On strongly coupled bosons in one dimension

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"The Art of Mathematical Physics" for Hubert's 60 years

Saclay, September 2021

Consider N bosons with positions $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N$ interacting via a potential $V(\boldsymbol{r})$. The wave function $\phi(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N | t)$ satisfies the Schrödinger equation

$$-i\partial_t \phi = \Delta\phi + \sum_{i<j} V(\boldsymbol{r}_i - \boldsymbol{r}_j)\phi$$

If the potential V varies on a much smaller length scale than the wavelength of the bosons, it can be approximated by

$$2c\delta(x)\,, \qquad \text{with } c = \frac{1}{2}\int V(y)\mathrm{d}y$$

In 1D, this is known as the Lieb-Liniger model with second quantized form

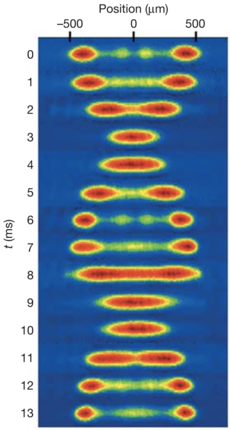
$$H = \int_0^L \mathrm{d}x \left[-\phi^\dagger(x) \frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2} + c\phi^{\dagger 2}(x)\phi^2(x) \right]$$

and it is exactly solvable.

The Lieb-Liniger model is realizable in cold atom experiments, such as the Quantum Newton's cradle [Kinoshita Weiss 06].

Consequences of integrability :

- Presence of infinitely many conserved charges : absence of thermalization at late times [Rigol et al 07]
- Emergence of simple macroscopic laws out of microscopic short-range interactions with Generalized Hydrodynamics [Castro-Alvaredo et al 16, Bertini et al 16]



Program : using exact solvability of the Lieb-Liniger model to prove these properties in a strong coupling expansion

- Correlation functions of local observables $\mathcal{O}$ at equilibrium

$$\langle \mathcal{O}^\dagger(x,t)\mathcal{O}(0,0)\rangle$$

- Time evolution of expectation values after a quantum quench $c \mapsto c'$, relaxation, thermalisation

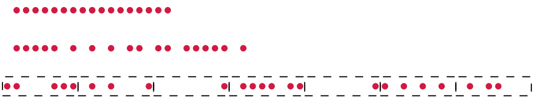
$$\langle \Psi(t)|\mathcal{O}|\Psi(t)\rangle$$

- Inhomogeneous initial states

$t = 0$

$t = 1.0$

$t = \ldots$



I will discuss two approaches :

- A strong-weak and boson-fermion duality
- A $1/c$ expansion of form factor sums

I - Strong-weak and boson-fermion duality

What about N fermions interacting via a short-range potential ?

$$-i\partial_t\psi = \Delta\psi + \sum_{i<j} W(\boldsymbol{r}_i - \boldsymbol{r}_j)\psi$$

The only non-trivial self-adjoint extension of W when its length scale becomes infinitesimal is $\eta(x)$ a "formal" potential that satisfies [Albeverio et al 98]

$$\phi''(x) + k^2\phi(x) = 2\gamma\delta(x)\phi(x)$$

$$\begin{cases}\phi(0^+) - \phi(0^-) = 0\\ \phi'(0^+) - \phi'(0^-) = 2\gamma\phi(0)\end{cases}$$

$$\psi''(x) + k^2\psi(x) = 2\beta\eta(x)\psi(x)$$

$$\begin{cases}\psi(0^+) - \psi(0^-) = 2\beta\psi'(0)\\ \psi'(0^+) - \psi'(0^-) = 0\end{cases}$$

The Girardeau mapping [Girardeau 60]

$$\psi(x_1, ..., x_n) = \left[\prod_{i<j} \mathrm{sgn}\,(x_i - x_j)\right] \phi(x_1, ..., x_n)$$

maps bosons with coupling γ to fermions with coupling $\beta = \frac{1}{\gamma}$. Remarkable boson-fermion and strong-weak duality [Cheon Shigehara 98].

