Bethe algebras and quantum separation of variables

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General aims for quantum integrable models:

- Algebraic tools to construct integrable models, i.e., having enough commuting conserved charges (Yang-Baxter algebra, Quantum groups,...)
- Methods to solve their spectrum (Bethe ansatz, Algebraic Bethe ansatz, ..., SoV, ...)
- Methods to compute their form factors, correlation functions, spectral functions,...

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Let's go!

The basic ingredient: a complete set of commuting conserved charges

- In classical case: complete set of independent conserved charges in involution > Liouville-Arnold theorem > action-angle variables with the intermediate fundamental simplifying step of separation of variables (in practice, soluble cases are those where we know separation of variables (SoV))
- In quantum case: no analogue of Liouville-Arnold theorem, but SoV scheme initiated by Sklyanin (separate variables from the Yang-Baxter algebra generators); works well for rank one algebras but problems show up for higher rank cases
- Purpose of my seminar: present a new approach to quantum SoV based on the knowledge of a *complete* set of conserved charges and the knowledge of their commutative (Bethe) algebra, in particular the corresponding structure constants

Based on a series of works with Giuliano Niccoli and more recently Louis Vignoli

Outline

Quantum separation of variables bases

• $T(\lambda)$ a one-parameter family of commuting conserved charges acting in $\mathcal{H}$ of finite dimension $d_{\mathcal{H}}$

• A set $\langle y_1, ..., y_{\mathrm{N}}|$, with $y_n = 0, 1, ..., d_n - 1$, $\prod_{n=1}^{N} d_n = d_{\mathcal{H}}$, is a **separation of variables basis** for $T(\lambda)$ if it is a basis of $\mathcal{H}^*$ and if all common eigenstates $|t\rangle$ of $T(\lambda)$, with eigenvalues $t(\lambda)$, have separate wave functions in that basis:

$$\langle y_1, ..., y_{\mathrm{N}}|t\rangle = \prod_{n=1}^{\mathrm{N}} \mathrm{Q}_t^{(n)}(y_n)$$

with $Q_t^{(n)}(\lambda)$ solutions of discrete difference equations of order d_n in y_n (with proper b.c.):

$$F_n[y_n, D_n^{\pm}, t(y_n)]\, Q_t^{(n)}(y_n) = 0, \quad \text{for all} \;\; n \in \{1, ..., \mathrm{N}\},$$

with $D_n^{\pm}$ the positive and negative shifts on the variable y_n.

• Proof of completeness of the $T(\lambda)$-spectrum needs the action of $T(\lambda)$ on the basis $\langle y_1, ..., y_{\mathrm{N}}|$ to be determined through the above discrete difference equations.

• The N quantum separate relations are the natural quantum analogue of the classical ones in the Hamilton-Jacobi's approach.

⇒ Construct separate bases $\langle y_1, ..., y_{\mathrm{N}}|$ for transfer matrices of quantum integrable models

Sov for quantum integrable models

- A quantum version of SoV invented by E. Sklyanin (1985) in an impressive series of works and applied to some important integrable quantum models, like Toda model and XXZ spin chain. The key point is to construct a couple of commuting operator families, $B(\lambda)$ and $A(\lambda)$, the separate basis being the eigenbasis of $B(\lambda)$ while the shift operator acting on it is obtained from $A(\lambda)$.
- Applied to various rank one models: Gutzwiller, Kharchev, Lebedev, Babelon, Smirnov (Toda model), Babelon, Bernard and Smirnov (sine-Gordon model), Derkachov, Korchemsky and Manashov (non-compact XXX chain), Lukyanov, Bytsko and Teschner (sinh-Gordon model), von Gehlen, Iorgov, Pakuliak, and Shadura (tau-2 and Bazhanov-Stroganov model), Frahm, Grelik, Seel and Wirth (SoV functional version for spin 1/2 XXX chain) etc.
- Generalization of Sklyanin approach to a large variety of compact integrable quantum models for rank one in a series of works by Niccoli, Grosjean, Maillet, Faldella, Kitanine, Levy-Bencheton, Terras, Pezelier.
- SoV for higher rank case more problematic: Introduced by Sklyanin for $Y(gl_3)$ (1996), see also Smirnov for $U_q(sl_n)$ (2001). But some problems already for fundamental representations (shift operator, quantum spectral curve,...)
- Interesting conjectures on SoV spectrum for higher rank by Gromov et al (2016), see Liashyk and Slavnov (2018) for NABA proof of Gromov et al conjecture on transfer matrix eigenvectors and by Volin et al (2019) using the new framework we developed for SoV bases (M, Niccoli) in 2018.
- **New quantum SoV method based on conserved charges only (M, Niccoli-2018)**

