

Four-point cluster connectivities in 2d percolation and LCFT

Yifei He

ENS Paris

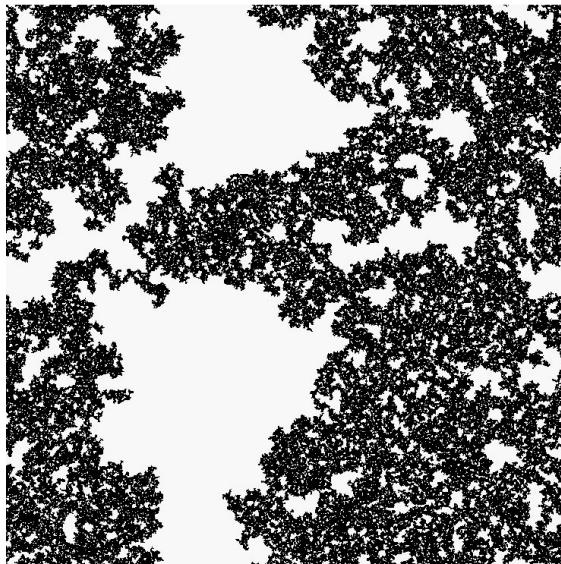
« Il faut traiter la nature par le cylindre, la sphère et le cône »

The art of mathematical physics

A conference to celebrate the 60th birthday of Hubert Saleur

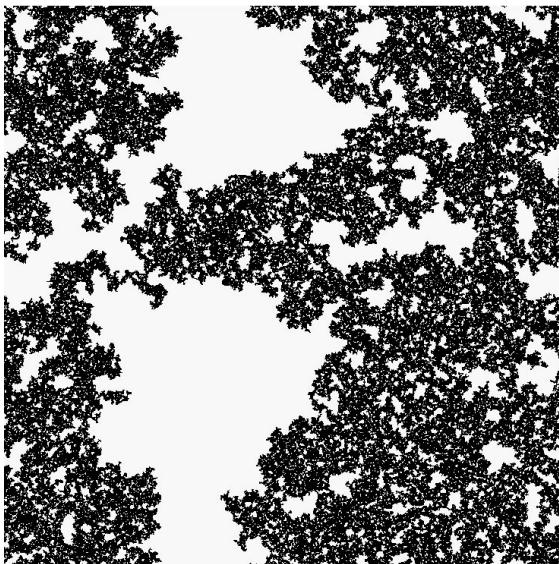
Institut de Physique Théorique, CEA-Saclay, 20-23 September 2021

Cluster connectivity in percolation



geometrical phase transition
described by **non-unitary** conformal field theory

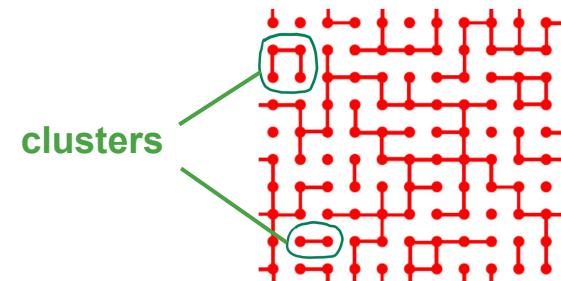
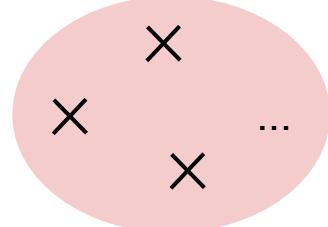
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cluster connectivities



Q-state Potts model

$$Z_{\text{spin}} \longrightarrow Z_{\text{cluster}}$$

[Fortuin, Kasteleyn, 1972]

$$Q \in \mathbb{R}$$

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critical cluster model $0 < Q < 4$  Potts CFT $-2 < c < 1$

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$Q \rightarrow 1$ ($c \rightarrow 0$) percolation

Q-state Potts model

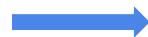
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Potts CFT

$$-2 < c < 1$$

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Potts spin: order parameter

cluster connectivities



CFT correlator of spin operator $\Phi_{\frac{1}{2}, 0}$

[Delfino, Viti, 2011]

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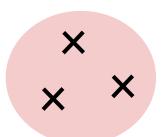
two-point
&
three-point

connectivities are understood

[Delfino, Viti, 2010]

[Picco, Santachiara, Viti, Delfino, 2013]

[Ikhlef, Jacobsen, Saleur, 2015]



Four-point connectivities

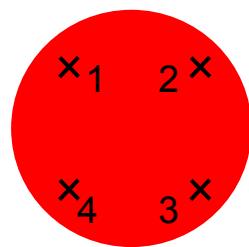
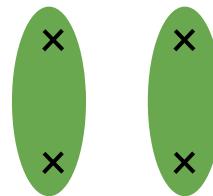
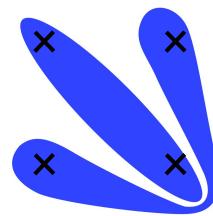
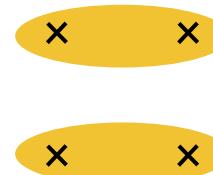
highly non-trivial, probe the spectrum of the CFT

$x_1 \quad 2 \quad x$

$x_4 \quad 3 \quad x$

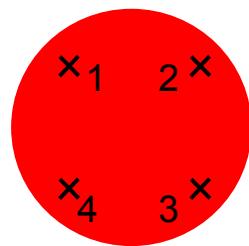
Four-point connectivities

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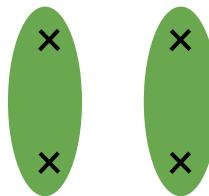
 P_{aaaa}  P_{aabb}  P_{abab}  P_{abba}

Four-point connectivities

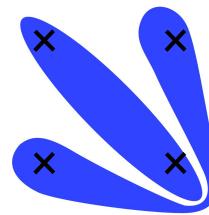
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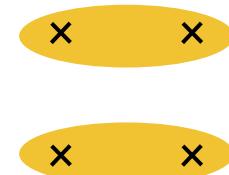
$$P_{aaaa}$$



$$P_{aabb}$$



$$P_{abab}$$

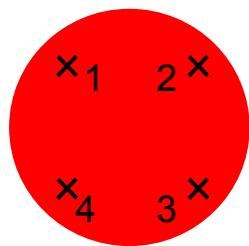


$$P_{abba}$$

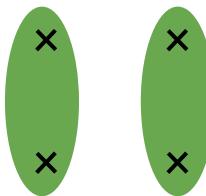
Potts CFT: $\langle \Phi_{\frac{1}{2},0} \Phi_{\frac{1}{2},0} \Phi_{\frac{1}{2},0} \Phi_{\frac{1}{2},0} \rangle$