This would yield a fermionic formulation of the Lieb-Liniger model

$$H=-\int_0^L \mathrm{d}x\psi^\dagger(x)\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2}+\beta\int_0^L \mathrm{d}x\mathrm{d}y\eta(x-y)\psi^\dagger(x)\psi^\dagger(y)\psi(y)\psi(x)$$

with $\beta=\frac{2}{c}$. But what is η?

Heuristic interpretation [Kurasov 96]

$$\eta(x)="\partial_x\delta(x)\partial_x"$$

requires distribution theory for discontinuous test functions, and second quantized form is unclear. Moreover, regularizing $\delta(x)$ into a smooth function does not yield the expected discontinuity.

Besides, known regularisations with $a>0$ a cut-off [Cheon Shigehara 98]

$$W_a(x)=\left[\frac{1}{2\beta}-\frac{1}{2a}\right](\delta(x+a)+\delta(x-a))$$

are still strongly coupled at large c / small β.

"Just" a 2-particle quantum mechanics problem : find a potential $V_{a,\beta}(x)$ that is proportional to β such that the odd solution $\psi_a(x)$ to the Schrödinger equation

$$\psi_a''(x) + k^2\psi_a(x) = V_{a,\beta}(x)\psi_a(x)\,,$$

satisfies in the limit $a \to 0$

$$\begin{cases} \psi''(x) + k^2\psi(x) = 0 & \text{for } x \neq 0 \\ \psi(0^+) - \psi(0^-) = 2\beta\psi'(0) \\ \psi'(0^+) - \psi'(0^-) = 0 \end{cases}$$

Solution found in [EG Bertini Essler 21] :

$$V_{a,\beta}(x) = \frac{\beta\sigma_a''(x)}{x + \beta\sigma_a(x)}$$

with "any" smooth function $\sigma_a(x) \to \operatorname{sgn}(x)$ when $a \to 0$.
Makes the strong-weak duality manifest. The discontinuity of the wavefunction is built perturbatively in the coupling constant, whereas in previous formulations it is a non-perturbative effect.

Allows for a diagrammatic expansion in second quantization.
Example of the correlation function at finite temperature

$$\langle \psi^\dagger(x,t)\psi(0,0)\rangle$$

or equivalently the spectral function $A(\omega, q)$, computed for all ω, q perturbatively in $\beta = \frac{2}{c}$ at order β^2.

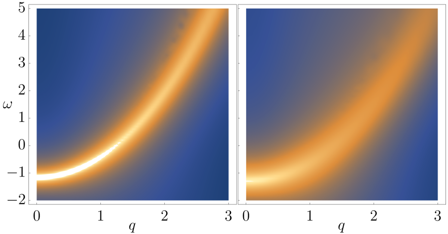


FIGURE – Spectral function $A(\omega, q)$ for $\beta = 2/c = 0.5$ (left) and $\beta = 1$ (right) in an equilibrium state at temperature $T = 1$ and chemical potential $\mu = 1$.

II - Strong coupling expansion of form factor sums

The Lieb-Liniger model is solvable by the Bethe ansatz. Eigenstates are parametrized by solutions $\boldsymbol{\lambda} = \{\lambda_j\}$ to the Bethe equations

$$e^{i\lambda_j L} = \prod_{k\neq j} \frac{\lambda_j - \lambda_k + ic}{\lambda_j - \lambda_k - ic}$$

Eigenstates, energy levels can be expressed in terms of them

$$E(\boldsymbol{\lambda}) = \sum_j \lambda_j^2$$

as well as form factors, i.e. the matrix elements of $\mathcal{O}$ between two eigenstates [Slavnov 90's]

$$\langle \boldsymbol{\lambda} | \mathcal{O} | \boldsymbol{\mu} \rangle = \det U(\boldsymbol{\lambda}, \boldsymbol{\mu})$$

In the thermodynamic limit $L \to \infty$, they are parametrized by a root density $\rho(\lambda)$ such that

$$\frac{1}{L} \sum_j f(\lambda_j) \to \int f(\lambda)\rho(\lambda)\mathrm{d}\lambda$$