SoV bases from transfer matrices - I

Definition: a family of commuting conserved charges $T(\lambda)$, $\lambda \in \mathbb{C}$, acting on the Hilbert space $\mathcal{H}$ of finite dimension $d = \prod_{n=1}^{N} d_n$ is "basis generating" if there exist $\langle L| \in \mathcal{H}^*$ and sets of conserved charges $T_{h_i}^{(i)}$ constructed from $T(\lambda)$, $i = 1, \ldots, N$ and $h_i = 0, \ldots, d_i - 1$, such that the following set is a basis of $\mathcal{H}^*$:

$$\langle h_1, ..., h_N| \equiv \langle L| \prod_{a=1}^{N} T_{h_a}^{(a)}$$

⇒ In all known quantum integrable lattice models we have been considering so far the above set is provided by the transfer matrix itself or by the hierarchy of fused transfer matrices. The spectrum is then determined by the complete set of fusion relations for the hierarchy of transfer matrices.

In particular for gl_n quasi-periodic higher rank models in fundamental representations the set:

$$\langle h_1, ..., h_N| \equiv \langle L| \prod_{a=1}^{N} T(\xi_a)^{h_a} \ , \ \{h_1, ..., h_N\} \in \{0, ..., n-1\}^{\otimes N}$$

is a basis of $\mathcal{H}^*$ for almost any choice of $\langle L|$ and of the inhomogeneity parameters $\{\xi_1, ..., \xi_N\}$ if the matrix $K \in End(\mathbb{C}^n)$ has simple spectrum. In particular, the covector $\langle L|$ can be chosen as a pure tensor product $\langle L| \equiv \bigotimes_{a=1}^{N} \langle L_a|$.

SoV bases from transfer matrices - II

$$\langle h_1, ..., h_N| \equiv \langle L| \prod_{a=1}^{N} T_{h_a}^{(a)} \;, \; \{h_1, ..., h_N\} \in \{0, ..., n-1\}^{\otimes N}$$

- The wave function $\Psi_t(h_1, ..., h_N)$ of any common eigenvector $|\, t\,\rangle$ of the commuting family $T(\lambda)$ factorizes in terms of one variable wave functions given by eigenvalues $t_{h_a}^{(a)}$:

$$\Psi_t(h_1, ..., h_N) \equiv \langle h_1, ..., h_N|t\rangle = \langle L|t\rangle \prod_{a=1}^{N} t_{h_a}^{(a)}$$

- As eigenvector's coordinates in the basis $\langle h_1, ..., h_N|$ are given in terms of eigenvalues, there is (up to normalization) a unique eigenvector corresponding to an eigenvalue $t(\lambda)$, hence, **the common spectrum of the family of conserved charges $T(\lambda)$ is simple**.

Matrices having simple spectrum....

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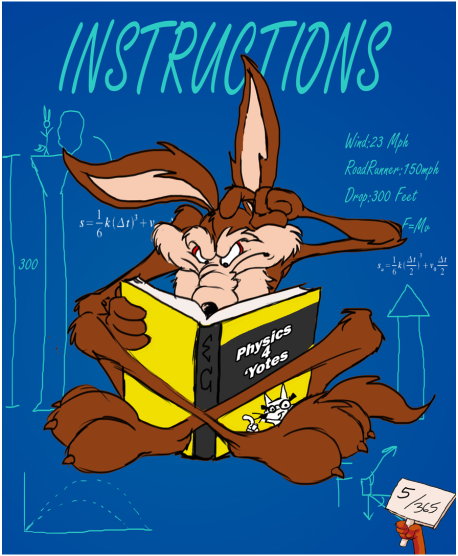

Matrices having simple spectrum-I

- If a $d \times d$ matrix X has simple spectrum then to any eigenvalue, root of its characteristic polynomial $P_X(t) = a_0 + ta_1 + t^2 a_2 + \cdots + t^{d-1} a_{d-1} + t^d$ corresponds a unique eigenvector array.
- Such a matrix is similar to its companion matrix C:

$$V_X X V_X^{-1} = C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{d-2} & -a_{d-1} \end{pmatrix}$$

- In the canonical basis $\langle e_j|$, $j = 1, \ldots, d$, $\langle e_j|C = \langle e_{j+1}|$ for any $j = 1, \ldots, d-1$, making the set $\langle e_1|C^k$, $k = 0, 1, \ldots, d-1$ a basis.
- Hence defining $\langle f_j| = \langle e_j|V_X$, the set $\langle f_1|X^k$, , $k = 0, 1, \ldots, d-1$, forms a basis. $\langle f_1|$ is called a cyclic vector.