Four-point connectivities

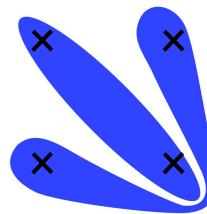
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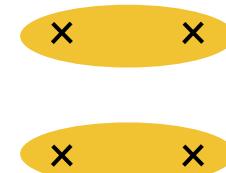
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$$P_{aabb}$$



$$P_{abab}$$



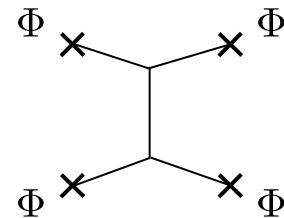
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Potts CFT: $\langle \Phi_{\frac{1}{2},0} \Phi_{\frac{1}{2},0} \Phi_{\frac{1}{2},0} \Phi_{\frac{1}{2},0} \rangle$

fractional Kac indices

cannot use BPZ

Conformal bootstrap approach



$$\sum_{(h, \bar{h}) \in s\text{-channel}} A(h, \bar{h}) \mathcal{F}^{(s)}$$

conformal block

Conformal bootstrap approach

A Feynman diagram showing a four-point interaction of scalar fields Φ . It consists of four external legs meeting at a central vertex. The top-left leg is labeled Φ , the top-right Φ , the bottom-left Φ , and the bottom-right Φ .

amplitudes

$$\sum_{(h, \bar{h}) \in s\text{-channel}} A(h, \bar{h}) \mathcal{F}^{(s)}$$

spectrum

conformal block

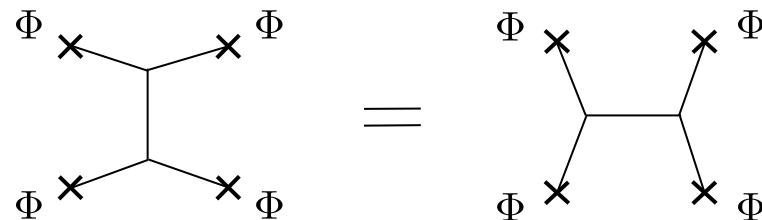
Conformal bootstrap approach

The diagram shows two Feynman-like graphs with four external legs, each labeled with a Φ . The left graph has a central vertical line with two diagonal lines meeting it at the top and bottom. The right graph has a horizontal central line with two diagonal lines meeting it from the left and right. An equals sign is placed between them.

Below the diagrams, a blue bracket labeled "amplitudes" spans both sides of the equals sign. A red bracket labeled "conformal block" is positioned under the central vertical line of the left diagram.

$$\sum_{(h, \bar{h}) \in s\text{-channel}} A(h, \bar{h}) \mathcal{F}^{(s)} = \sum_{(h, \bar{h}) \in t\text{-channel}} A(h, \bar{h}) \mathcal{F}^{(t)}$$

Conformal bootstrap approach



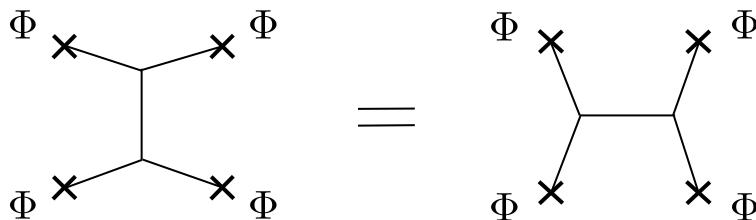
★ amplitudes

★ spectrum

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conformal block ✓

Conformal bootstrap approach

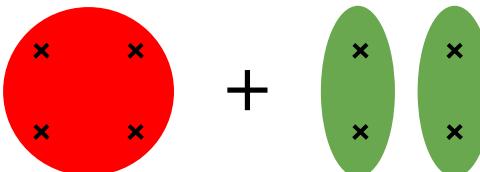


★ amplitudes

★ spectrum

first attempt to
bootstrap connectivity:

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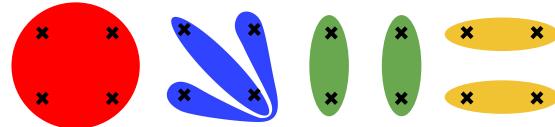
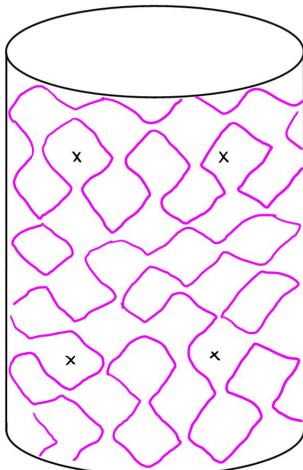


[Picco, Ribault, Santachiara, 2016]

Spectrum of connectivities

[Jacobsen, Saleur, 2018]

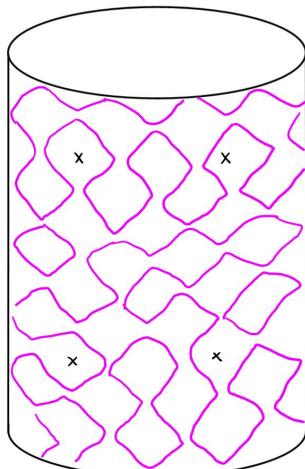
equivalent loop representation



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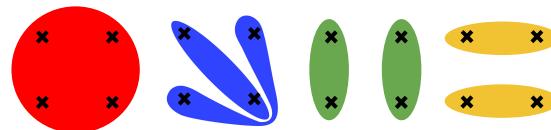


eigenvalues λ_i of transfer matrix



*continuum
limit*

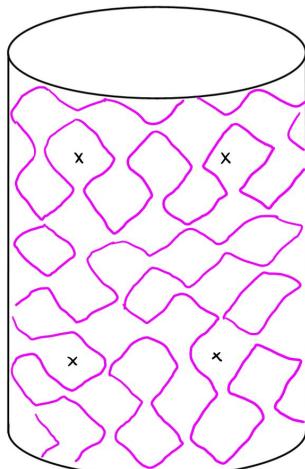
scaling dimensions (h, \bar{h}) in the spectrum



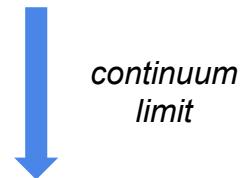
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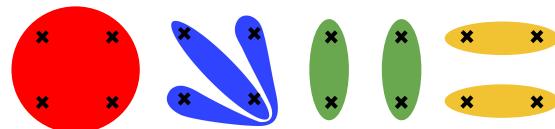


irreducible module of affine Temperley-Lieb algebra

\mathcal{W}

infinite tower of Virasoro conformal family

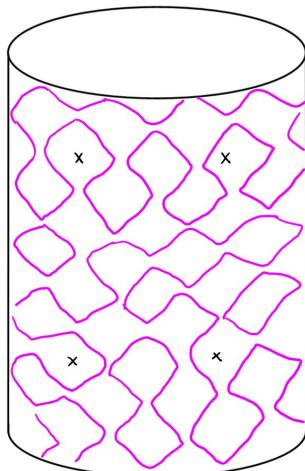
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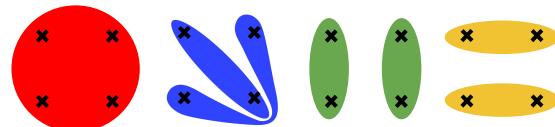


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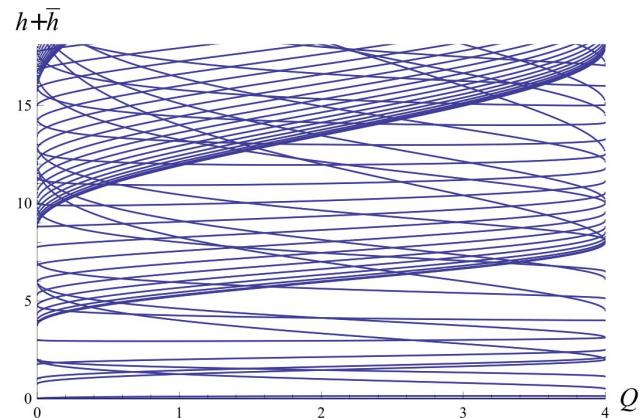
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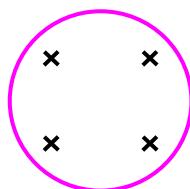
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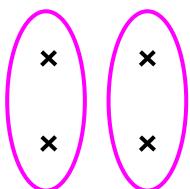