Correlation functions can be written as sums over form factors

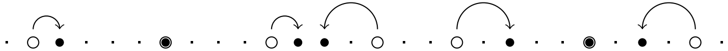
- Ground state correlations $\langle \mathrm{GS}|\mathcal{O}(x,t)\mathcal{O}(0,0)|\mathrm{GS}\rangle$
- Typicality formulation of finite-temperature averages, with $\boldsymbol{\lambda}$ a "thermal" eigenstate

$$\begin{aligned}\frac{\mathrm{tr}\left[\mathcal{O}(x,t)\mathcal{O}(0,0)e^{-\beta H}\right]}{\mathrm{tr}\left[e^{-\beta H}\right]} &= \frac{\sum_{\boldsymbol{\nu}}\langle\boldsymbol{\nu}|\mathcal{O}(x,t)\mathcal{O}(0,0)e^{-\beta H}|\boldsymbol{\nu}\rangle}{\sum_{\boldsymbol{\nu}}\langle\boldsymbol{\nu}|e^{-\beta H}|\boldsymbol{\nu}\rangle} \\ &\approx \frac{e^S\langle\boldsymbol{\lambda}|\mathcal{O}(x,t)\mathcal{O}(0,0)|\boldsymbol{\lambda}\rangle}{e^S\langle\boldsymbol{\lambda}|\boldsymbol{\lambda}\rangle} \qquad (e^S \text{ entropic factor}) \\ &= \langle\boldsymbol{\lambda}|\mathcal{O}(x,t)\mathcal{O}(0,0)|\boldsymbol{\lambda}\rangle \\ &= \sum_{\boldsymbol{\mu}} |\langle\boldsymbol{\lambda}|\mathcal{O}(0,0)|\boldsymbol{\mu}\rangle|^2 e^{it[E(\boldsymbol{\mu})-E(\boldsymbol{\lambda})]+ix[P(\boldsymbol{\lambda})-P(\boldsymbol{\mu})]}\end{aligned}$$

- Out-of-equilibrium correlation functions can also be written as sums over form factors

$$\langle\Psi(t)|\mathcal{O}|\Psi(t)\rangle = \sum_{\boldsymbol{\nu},\boldsymbol{\mu}}\langle\Psi(0)|\boldsymbol{\nu}\rangle\langle\boldsymbol{\nu}|\mathcal{O}|\boldsymbol{\mu}\rangle\langle\boldsymbol{\mu}|\Psi(0)\rangle e^{it(E(\boldsymbol{\mu})-E(\boldsymbol{\nu}))}$$

$$\langle \mathcal{O}(x,t)\mathcal{O}(0,0)\rangle = \sum_{\boldsymbol{\mu}} |\langle \boldsymbol{\lambda}|\mathcal{O}(0,0)|\boldsymbol{\mu}\rangle|^2 e^{it[E(\boldsymbol{\mu})-E(\boldsymbol{\lambda})]+ix[P(\boldsymbol{\lambda})-P(\boldsymbol{\mu})]}$$



Difficulties in evaluating these spectral sums :

- Spectral sums contain an exponential number of terms in the system size L
- Form factors $|\langle \boldsymbol{\lambda}|\mathcal{O}|\boldsymbol{\mu}\rangle|^2$ have non-integrable poles in $\frac{1}{(\lambda_i-\mu_j)^2}$

Known cases

- Free fermion case with $U(1)$ symmetry [Korepin Slavnov 90] or not [EG Dreyer Essler 21]
- Ground state case : $\boldsymbol{\lambda}$ "zero-entropy" state [Maillet et al 11]

$$|\mathrm{GS}\rangle \;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\odot\;\odot\;\odot\;\odot\;\odot\;\odot\;\odot\;\odot\;\odot\;\odot\;\odot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot$$

- Static case $t=0$ with different methods [Patu Klümper 12]

Expansion in $1/c$ developed in [EG Essler 20]. Shown to be well-defined in the thermodynamic limit and uniform in x, t !

Exact expression for density $\sigma = \psi^\dagger \psi$ correlations $\langle \sigma(x,t)\sigma(0,0)\rangle$ and dynamical structure factor $S(q,\omega)$ at order $1/c^2$ that :

- Agrees with CFT and non-linear Luttinger liquid predictions for ground state correlations, in particular critical exponents
- Agrees with GHD predictions for leading term $1/|t|$ of finite temperature correlations [Doyon Spohn 17]
- Recovers universal behaviours such as high frequency tails [Wong Gould 74]

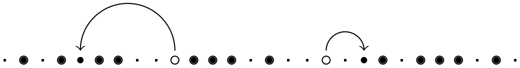
$$S(q,\omega) \propto \frac{q^4}{\omega^{7/2}} \qquad \text{at large } \omega$$

and obtain all corrections to these (at order $1/c^2$).