Matrices having simple spectrum-II

- For any eigenvalue λ, the unique (up to trivial scalar multiplication) eigenvector $|\Lambda\rangle$ has coordinates:

$$\langle f_n|\Lambda\rangle = \lambda^{n-1}$$

  To prove this, the set $\langle f_n|$ being a basis, we need to show that $\langle f_n|X|\Lambda\rangle = \lambda\langle f_n|\Lambda\rangle$, $1 \leq n \leq d$.

- If $n \leq d-1$, $\langle f_n|X|\Lambda\rangle = \langle f_{n+1}|\Lambda\rangle = \lambda^n\langle f_1|\Lambda\rangle = \lambda\langle f_n|\Lambda\rangle$.

- For $n = d$, $\langle f_d|X = \langle f_1|X^d$ is not a vector of the basis, but we know that $P_X(X) = 0$, hence X^d decomposes on lower powers of X:

$$\langle f_d|X|\Lambda\rangle = \langle f_1|X^d|\Lambda\rangle = -\langle f_1|\sum_{n=0}^{d-1} a_n X^n|\Lambda\rangle = -\langle f_1|\Lambda\rangle\sum_{n=0}^{d-1} a_n\lambda^n = \langle f_1|\Lambda\rangle\lambda^d = \lambda\langle f_d|\Lambda\rangle$$

  using $P_X(X) = 0$ and $P_X(\lambda) = 0$, $P_X(t) = a_0 + ta_1 + t^2a_2 + \cdots + t^{d-1}a_{d-1} + t^d$.

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- The vector space $\mathcal{C}_X$ of matrices commuting with X has dimension d with basis X^k, $0 \le k \le d-1$. Hence any matrix commuting with X can be decomposed linearly on that basis. The commutative algebra $\mathcal{C}_X$ has structure constants given by the coefficients of the characteristic polynomial.

SoV bases from transfer matrices - III

- If the set $\langle h_1, ..., h_N| \equiv \langle L| \prod_{a=1}^{N} T_{h_a}^{(a)}$ is a basis of $\mathcal{H}^*$ then the set $T_{\mathbf{h}} \equiv \prod_{a=1}^{N} T_{h_a}^{(a)}$ with $\mathbf{h} = (h_1, ..., h_N)$ is a basis of the Bethe algebra $\mathcal{C}_{T(\lambda)}$ considered as a vector space.
- Therefore there exist sets $C_{\mathbf{h}\mathbf{h}'}(\lambda)$ and $C_{\mathbf{h}\mathbf{h}'}^{\mathbf{h}''}$ depending sets of parameters $\mathbf{h}$, $\mathbf{h}'$, $\mathbf{h}''$ and on λ giving the closure relations:

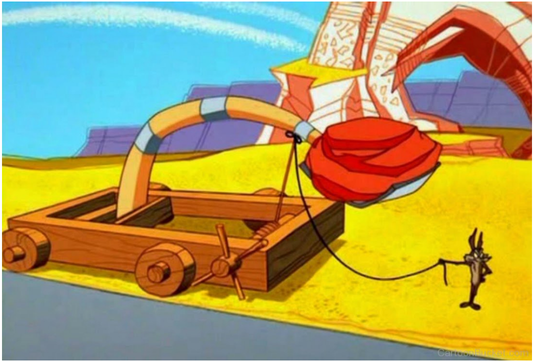
$$T_{\mathbf{h}} \cdot T(\lambda) = \sum_{\mathbf{h}'} C_{\mathbf{h}\mathbf{h}'}(\lambda)\ T_{\mathbf{h}'}$$

  and the $\mathcal{C}_{T(\lambda)}$ commutative algebra structure constants:

$$T_{\mathbf{h}} \cdot T_{\mathbf{h}'} = \sum_{\mathbf{h}''} C_{\mathbf{h}\mathbf{h}'}^{\mathbf{h}''}\ T_{\mathbf{h}''}$$

- These relations hold also for the transfer matrix eigenvalues and enable to compute the action of the transfer matrix on the SoV basis. They characterize the complete transfer matrix spectrum.
- These relations are in general direct consequences of the fusion relations satisfied by the transfer matrices, which therefore also fully characterize the spectrum. In all considered cases they are equivalent to a quantum spectral curve equation.
- Link to the N. Reshetikhin very nice idea of analytic Bethe ansatz. But here it's no longer an ansatz.
- **Key question: what is the optimal choice for the set of conserved charges to construct the above bases? For the gl_2 case it is given by Baxter Q operator. Higher rank cases seem more involved.**
- Related recent works by D. Volin and collaborators (2020, 2021).