Minimal models on the lattice

$$\langle \Phi_\Delta \Phi_\Delta \Phi_\Delta \Phi_\Delta \rangle \underset{\text{Monte-Carlo}}{\propto} \Delta = \Delta_{\frac{1}{2}, 0}$$



$$+ \frac{2}{Q-2}$$

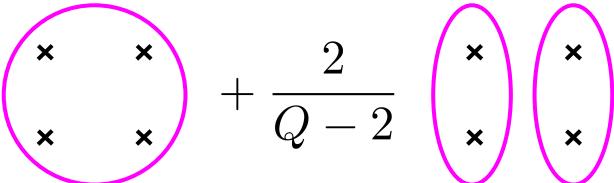


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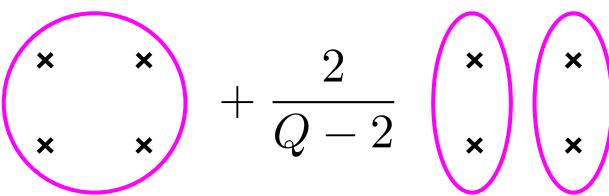
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Bootstrap: simple spectrum (subset of Potts)

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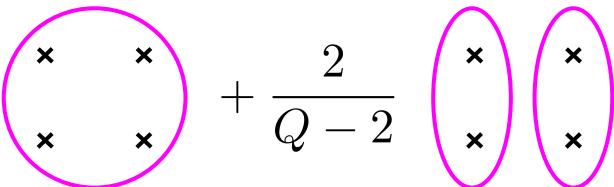


The diagram illustrates the lattice configurations used in the Monte-Carlo simulation. On the left, a single circle contains four spins arranged in a square pattern: top-left (x), top-right (x), bottom-left (x), and bottom-right (x). On the right, there are two separate circles, each containing two spins: top (x) and bottom (x).

[Picco, Ribault, Santachiara, 2016]

Bootstrap: simple spectrum (subset of Potts) \longrightarrow analytic continuation of MM Kac table to generic c
[Migliaccio, Ribault, 2018]

Minimal models on the lattice

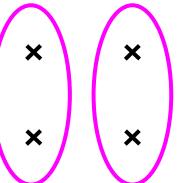
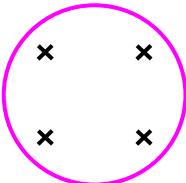
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MM on the lattice: ADE RSOS model [Pasquier, 1987] [Kostov, 1989]

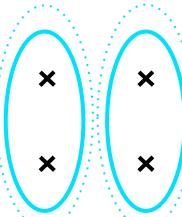
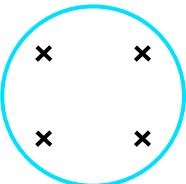
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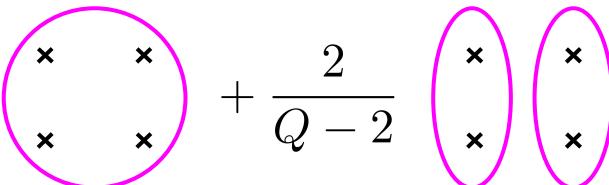
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[YH, Grans-Samuelsson, Jacobsen, Saleur, 2020]

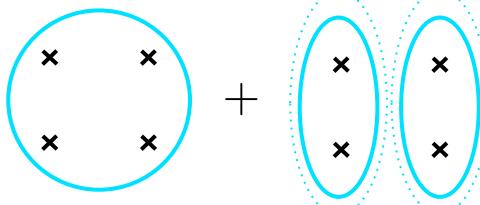
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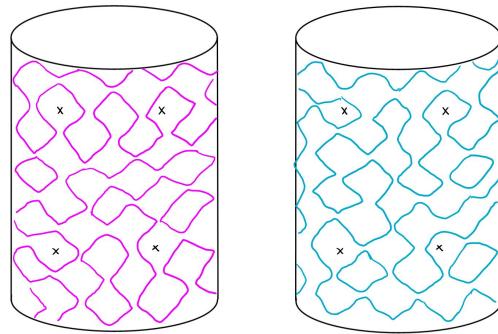
different weights depending on topology

[YH, Grans-Samuelsson, Jacobsen, Saleur, 2020]

pseudo-probabilities \tilde{P}

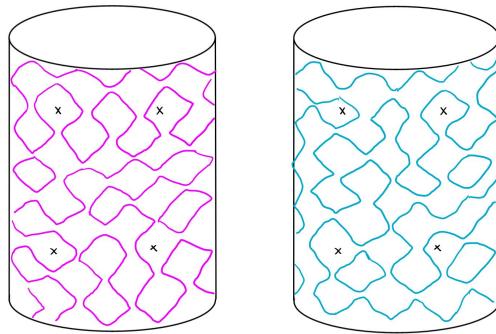
Universal amplitude ratios on the lattice

[YH, Grans-Samuelsson, Jacobsen, Saleur, 2020]



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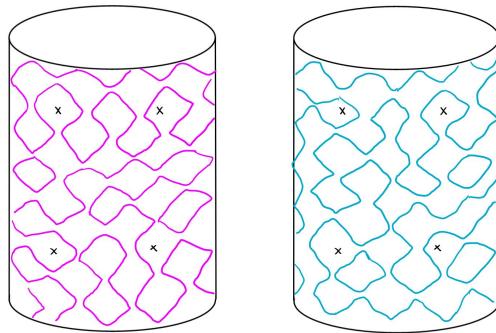


\tilde{P} spectrum $\sim P$ spectrum

affine Temperley-Lieb modules \mathcal{W}

Universal amplitude ratios on the lattice

[YH, Grans-Samuelsson, Jacobsen, Saleur, 2020]



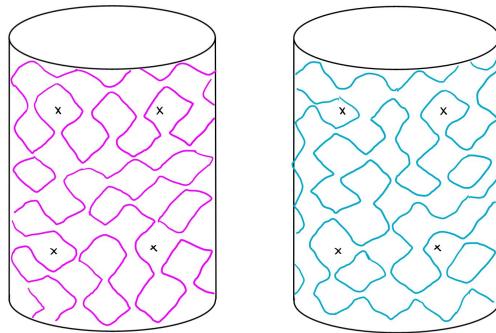
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$$\frac{A_{abab}}{A_{aaaa}}(\lambda_i), \quad \frac{\tilde{A}_{abab}}{A_{abab}}(\lambda_i), \quad \dots$$