The main difficulties appearing at higher orders are already present at order $1/c^2$.

First step : compute the $1/c$ expansion of the form factors from exact expressions [Slavnov 90's].

Second step : identify the contributing states $\boldsymbol{\mu}$ at a given order in $1/c$. Reduction from exponential number of states in L to polynomial in L since the density form factor is of order c^{1-n} for n particle-hole excitations.



Also needs to truncate the spectral sum with a cut-off Λ

$$\langle \sigma(x,t)\,\sigma(0,0)\rangle_\Lambda = \sum_{\substack{\boldsymbol{\mu} \\ \forall i,\, |\mu_i|<\Lambda}} |\langle \boldsymbol{\lambda}|\sigma(0)|\boldsymbol{\mu}\rangle|^2\, e^{it(E(\boldsymbol{\lambda})-E(\boldsymbol{\mu}))+ix(P(\boldsymbol{\mu})-P(\boldsymbol{\lambda}))}$$

Order of limits : expansion around $c\to\infty$, then $L\to\infty$, then $\Lambda\to\infty$. Limits are meaningful in a distribution sense

$$\int_{-\Lambda}^{\Lambda} \mu^2 e^{-it\mu^2+ix\mu}\mathrm{d}\mu \to \left(\left(\frac{x}{2t}\right)^2+\frac{1}{2it}\right)\int_{-\infty}^{\infty} e^{-it\mu^2+ix\mu}\mathrm{d}\mu$$

Third step is to decompose all summands into partial fractions.
Example of two-particle-hole contributions

$$\mathcal{C}_2^{\Lambda}(x,t) = \frac{1}{c^2L^4}\sum_{a\neq b}\sum_{\substack{n\\ \forall i,\, \lambda_{a,n}\neq\lambda_i\\ |\lambda_{a,n}|<\Lambda}}\sum_{\substack{m\\ \forall i,\, \lambda_{b,m}\neq\lambda_i\\ \lambda_{b,m}\neq\lambda_{a,n}\\ |\lambda_{b,m}|<\Lambda}}$$

$$\left(\frac{2\pi n}{L}\right)^2\left[\frac{1}{(\frac{2\pi m}{L})^2}+\frac{2}{(\lambda_a-\lambda_b)\frac{2\pi m}{L}}+\frac{1}{(\lambda_a-\lambda_b-\frac{2\pi m}{L})^2}+\frac{2}{(\lambda_a-\lambda_b)(\lambda_a-\lambda_b-\frac{2\pi m}{L})}\right]$$
$$+\frac{2\pi n}{L}\left[\frac{2}{\frac{2\pi m}{L}}+\frac{2(\lambda_a-\lambda_b)}{(\lambda_a-\lambda_b-\frac{2\pi m}{L})^2}+\frac{2}{\lambda_a-\lambda_b-\frac{2\pi m}{L}}\right]$$
$$+\left[\frac{2(\lambda_a-\lambda_b)}{\frac{2\pi m}{L}}+\frac{(\lambda_a-\lambda_b)^2}{(\lambda_a-\lambda_b-\frac{2\pi m}{L})^2}+\frac{2(\lambda_a-\lambda_b)}{\lambda_a-\lambda_b-\frac{2\pi m}{L}}\right]$$
$$+\left(\frac{2\pi n}{L}\right)^{-1}\left[-2(\lambda_a-\lambda_b)+2\frac{2\pi m}{L}-\frac{2(\frac{2\pi m}{L})^2}{\lambda_a-\lambda_b}+\frac{2(\lambda_a-\lambda_b)^2}{\lambda_a-\lambda_b-\frac{2\pi m}{L}}\right]$$
$$+\left(\frac{2\pi n}{L}\right)^{-2}\left(\frac{2\pi m}{L}\right)^2$$
$$+\left(\lambda_a-\lambda_b+\frac{2\pi n}{L}\right)^{-1}\left[2(\lambda_a-\lambda_b)-\frac{2(\lambda_a-\lambda_b)^2}{\frac{2\pi m}{L}}-2\frac{2\pi m}{L}+\frac{2(\frac{2\pi m}{L})^2}{\lambda_a-\lambda_b}\right]$$
$$+\left(\lambda_a-\lambda_b+\frac{2\pi n}{L}\right)^{-2}\left[(\lambda_a-\lambda_b)^2-2(\lambda_a-\lambda_b)\frac{2\pi m}{L}+\left(\frac{2\pi m}{L}\right)^2\right]$$
$$\times\exp\left(it\left[\lambda_a^2-\lambda_{a,n}^2+\lambda_b^2-\lambda_{b,m}^2\right]+ix'[\lambda_{a,n}-\lambda_a+\lambda_{b,m}-\lambda_b]\right)$$