SoV bases...



SoV bases for quasi-periodic $Y(gl_n)$ fundamental models

We consider the gl_n invariant R-matrix: $R_{ab}(\lambda_a - \lambda_b) = (\lambda_a - \lambda_b)\mathbb{I}_{ab} + \eta\mathbb{P}_{ab}$. Then any matrix $K \in (\mathbb{C}^n)$ satisfies:

$$R_{ab}(\lambda_a - \lambda_b)K_a K_b = K_b K_a R_{ab}(\lambda_a - \lambda_b)$$

The K-twisted monodromy matrix of an integrable models in $\mathcal{H} \equiv \otimes_{l=1}^{N} V_l$ of dimension $d = n^N$:

$$M_a^{(K)}(\lambda, \{\xi_1, ..., \xi_N\}) \equiv K_a R_{aN}(\lambda - \xi_N) \cdots R_{a1}(\lambda - \xi_1)$$

satisfies the Yang-Baxter algebra and defines a commuting family of transfer matrices:

$$T^{(K)}(\lambda, \{\xi\}) \equiv tr_{V_a} M_a^{(K)}(\lambda, \{\xi\})$$

The gl_n SoV basis:

$$\langle h_1, ..., h_N| \equiv \langle L| \prod_{a=1}^{N} (T^{(K)}(\xi_a, \{\xi\}))^{h_a} \ \forall \{h_1, ..., h_N\} \in \{0, ..., n-1\}^{\otimes N}$$

is a covector basis of $\mathcal{H}^*$ for almost any choice of the covector $\langle L|$ and of the inhomogeneity parameters $\{\xi_1, ..., \xi_N\}$ if $K \in End(\mathbb{C}^n)$ has simple spectrum. In particular we can chose:

$$\langle L| \equiv \bigotimes_{a=1}^{N} \langle L_a|$$

with $\langle L_a|$ a local covector in V_a^* such that

$$\langle L_a| K_a^h \text{ with } h \in \{0, ..., n-1\}$$

is a covector basis for V_a for any $a \in \{1, ..., N\}$. Moreover if K is diagonalizable with simple spectrum then $T^{(K)}(\lambda, \{\xi\})$ has the same property. The proofs use the polynomial properties of all the objects involved.

SoV basis for quasi-periodic $Y(gl_2)$ models - I

The basis writes for chain with arbitrary spin-s_n representation in different lattice sites $n = 1, \ldots, \mathrm{N}$.

$$\langle h_1, ..., h_{\mathrm{N}}| \equiv \langle O| \prod_{n=1}^{\mathrm{N}} \frac{\mathrm{T}^{(K|h_n)}(\xi_n^{(h_n-1)})}{\mathrm{k}_1^{h_n} \prod_{k=0}^{h_n-1} a(\xi_n^{(k)})} \quad \forall h_a \in \{0, ..., 2s_a\}, a \in \{1, ..., \mathrm{N}\}$$

$$a(\lambda) = \prod_{n=1}^{\mathrm{N}} \left(\lambda - \xi_n^{(2s_n)}\right) \text{ and } d(\lambda) = \prod_{n=1}^{\mathrm{N}} \left(\lambda - \xi_n^{(0)}\right)$$

k_1 and k_2 the eigenvalues of the 2×2 twist matrix K, $\mathrm{k}_1 \neq \mathrm{k}_2$ and $\mathrm{k}_1.\mathrm{k}_2 \neq 0$, so $K \neq \alpha I$ and $\xi_n^{(k_n)} \equiv \xi_n - \eta/2 + (s_n - k_n)\eta$.