Universal amplitude ratios on the lattice

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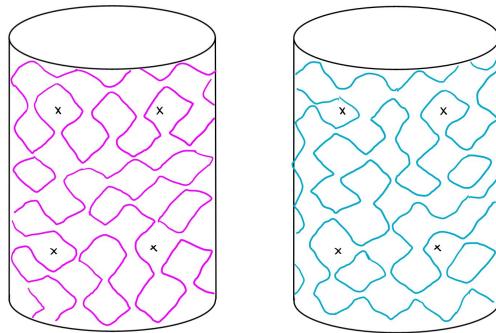
$$\frac{A_{abab}}{A_{aaaa}}(\lambda_i), \quad \frac{\tilde{A}_{abab}}{A_{abab}}(\lambda_i), \quad \dots$$

depend only on Q and $\lambda_i \in \mathcal{W}$

do not depend on lattice size

Universal amplitude ratios on the lattice

[YH, Grans-Samuelsson, Jacobsen, Saleur, 2020]



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\tilde{P} spectrum $\sim P$ spectrum

affine Temperley-Lieb modules \mathcal{W}

$\frac{\tilde{A}_{abab}}{A_{abab}}(\mathcal{W}_{2,-1})$	$=$	$\frac{2}{Q-2}$	$\frac{\tilde{A}_{abab}}{A_{abab}}(\mathcal{W}_{4,-1})$	$=$	$-\frac{4}{(Q-1)(Q-2)(Q^2-4Q+2)}$
$\frac{A_{aabbb}}{A_{aaaa}}(\mathcal{W}_{2,1})$	$=$	$\frac{1}{1-Q}$	$\frac{A_{abab}}{A_{aaaa}}(\mathcal{W}_{4,-1})$	$=$	$\frac{(Q-1)(Q-4)}{4}$
$\frac{A_{abab}}{A_{aaaa}}(\mathcal{W}_{2,1})$	$=$	$\frac{2-Q}{2}$	$\frac{A_{aabbb}}{A_{aaaa}}(\mathcal{W}_{4,1})$	$=$	$-\frac{Q^5 - 7Q^4 + 15Q^3 - 10Q^2 + 4Q - 2}{2(Q^2 - 3Q + 1)}$
$\frac{A_{aabbb}}{A_{aaaa}}(\mathcal{W}_{4,-1})$	$=$	$\frac{2-Q}{2}$	$\frac{A_{abab}}{A_{aaaa}}(\mathcal{W}_{4,1})$	$=$	$-\frac{(Q^2 - 4Q + 2)(Q^2 - 3Q - 2)}{4}$

Interchiral conformal blocks

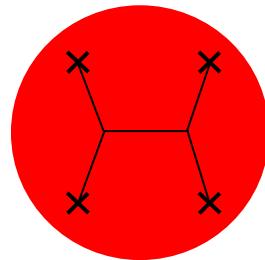
[YH, Jacobsen, Saleur, 2020]

$$\lambda_i \xrightarrow{\text{continuum}} (h, \bar{h}) \quad \text{organize the CFT states according to } \mathcal{W}$$

Interchiral conformal blocks

[YH, Jacobsen, Saleur, 2020]

$\lambda_i \xrightarrow{\text{continuum}} (h, \bar{h})$ organize the CFT states according to \mathcal{W}



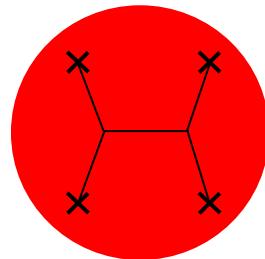
$$= \sum_{(h, \bar{h}) \in s\text{-channel}} A_{aaaa}(h, \bar{h}) \mathcal{F}_{h, \bar{h}}$$

P_{aaaa}

Interchiral conformal blocks

[YH, Jacobsen, Saleur, 2020]

$\lambda_i \xrightarrow{\text{continuum}} (h, \bar{h})$ organize the CFT states according to \mathcal{W}

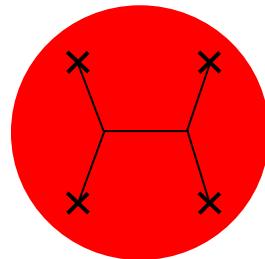


$$\begin{aligned} P_{aaaa} &= \sum_{(h, \bar{h}) \in s\text{-channel}} A_{aaaa}(h, \bar{h}) \mathcal{F}_{h, \bar{h}} \\ &= \sum_{\mathcal{W} \in s\text{-channel}} A_{aaaa}(\mathcal{W}) \sum_{(h, \bar{h}) \in \mathcal{W}} \frac{A(h, \bar{h})}{A_{aaaa}(\mathcal{W})} \mathcal{F}_{h, \bar{h}} \end{aligned}$$

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P_{aaaa}

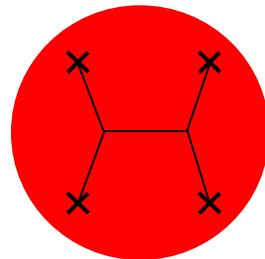
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interchiral conformal block

Interchiral conformal blocks

[YH, Jacobsen, Saleur, 2020]

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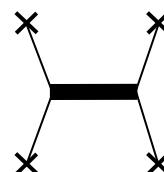
P_{aaaa}

$$= \sum_{(h, \bar{h}) \in s\text{-channel}} A_{aaaa}(h, \bar{h}) \mathcal{F}_{h, \bar{h}}$$

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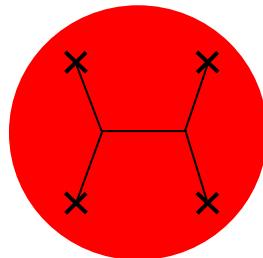
interchiral conformal block



Interchiral conformal blocks

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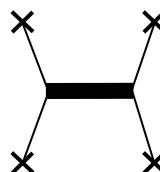
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interchiral conformal block

interchiral algebra

[Gainutdinov, Read, Saleur, 2012]



Interchiral conformal bootstrap

[YH, Jacobsen, Saleur, 2020]

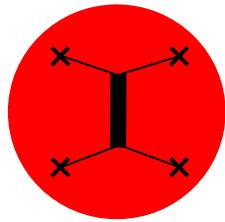
construct $\mathbb{F}_{\mathcal{W}}$: degeneracy \longrightarrow recursion

technique in Liouville bootstrap
[Zamolodchikov^2, 1995] [Teschner, 1995]
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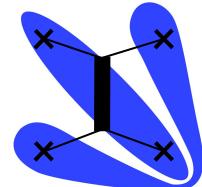
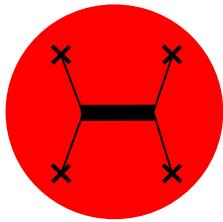
Interchiral conformal bootstrap

[YH, Jacobsen, Saleur, 2020]

