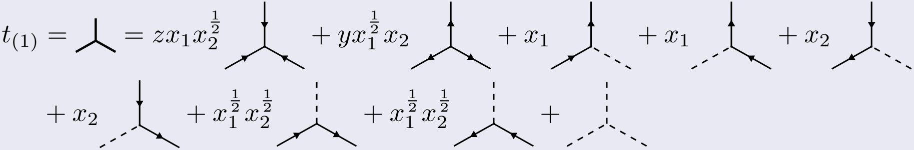
  with matrix elements = local weights

- $t = t_{(2)} t_{(1)} : \quad \mathscr{H} \otimes \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$
-

$$T = \left(\prod_{k=0}^{L-1} t_{2k+1} \right) \left(\prod_{k=1}^{L-1} t_{2k} \right) \qquad \text{so that} \quad Z_{\mathrm{K}} = \langle \, T^M \, \rangle$$

Double row transfer matrix

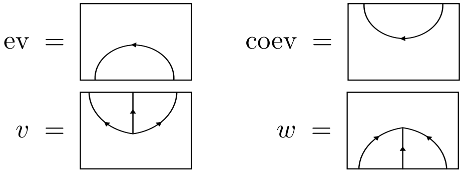
Transfer matrix T is $U_q sl(3)$ intertwiner!



and similarly for $t_{(2)}$

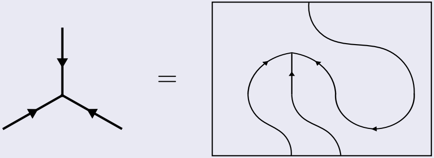
Double row transfer matrix

All terms are compositions/juxtapositions of elementary $U_q sl(3)$-intertwiners:



plus those with reversed arrows.

Example



So ...

Summary for the strip/cylinder geometry

- Elementary blocks of T and T itself are $U_q sl(3)$-intertwiners and it is very simple to compute them
- We have also construction of T_{cyl} in the periodic case or cylinder geometry
- In the periodic case, again, elementary blocks of T_{cyl} are $U_q sl(3)$-intertwiners (however T is not)
- We can now diagonalise T_{cyl} numerically and study phase diagrams!

Phase diagram of Web model

Effective central charge c_{eff}

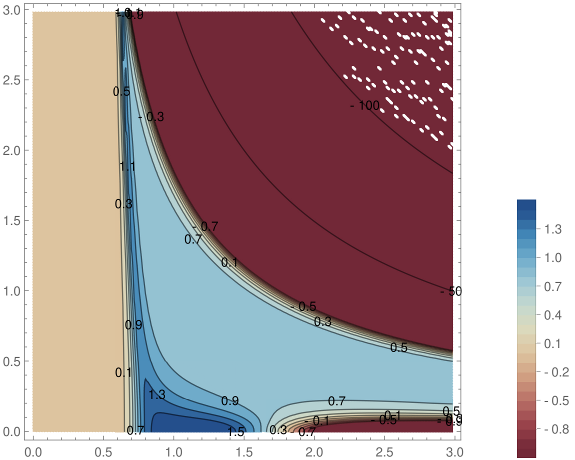
- Free energy density f_L on a cylinder with circumference of L hexagons

$$f_L = -\frac{2}{\sqrt{3}L} \log(\Lambda_{\text{max}})$$

- Calculate c_{eff} using finite-size scaling:

$$f_L = f_\infty - \frac{\pi c_{\text{eff}}}{6L^2} + o\left(\frac{1}{L^2}\right)$$

Phase diagram for $q = e^{i\pi/5}$ in the $(\sqrt{x}, y)$ plane



- Dilute phase $c = \frac{6}{5}$ and dense phase $c = \frac{4}{5}$
- At $y = 0$ (horizontal axis) we get $O(N)$ loop model with $N = 2[3]_q$

Summary

- Web models generalise the $U_q sl(2)$-invariant loop model to $U_q sl(n)$-invariant models
- Geometrical content with applications to $\mathbb{Z}_n$ spin interfaces
- In $U_q sl(3)$ case, we identified dense and dilute critical points for $q = e^{i\gamma}$ and $\gamma \in (0, \pi)$
- Also, we found precursors of electro-magnetic operators $\rightarrow$ starting point for Coulomb gas calculations and CFT identification

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What next?

- Coulomb gas description and fractal dimension of defects
- Statistical models for other spiders of types $U_q so(n)$ and $U_q sp(2n)$
- Detailed representation theoretical study, eg standard modules
- Link to integrable models?
- SLE-like description of branching curves?

Happy Birthday, Hubert!