Connectivity operators of 2d loop models

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Saclay

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Outline

1. Introduction
2. OPE structure constants
3. Fusion on the lattice

1. Introduction

Critical random curves

- Classical 2d critical system $\rightarrow$ random fractal curves

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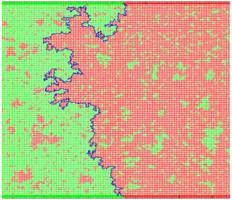

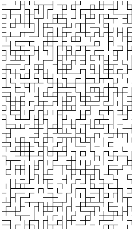

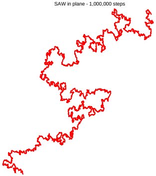

pictures courtesy of Tom Kennedy (University of Arizona)

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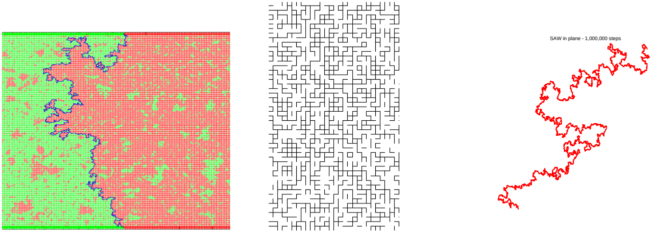

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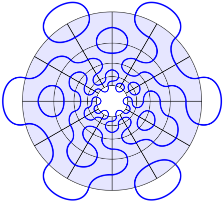
- Non-trivial fractal dimensions

  $d_{\text{Ising DW}} = 11/8 \quad d_{\text{perco contour}} = 7/4 \quad d_{\text{SAW}} = 4/3$

  [# disks of radius ϵ needed to cover the curve $= N(\epsilon) \propto 1/\epsilon^{d_f}$]

The dense O(n) loop model

- Dense loops configurations on the square lattice:



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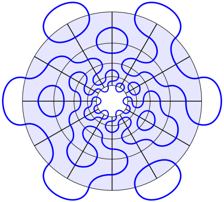

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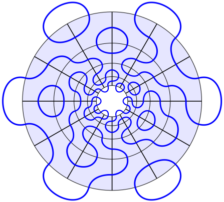

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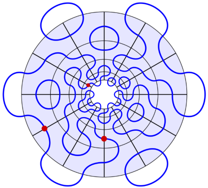
[Example: at $n = 1$, loops $\equiv$ percolation cluster contours]

- There exists a dilute variant $\Rightarrow$ critical polymers, domain walls, . . .

Connectivity operators

- Examples of correlation functions:
  - $\mathbb{P}[r_1, \ldots r_p$ sit on the same loop$] =$ 
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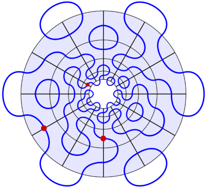
  - $\mathbb{P}[r_1, \ldots r_p$ sit on the same loop$] =$ 
  - $\mathbb{P}[r_1, \ldots r_p$ sit on the same FK cluster], etc.
- "Local operator" $=$ insertion of a marker (like $\bullet$)

Theories for loop correlations

- Description of the operator spectrum

	generating alg.	**operator content**
lattice	periodic TL	(quotients of) standard modules
continuum	$\mathrm{Vir} \otimes \overline{\mathrm{Vir}}$	discrete set of primary ops.

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- Universal correlation functions? $\langle \phi_1(r_1) \ldots \phi_p(r_p) \rangle$

2. OPE structure constants

Scaling theory of the O(n) loop model

[Nienhuis 84, Dotsenko-Fateev 84, Di Francesco-Saleur-Zuber 87, Nienhuis-Foda 89]

- "Coulomb-Gas" = Imaginary Liouville action, compact field

$$A[\phi] = \int d^2r \, \frac{\sqrt{g}}{4\pi} \left(\partial_\mu \phi \partial^\mu \phi + iQ\mathcal{R}\phi + \kappa \, e^{i\phi/b} \right) , \quad \phi \equiv \phi + 2\pi b$$

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- Sector of defect charge $\delta\phi = 2\pi bm$, with $m \in \mathbb{Z}^\times$:
  $\widehat{\Phi}_{em}$ with $(h, \bar{h}) = (h_{em}, h_{e,-m})$ for $e \in \mathbb{Z}/m$

[Kac notation: $h_{rs} = \frac{(r/b - sb)^2 - (1/b - b)^2}{4}$]