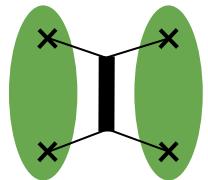
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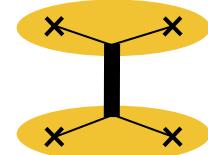
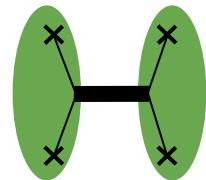
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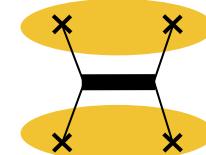
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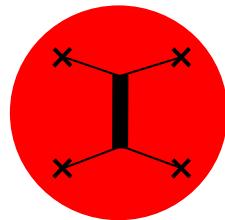
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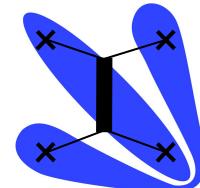
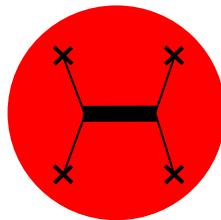
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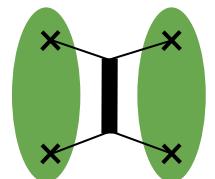
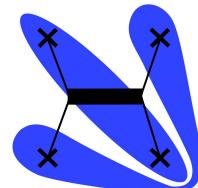
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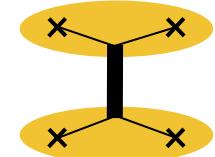
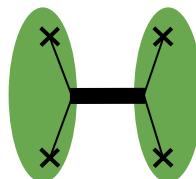
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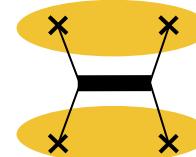
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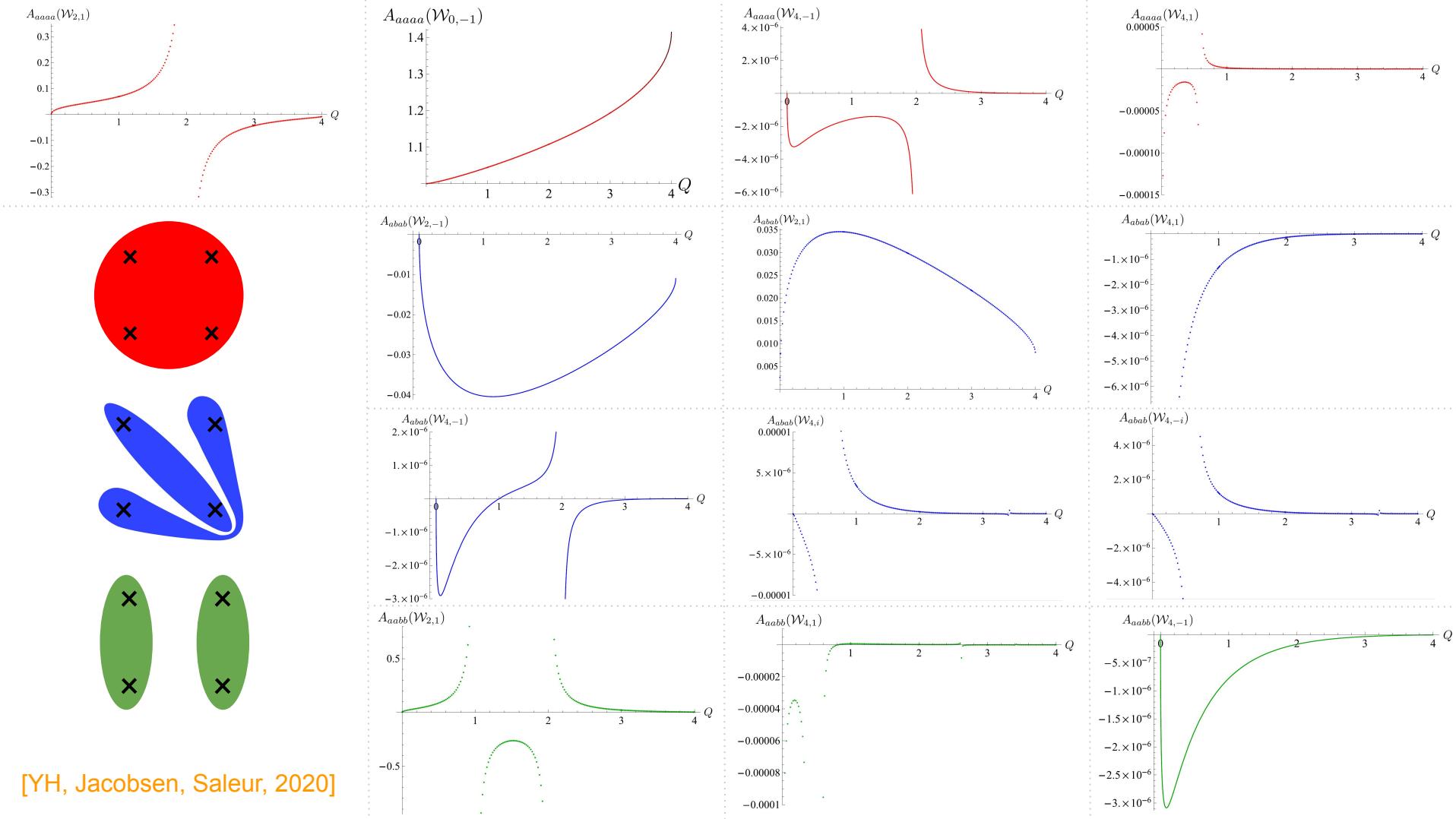
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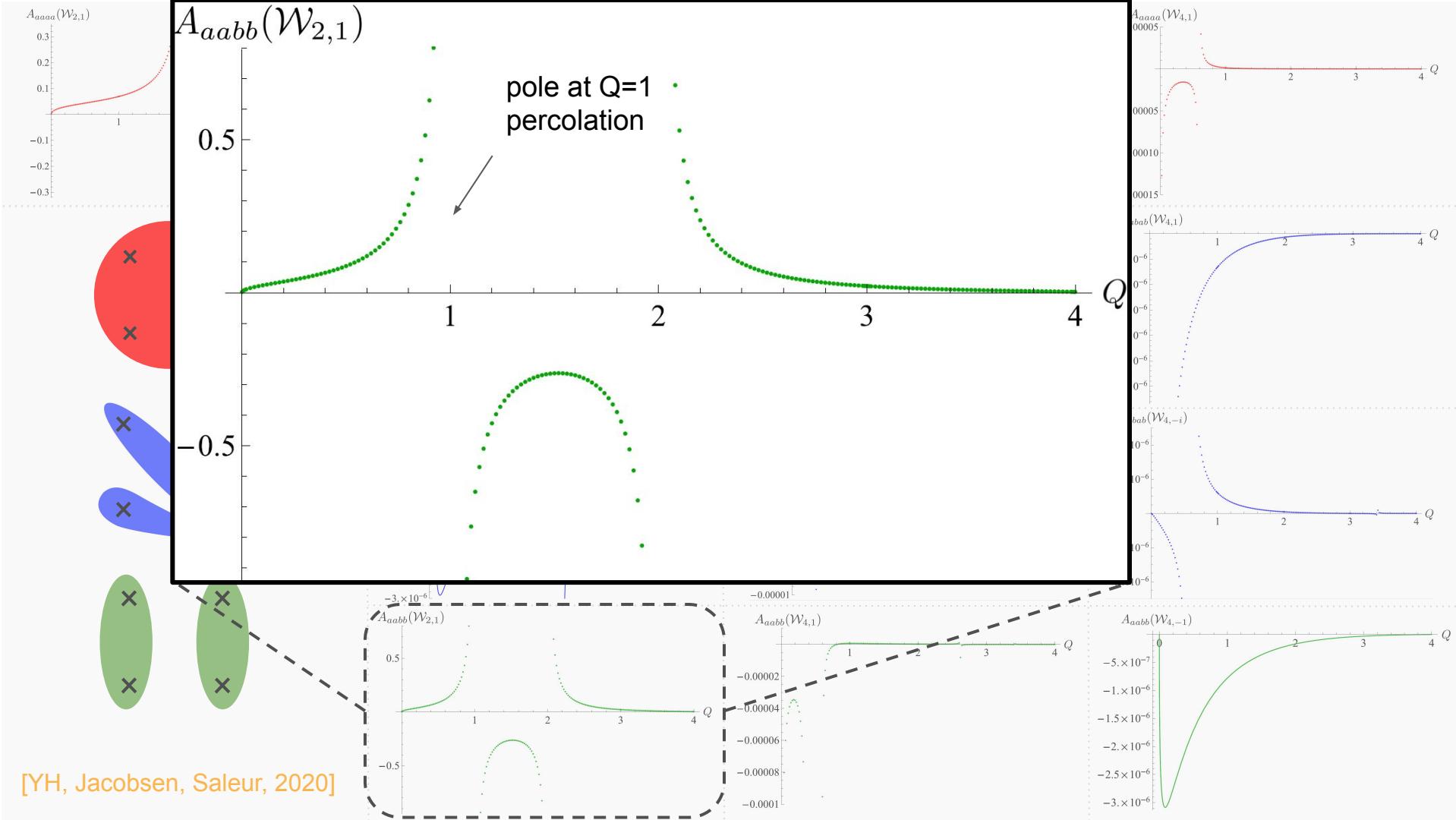