Fused transfer matrices

$$\mathrm{T}^{(K|l+1)}(\lambda) = \mathrm{T}^{(K)}(\lambda + l\eta)\mathrm{T}^{(K|l)}(\lambda) - \Delta_\eta^{(K)}(\lambda + l\eta)\mathrm{T}^{(K|l-1)}(\lambda)$$

Quantum determinant

$$\Delta_\eta^{(K)}(\lambda) = \mathrm{k}_1\mathrm{k}_2 a(\lambda) d(\lambda - \eta)$$

It coincides with Sklyanin's SoV basis (when it exists) for a special choice of the co-vector $\langle O|$

Action of the transfer matrix on the basis

$$\langle h_1, ..., h_{\mathrm{N}}|\mathrm{T}^{(K)}(\xi_n^{(h_n)}) = \mathrm{k}_1 a(\xi_n^{(h_n)})\langle h_1, ..., h_n + 1, ..., h_{\mathrm{N}}| + \mathrm{k}_2 d(\xi_n^{(h_n)})\langle h_1, ..., h_n - 1, ..., h_{\mathrm{N}}|$$

SoV basis for quasi-periodic $Y(gl_2)$ models - II

Spectrum of $\mathsf{T}^{(K)}(\lambda)$ coincides with the set of polynomials:

$$\Sigma_{\mathsf{T}^{(K)}} = \left\{ t(\lambda) : t(\lambda) = (\mathsf{k}_1 + \mathsf{k}_2) \prod_{a=1}^{\mathsf{N}} (\lambda - \xi_n^{(0)}) + \sum_{a=1}^{\mathsf{N}} g_a(\lambda) x_a, \quad \forall \{x_1, ..., x_{\mathsf{N}}\} \in D_{\mathsf{T}^{(K)}} \right\},$$

$g_a(\lambda) = \prod_{b \neq a, b=1}^{N} \frac{\lambda - \xi_b}{\xi_a - \xi_b}$ and $D_{\mathsf{T}^{(K)}}$ is the set of N-tuples $\{x_1, ..., x_{\mathsf{N}}\}$ solutions to :

$$t^{(K|2s_n+1)}(\xi_n^{(2s_n)}) = 0, \quad \forall n \in \{1, ..., \mathsf{N}\},$$

each of which is a degree $2s_n + 1$ polynomial equation in the N unknowns. Existence of Q-operator with degree less than N satisfying $T - Q$ equation is equivalent to the tower of fusion relations:

$$\mathsf{T}^{(K)}(\lambda) Q(\lambda) = \mathsf{k}_1 a(\lambda) Q(\lambda - \eta) + \mathsf{k}_2 d(\lambda) Q(\lambda + \eta)$$

Basis from Q-operator:

$$\langle h_1, ..., h_{\mathsf{N}} | = \langle L | \prod_{a=1}^{\mathsf{N}} \mathsf{Q}(\xi_a^{(h_a)})$$

Gives a clear understanding of the linear action of $\mathsf{T}^{(K)}(\lambda)$ on the basis: the spectrum characterization in terms of $T - Q$ equation also serves as closure relation for the action of $\mathsf{T}^{(K)}(\lambda)$ on our SoV basis

$$\mathsf{T}^{(K)}(\xi_a^{(h_a)}) Q(\xi_a^{(h_a)}) = \mathsf{k}_1 a(\xi_a^{(h_a)}) Q(\xi_a^{(h_a+1)}) + \mathsf{k}_2 d(\xi_a^{(h_a)}) Q(\xi_a^{(h_a-1)}) .$$

Properties of transfer matrices for $Y(gl_3)$

The commuting fused transfer matrices:

$$T_1^{(K)}(\lambda) \equiv \text{tr}_a M_a^{(K)}(\lambda) \quad T_2^{(K)}(\lambda+\eta) \equiv 3\text{tr}_{abc} P_{abc}^{-} M_a^{(K)}(\lambda) M_b^{(K)}(\lambda+\eta)$$

The quantum determinant is central:

$$q\text{-det}M^{(K)}(\lambda) \equiv \text{tr}_{123}(P_{123}^{-} M_1^{(K)}(\lambda) M_2^{(K)}(\lambda+\eta) M_3^{(K)}(\lambda+2\eta))$$

The quantum spectral invariants have the following polynomial form:
i) $T_1^{(K)}(\lambda)$ is a degree N polynomial in λ with tr$K$ as leading coefficient.
ii) $T_2^{(K)}(\lambda)$ is a degree $2N$ polynomial in λ with the following N central zeros and asymptotic:

$$T_2^{(K)}(\xi_a+\eta) = 0, \ \lim_{\lambda\to\infty} \lambda^{-2N} T_2^{(K)}(\lambda) = \frac{(\text{tr}K)^2 - \text{tr}K^2}{2}$$

iii) the quantum determinant $q\text{-det}M^{(K)}(\lambda) = \det K \ \prod_{b=1}^{N}(\lambda-\xi_b)(\lambda+\eta-\xi_b)(\lambda+3\eta-\xi_b)$