OPEs for operators in the discrete spectrum

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- Results from conformal bootstrap on 4-pt functions:

$$\boxed{\begin{aligned}
&C(\varepsilon_j, \varepsilon_j, \varepsilon_{2k}) = c_{j+1,0,k} \\
&C(\widehat{\Phi}_{em}, \widehat{\Phi}_{e,-m}, \varepsilon_{2k}) = \sqrt{c_{e,m,k}\, c_{e,-m,k}}
\end{aligned}}$$

with: $$c_{j,m,k} = \prod_{\ell=1}^{k} \frac{\gamma(\rho\ell-m)\gamma(\rho\ell+m)\sqrt{\gamma[2-\rho(2\ell-1)]\gamma[2-\rho(2\ell+1)]}}{\gamma[2-\rho(j+\ell)]\gamma[\rho(j-\ell)]\gamma[\rho(2\ell-1)]}$$

$$\rho = b^{-2}\,, \qquad \gamma(x) = \Gamma(x)/\Gamma(1-x)\,.$$

The case of loop-weighting operators

[Delfino and Viti, J. Phys. A 44, 032001 (2011)]

- FK cluster connectivity:

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  $\langle \ldots V_\alpha(r_j) \ldots \rangle$ gives weight $n_\alpha = 2\cos 2\pi b(Q-\alpha)$ to loops encircling only r_j.
  Conformal dimensions $h = \bar{h} = h_\alpha = \alpha(\alpha - Q)$
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  FK "Spin" operator : V_{α_σ} with $\alpha_\sigma = Q + b^{-1}/4$
- DV's argument, supported by Monte-Carlo on percolation:

$$\mathbb{P}_{\text{cluster}}[r_1, r_2, r_3] = \langle V_{\alpha_\sigma}(r_1) V_{\alpha_\sigma}(r_2) V_{\alpha_\sigma}(r_3) \rangle_{O(n)}$$
$$= \frac{C_{\text{IL}}(\alpha_\sigma, \alpha_\sigma, \alpha_\sigma)}{|r_1 - r_2|^{2h_\sigma} \, |r_2 - r_3|^{2h_\sigma} \, |r_1 - r_3|^{2h_\sigma}}$$

$C_{\text{IL}}(\alpha_1, \alpha_2, \alpha_3)$ = OPE constants for imaginary Liouville CFT
[Zamolodchikov '05, Kostov-Petkova '06]

Extension of the Delfino-Viti result

[YI, J. L. Jacobsen, and H. Saleur, PRL 116, 130601 (2016)]

- Define three-point function $Z_{n_1,n_2,n_3}(r_1,r_2,r_3) = \sum_{\text{configs}} n^{\ell_0}\, n_1^{\ell_1}\, n_2^{\ell_2}\, n_3^{\ell_3}$

$\ell_0 = \#$ trivial loops
$\ell_1 = \#$ loops encircling only r_1
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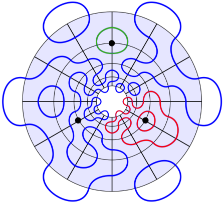

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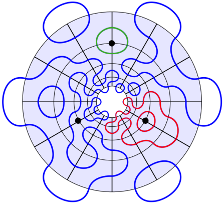
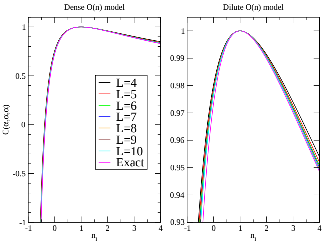
$$\begin{aligned} \ell_0 &= \#\ \text{trivial loops} \\ \ell_1 &= \#\ \text{loops encircling only } r_1 \\ \ell_2 &= \#\ \text{loops encircling only } r_2 \\ \ell_3 &= \#\ \text{loops encircling only } r_3 \end{aligned}$$



- Numerics on the cylinder: Z_{n_1,n_2,n_3} matches $C_{\mathrm{IL}}(\alpha_1,\alpha_2,\alpha_3)$ on large range of b and α_j's.