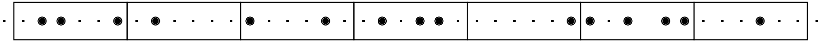
Fourth step is to perform carefully the sum over the Bethe roots.

$$\frac{1}{L^3}\sum_{\substack{i,j,k\\ i\neq j\\ j\neq k}} \frac{g(\lambda_i,\lambda_j,\lambda_k)}{(\lambda_i-\lambda_j)(\lambda_j-\lambda_k)} = \int \frac{g(\lambda,\mu,\nu)\rho(\lambda)\rho(\mu)\rho(\nu)}{(\lambda-\mu)(\mu-\nu)}\mathrm{d}\lambda\mathrm{d}\nu\mathrm{d}\mu$$

$$+\int_{-\infty}^{\infty} g(\lambda,\lambda,\lambda)\left[\frac{\pi^2}{3}\rho(\lambda)^3-\gamma_{-2}(\lambda)\right]\mathrm{d}\lambda$$

$$+\mathcal{O}(L^{-1})$$

Sums with singularities still depend on the choice of the representative state $\boldsymbol{\lambda}$ of the root density $\rho(\lambda)$ in the thermodynamic limit !



$\gamma_{-2}(\lambda)$ measures a dispersion of roots in the boxes in the thermodynamic limit.

Correlation at order c^{-2}

$$\langle\sigma(x,t)\sigma(0,0)\rangle = D^2 + C_1(x,t) + C_2(x,t) + \mathcal{O}(c^{-3})$$

with $C_{1,2}(x,t)$ the contribution for one- and two-particle-hole excitations.

Both diverge in the thermodynamic limit and depend on the representative state

$$C_{1,2}(x,t) = LA_{1,2} + F_{1,2}(\rho,\gamma_{-2}) + \mathcal{O}(L-1)$$

but their sum is finite and depends only on ρ.

One-particle-hole excitations contribution to the dynamical structure factor

$$S^{(1)}(q,\omega)=2\pi^2\big(1+\frac{2D}{c}\big)\rho\big(\tfrac{\omega'-q'^2}{2q'}\big)\rho_h\big(\tfrac{\omega'+q'^2}{2q'}\big)\Bigg[\frac{1}{|q'|}$$
$$-\frac{4\,\mathrm{sgn}\,(q')}{c}\big(\tilde{\rho}\big(\tfrac{\omega'+q'^2}{2q'}\big)-\tilde{\rho}\big(\tfrac{\omega'-q'^2}{2q'}\big)\big)+\frac{8|q'|}{c^2}\big(\tilde{\rho}\big(\tfrac{\omega'+q'^2}{2q'}\big)-\tilde{\rho}\big(\tfrac{\omega'-q'^2}{2q'}\big)\big)^2$$
$$+\frac{4\pi^2|q'|}{c^2}\rho\big(\tfrac{\omega'+q'^2}{2q'}\big)\rho_h\big(\tfrac{\omega'+q'^2}{2q'}\big)\Bigg] \qquad \text{(finite !)}$$

There is a piece resulting from the cross cancelations of representative state dependent parts : 'dressing' of 1ph by 2ph excitations.