solve $A_{aaaa}(\mathcal{W}), A_{abab}(\mathcal{W}), A_{aabb}(\mathcal{W}), A_{abba}(\mathcal{W})$

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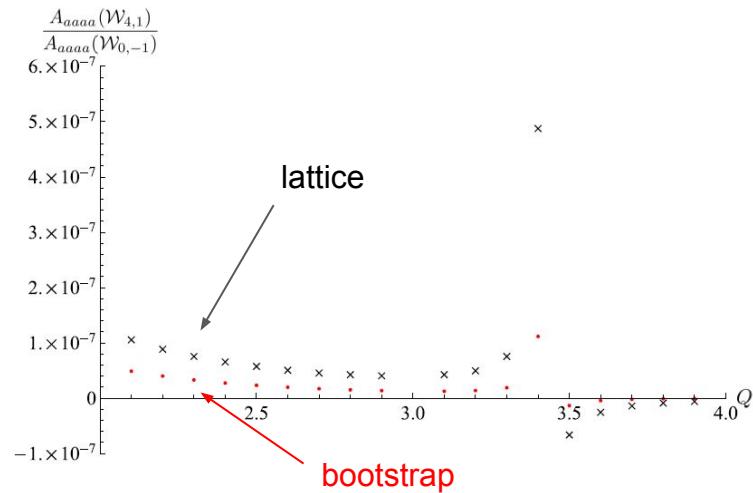
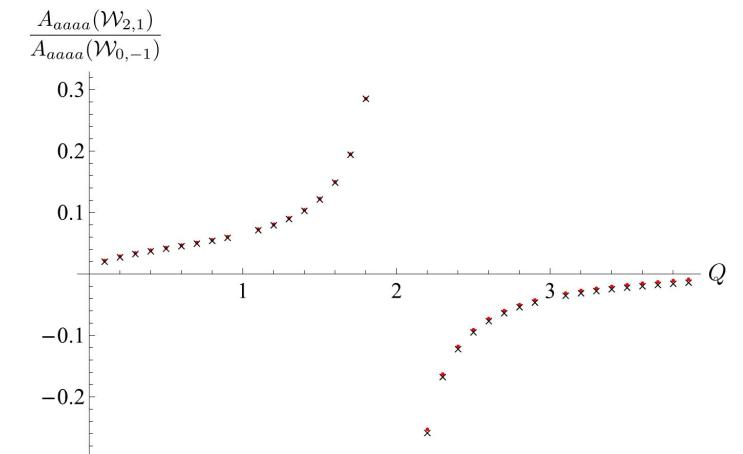
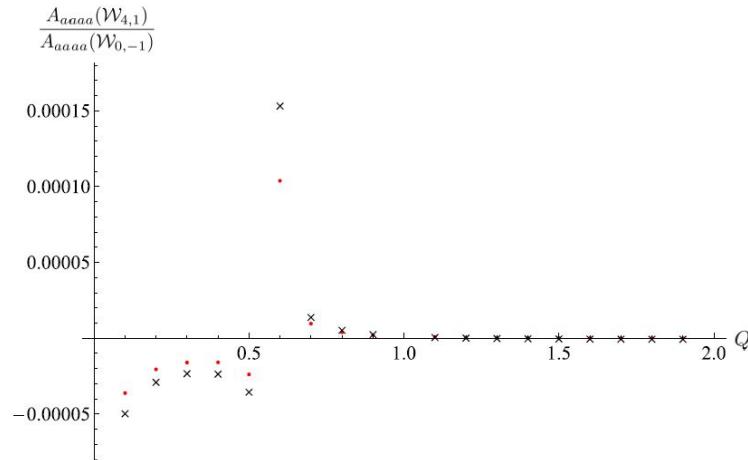




Comparison with lattice

[YH, Jacobsen, Saleur, 2020]

- *order of magnitude*
- *behavior as a function of Q*
- *analytic structure*



“Renormalized” Liouville recursion

[YH, Jacobsen, Saleur, 2020]

in Liouville
(and its non-diagonal generalization)

degenerate
 $\Phi_{1,2}, \Phi_{2,1}$



amplitude recursions
analytic bootstrap solution

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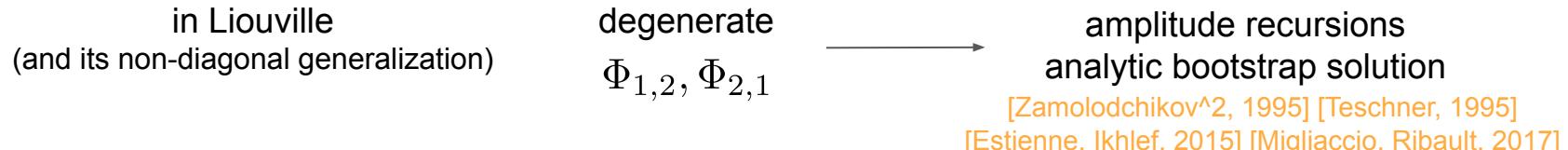
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Potts: only $\Phi_{2,1}$ degenerate (energy operator)

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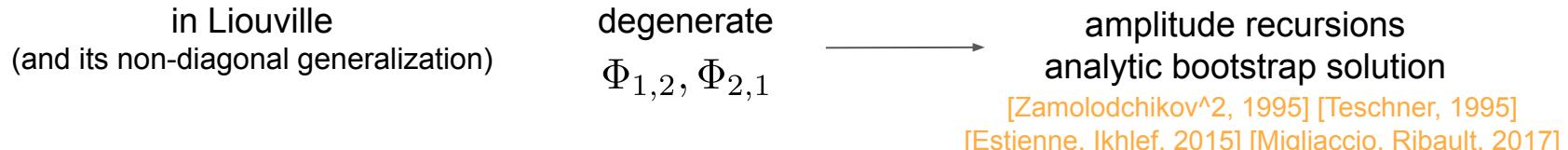