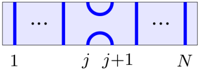


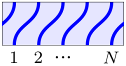
3. Fusion on the lattice

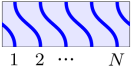
Background on the periodic Temperley-Lieb algebra (1/3)

Basic facts

- Generators of $\mathrm{PTL}_N(n)$

$e_j =$ 

$\Omega =$ 

$\Omega^{-1} =$ 

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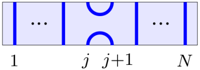
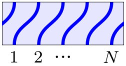
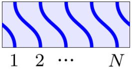
$e_j =$ 

$\Omega =$ 

$\Omega^{-1} =$ 

- Relations

$$e_j^2 = n\, e_j\,, \qquad e_j e_{j\pm 1} e_j = e_j\,, \qquad e_i e_j = e_j e_i \quad |i-j|>1\,,$$

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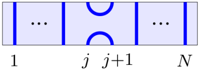
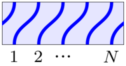
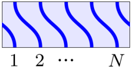
$e_j =$ 

$\Omega =$ 

$\Omega^{-1} =$ 

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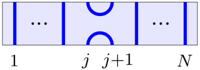
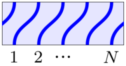
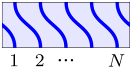
$e_j =$ 

$\Omega =$ 

$\Omega^{-1} =$ 

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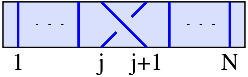
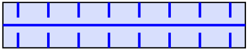
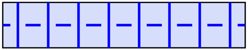
- Braid operators $b_j =$ 

$:= q^{1/2}1 + q^{-1/2} e_j$

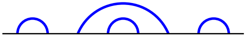
- Central elements

$F =$  $\bar{F} =$ 

Background on the periodic Temperley-Lieb algebra (2/3)

Link-state representations

- Vacuum module $V(N)$: simple link states

[Ex: ]

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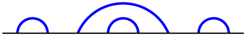
[Ex: ]

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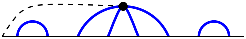
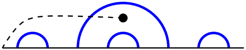
[Ex:  , ]

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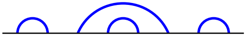
  [Ex: ]

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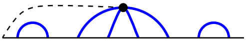
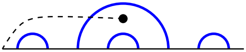
  [Ex:  , ]

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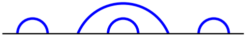
  [Ex: ]

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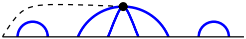
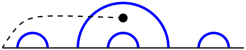
  [Ex:  , ]

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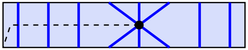
$\mathcal{O}_{k,z}(j) =$ 

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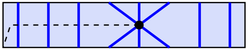
$\mathcal{O}_{k,z}(j) =$ 

- How to define the fusion $W_{k,x} \times W_{\ell,y}$?
  [Gainutdinov-Jacobsen-Saleur 16-18]
  [Belletête–Saint-Aubin 18]

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[YI–Morin-Duchesne 21]

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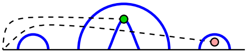
[Ex: ]

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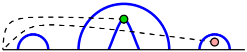
  [Ex: ]

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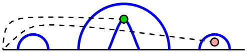
  [Ex: ]

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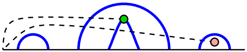
  [Ex: ]

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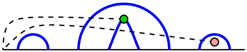
  [Ex: ]

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- **Result 2**: For z generic [and q not a root of unity]:

$$X_{k,\ell,x,y,z}(N) \simeq W_{k-\ell,z}(N) \oplus \bigoplus_{m=k-\ell+1}^{N/2} \bigoplus_{r=0}^{2m-1} W_{m,z^{(k-\ell)/m}e^{i\pi r/m}}(N)$$

for $k \geq \ell$. [Proof based on the properties of $F, \bar{F}$.]

Correlation functions

- Example four-point function of connectivity operators on an infinite cylinder of circumference N with $k \neq \ell$:

$$G = \langle \mathcal{O}_{k,x}(r_1) \mathcal{O}_{\ell,y}(r_2) \mathcal{O}_{\ell,y}(r_3) \mathcal{O}_{k,x}(r_4) \rangle_{\mathrm{cyl}}$$

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- "PTL block" decomposition [from structure of $X_{k,\ell,x,y,1}(N)$]:

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$$G_{m,\omega} = \sum_j \langle \mathcal{O}_{\ell,y}(r_2)\mathcal{O}_{k,x}(r_1)v, u_{m,\omega,j}\rangle\langle u_{m,\omega,j}, \mathcal{O}_{\ell,y}(r_3)\mathcal{O}_{k,x}(r_4)v\rangle$$

  $\{u_{m,\omega,j}\}$: orthonormal basis of $W_{m,\omega} \subset X_{k,\ell,x,y,1}$.

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Thank you for your attention!