Two-particle-hole excitations contribution

$$S^{(2)}(q,\omega)=\frac{8\pi^2}{c^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho(\lambda)\rho_h(\mu)\rho(\bar{\lambda})\rho_h(\bar{\mu})\frac{\frac{2(\lambda-\mu)^2}{\lambda-\bar{\lambda}}-\frac{(\lambda-\mu)^2}{\mu-\bar{\lambda}}+3\mu-2\lambda-\bar{\lambda}}{(q'+\lambda-\mu)|q'+\lambda-\mu|}\mathrm{d}\lambda\mathrm{d}\mu$$
$$+\frac{8\pi^2}{c^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{|q'+\lambda-\mu|^3}\Big[\rho(\lambda)\rho_h(\mu)\rho(\bar{\lambda})\rho_h(\bar{\mu})(\lambda-\mu)^2$$
$$-|q'(\lambda-\mu)|\rho(\bar{\lambda})\rho_h(\bar{\mu})\rho\big(\tfrac{\omega'-q'^2}{2q'}\big)\rho_h\big(\tfrac{\omega'+q'^2}{2q'}\big)\Big]\mathrm{d}\lambda\mathrm{d}\mu$$

Plots of dynamical structure factor at finite temperature

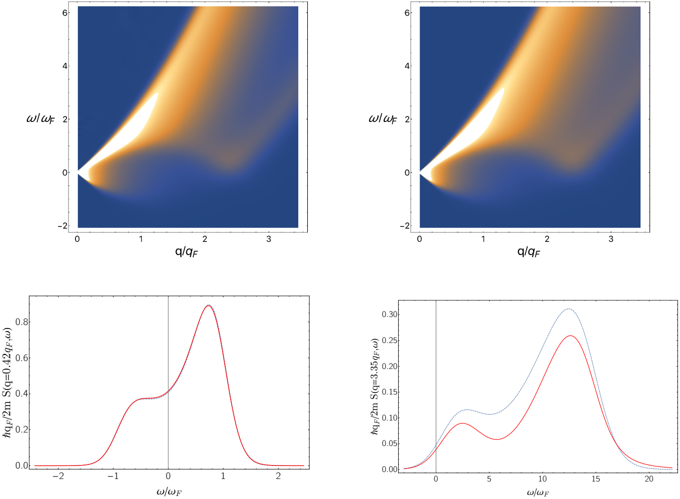


FIGURE – Top : One-particle-hole excitations (left) and one+two-particle-hole excitations (right), for a thermal state. Bottom : Cuts at fixed q, for small q (left) and larger q (right).

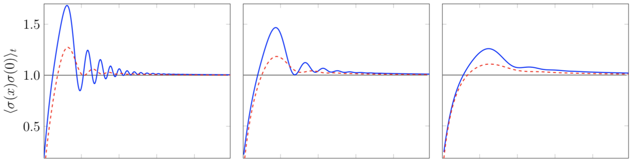
What about out-of-equilibrium settings ?

$$\langle \Psi(t)|\mathcal{O}|\Psi(t)\rangle = \sum_{\boldsymbol{\nu},\boldsymbol{\mu}} \langle \Psi(0)|\boldsymbol{\nu}\rangle\langle\boldsymbol{\nu}|\mathcal{O}|\boldsymbol{\mu}\rangle\langle\boldsymbol{\mu}|\Psi(0)\rangle e^{it(E(\boldsymbol{\mu})-E(\boldsymbol{\nu}))}$$

Quantum quench : the system is prepared in the ground state of the model in absence of interactions $c=0$ at $t=0$, and then time-evolved with interaction $c>0$ at $t>0$.

$$|\Psi(t)\rangle = e^{-iH(c)t}|\mathrm{GS}_{c=0}\rangle$$

Same method to compute out-of-equilibrium correlations in a $1/c$ expansion [EG Essler 21]. Again typicality assumptions [Caux Essler 13] hold and the expansion is uniform in x, t.



Summary and perspectives

Two techniques for strongly coupled bosons

- Fermionic formulation of the Lieb-Liniger model with the potential $V_{a,\beta}$ that allows perturbative calculations for strongly coupled bosons
- Form factor sums possess a well-controlled $1/c$ expansion

Future directions

- Using the duality to investigate out-of-equilibrium physics perturbatively
- Applying the $1/c$ expansion to higher orders and other observables (sub diffusion effects, etc)