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$$\begin{aligned}\frac{A_{aaaa}(\mathcal{W}_{4,-1})}{A_{aaaa}(\mathcal{W}_{0,-1})} &= \frac{(Q-2)(Q^2-4Q+2)}{Q(Q-3)^2} \frac{A^L(\mathcal{W}_{4,-1})}{A^L(\mathcal{W}_{0,-1})} \\ \frac{A_{abab}(\mathcal{W}_{4,-1})}{A_{abab}(\mathcal{W}_{2,-1})} &= \frac{(Q-1)(Q-4)(Q^2-4Q+2)}{2Q(Q-3)^2} \frac{A^L(\mathcal{W}_{4,-1})}{A^L(\mathcal{W}_{2,-1})} \\ \frac{A_{aaaa}(\mathcal{W}_{4,1})}{A_{aaaa}(\mathcal{W}_{2,1})} &= \frac{(Q-2)^2}{(Q-1)^2(Q^2-4Q+2)} \frac{A^L(\mathcal{W}_{4,1})}{A^L(\mathcal{W}_{2,1})}\end{aligned}$$

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 \end{aligned}$$

Liouville recursion
IF $\Phi_{1,2}$ is degenerate

“renormalization” factors -- rational functions of Q

“ $c \rightarrow 0$ catastrophe”

$$\Phi_{\Delta}(z, \bar{z}) \times \Phi_{\Delta}(0, 0) \sim 1 + \frac{2\Delta}{c} (z^2 T + \bar{z}^2 \bar{T}) + \dots \quad [\text{Gurarie, 1998}]$$

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logarithmic partner t

$$\langle t(z, \bar{z}) t(0, 0) \rangle = \frac{\theta - 2b \ln |z|^2}{z^4}$$

$$\begin{aligned}\langle T(z) t(0, 0) \rangle &= \frac{b}{z^4} \\ \langle T(z) T(0) \rangle &= 0\end{aligned}$$

b-number

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lattice measurement

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Coulomb gas:

$$T(z) \text{ & } X(z, \bar{z}) \longrightarrow t(z, \bar{z})$$

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$$(h_X, \bar{h}_X)|_{c=0} = (2, 0)$$

2-leg operator in $O(n)$
4-leg operator in Potts

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Potts and O(n) logarithmic CFT

Potts and O(n) CFT is logarithmic at generic c

[Estienne, Ikhlef, 2015] [Gorbenko, Zan, 2020]

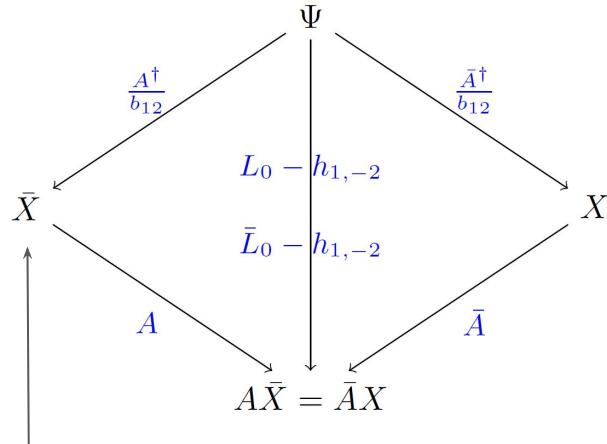
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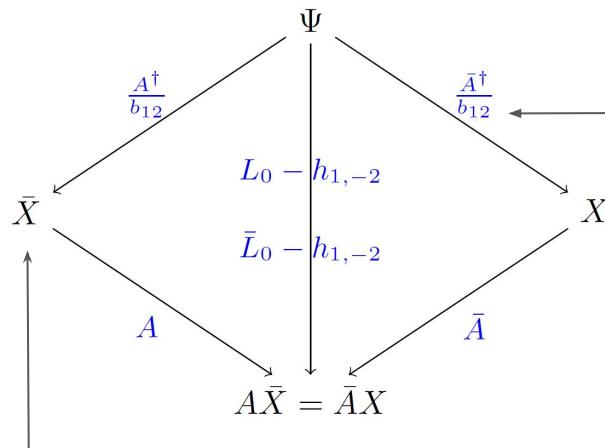


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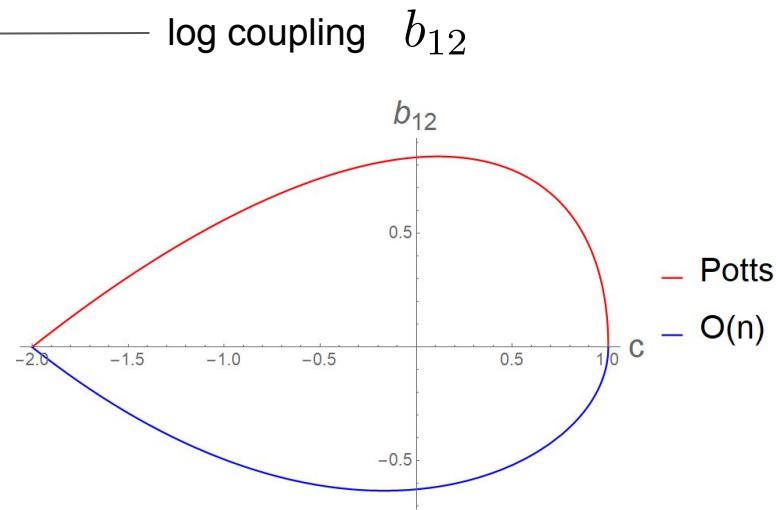
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2-leg operator in O(n)
4-leg operator in Potts



$c=0$ log OPE and rank-3 Jordan cell [YH, Saleur, 2021]

a generic c OPE in Potts or $O(n)$:

$$\Phi_{\Delta}(z, \bar{z})\Phi_{\Delta}(0, 0) \sim 1 + \frac{2\Delta}{c}(z^2 T + \bar{z}^2 \bar{T}) + \frac{4\Delta}{c^2}(z\bar{z})^2 T\bar{T} + (z\bar{z})^{h_{1,2}} \mathcal{A}\left(\bar{z}^2 \bar{X} + z^2 X + (z\bar{z})^{h_{-1,2}} (\Psi + \ln(z\bar{z}) A\bar{X})\right) + \dots$$

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\downarrow $c \rightarrow 0$ limit finite

$c=0$ log OPE and rank-3 Jordan cell

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log OPE at $c=0$:

$$\downarrow c \rightarrow 0 \text{ limit finite}$$

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rank-3 Jordan cell

$$\Psi_0 = T\bar{T}$$

$c=0$ log OPE and rank-3 Jordan cell

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$$a_0 = -\frac{25}{48}$$

rank-3 Jordan cell

$$\Psi_0 = T\bar{T}$$

identical for percolation and polymers

$c=0$ log OPE and rank-3 Jordan cell

[YH, Saleur, 2021]

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singularity cancellations

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$$\Psi_0 = T\bar{T}$$

identical for percolation and polymers

$c=0$ log OPE and rank-3 Jordan cell

[YH, Saleur, 2021]

a generic c OPE in Potts or $O(n)$:

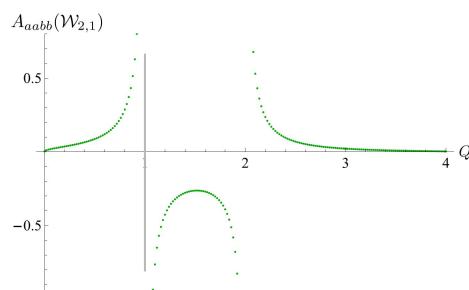
$$\Phi_{\Delta}(z, \bar{z})\Phi_{\Delta}(0, 0) \sim 1 + \frac{2\Delta}{c}(z^2 T + \bar{z}^2 \bar{T}) + \frac{4\Delta}{c^2}(z\bar{z})^2 T\bar{T} + (z\bar{z})^{h_{1,2}} \mathcal{A}\left(\bar{z}^2 \bar{X} + z^2 X + (z\bar{z})^{h_{-1,2}} (\Psi + \ln(z\bar{z}) A\bar{X})\right) + \dots$$

log OPE at $c=0$:

$$\downarrow c \rightarrow 0 \text{ limit finite}$$

$$\Phi_{\Delta}(z, \bar{z})\Phi_{\Delta}(0, \bar{0}) = (z\bar{z})^{-2\Delta} \left[1 + z^2 \frac{\Delta}{b} (t + T \ln(z\bar{z})) + \bar{z}^2 \frac{\Delta}{b} (\bar{t} + \bar{T} \ln(z\bar{z})) + (z\bar{z})^2 \frac{\Delta^2}{a_0} (\Psi_2 + \ln(z\bar{z}) \Psi_1 + \frac{1}{2} \ln^2(z\bar{z}) \Psi_0) + \dots \right]$$

singularity cancellations $\Delta = \Delta_{\frac{1}{2}, 0}$ ✓



$$a_0 = -\frac{25}{48}$$

rank-3 Jordan cell

$$\Psi_0 = T\bar{T}$$

identical for percolation and polymers

Virasoro structure

[YH, Saleur, 2021]

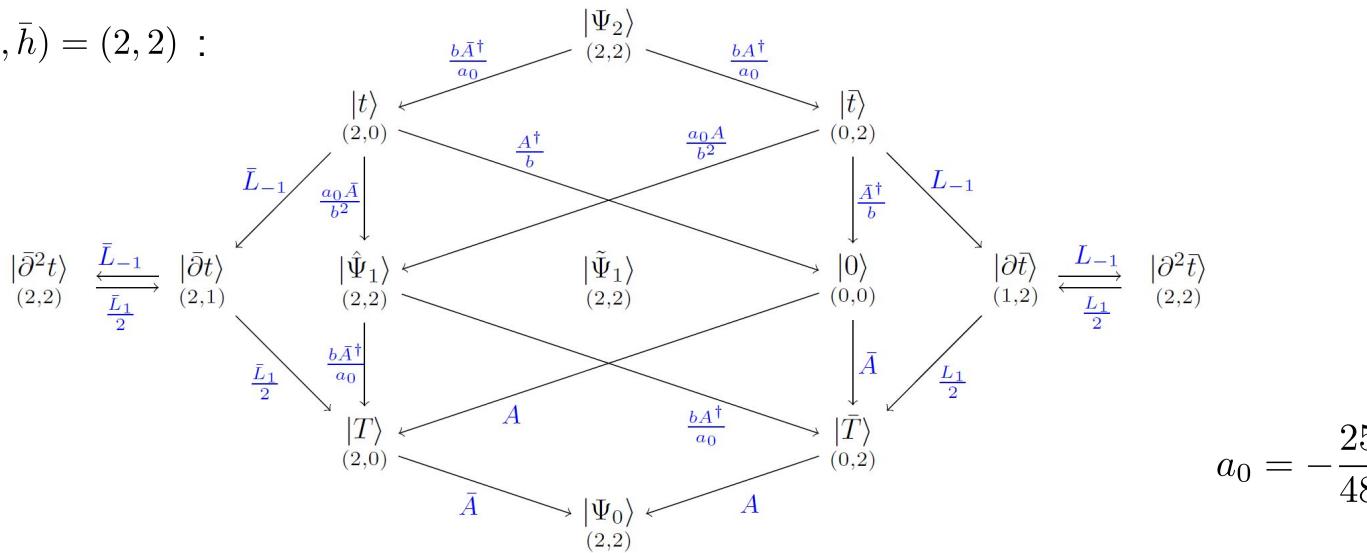
based on the log OPE: **conformal invariance + self-duality**

Virasoro structure

[YH, Saleur, 2021]

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states up to $(h, \bar{h}) = (2, 2)$:

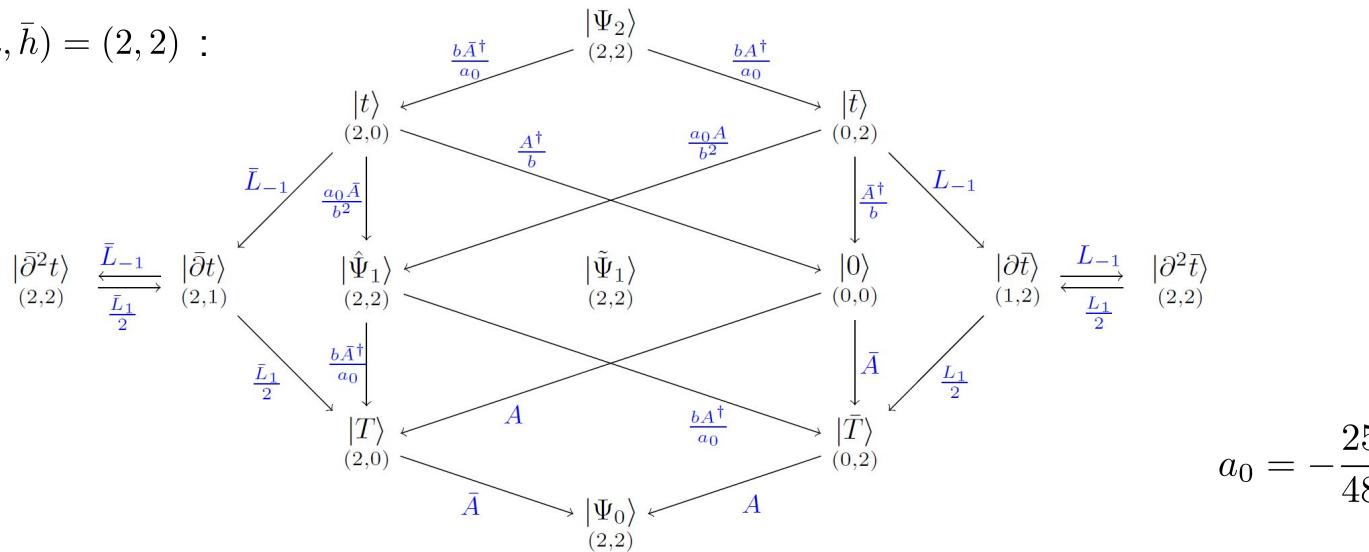


Virasoro structure

[YH, Saleur, 2021]

based on the log OPE: **conformal invariance + self-duality**

states up to $(h, \bar{h}) = (2, 2)$:



different fields and log structures at generic c \longrightarrow identical structure for percolation and polymers at $c=0$

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- conformal bootstrap approach + lattice algebra
numerically determined four-point cluster connectivities

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numerically determined four-point cluster connectivities
- further study the “renormalized” Liouville recursion
analytic bootstrap solutions?
- identity Virasoro module identical for percolation and polymers
much more to understand about $c=0$ LCFT!



Bon anniversaire Hubert!