Fusion identities:

$$\begin{aligned} T_1^{(K)}(\xi_a)T_2^{(K)}(\xi_a-\eta) &= q\text{-det}M^{(K)}(\xi_a-2\eta) \\ T_1^{(K)}(\xi_a-\eta)T_1^{(K)}(\xi_a) &= T_2^{(K)}(\xi_a) \end{aligned}$$

Then, defining $f_{a,\mathbf{h}}(\lambda) = g_{a,\mathbf{h}}(\lambda)\prod_{b=1}^{N} \frac{\lambda-\xi_b}{\xi_a^{(ha)}-\xi_b}$ and $g_{a,\mathbf{h}}(\lambda) = \prod_{b\neq a, b=1}^{N} \frac{\lambda-\xi_b^{(h)}}{\xi_a^{(h)}-\xi_b^{(h)}}$, $\xi_b^{(h)} = \xi_b - h\eta$:

$$T_2^{(K)}(\lambda+\eta) = T_{2,\mathbf{h}=\mathbf{1}}^{(K,\infty)}(\lambda+\eta) + \sum_{a=1}^{N} f_{a,\mathbf{h}=\mathbf{1}}(\lambda) T_1^{(K)}(\xi_a-\eta) T_1^{(K)}(\xi_a)$$

The transfer matrix spectrum for $Y(gl_3)$

The spectrum of $T_1^{(K)}(\lambda)$ is characterized by:

$$\Sigma_{T^{(K)}} = \left\{ t_1(\lambda) : t_1(\lambda) = \text{tr}\, K \prod_{a=1}^{N} (\lambda - \xi_a) + \sum_{a=1}^{N} g_{a,\mathbf{h}=0}(\lambda) x_a, \;\; \forall \{x_1, ..., x_N\} \in \Sigma_T \right\},$$

Σ_T is the set of solutions to the following inhomogeneous system of N cubic equations:

$$x_a [T_{2,\mathbf{h}=\mathbf{1}}^{(K,\infty)}(\xi_a - \eta) + \sum_{n=1}^{N} f_{n,\mathbf{h}=\mathbf{1}}(\xi_a - 2\eta) t_1(\xi_n - \eta) x_n] = q\text{-det} M^{(K)}(\xi_a - 2\eta),$$

in N unknown $\{x_1, ..., x_N\}$. Moreover, $T_1^{(K)}(\lambda, \{\xi\})$ has simple spectrum and for any $t_1(\lambda) \in \Sigma_{T^{(K)}}$ the associated unique (up-to normalization) eigenvector $|t\rangle$ has the following wave-function in the left SoV basis:

$$\langle h_1, ..., h_N | t \rangle = \prod_{n=1}^{N} t_1^{h_n}(\xi_n)$$

Action of the transfer matrix using the fusion relations until we get the quantum determinant which acts trivially on any covector. We use interpolation formulae for the transfer matrix and the fact that the same fusion relations and interpolation formulae are true for the eigenvalues of the transfer matrices, hence giving the possibility to reverse the process and to reconstruct it in the necessary points (very much like the use of the characteristic polynomial in standard matrix case).

Quantum spectral curve

Let us assume that the twist matrix K has at least one nonzero eigenvalue γ_0 then the entire functions $t_1(\lambda)$ is a $T_1^{(K)}(\lambda)$ transfer matrix eigenvalue if and only if there exists a unique polynomial:

$$Q_t(\lambda) = \prod_{a=1}^{\mathrm{M}} (\lambda - \lambda_a) \text{ with } \mathrm{M} \leq N \text{ and } \lambda_a \neq \xi_n \ \forall (a, n) \in \{1, ..., \mathrm{M}\} \times \{1, ..., N\}$$

satisfying with

$$t_2(\lambda + \eta) = T_{2,\mathbf{h}=\mathbf{1}}^{(K,\infty)}(\lambda + \eta) + \sum_{n=1}^{N} f_{n,\mathbf{h}=\mathbf{1}}(\lambda) t_1(\xi_n - \eta) t_1(\xi_n),$$

the following quantum spectral curve equation:

$$\alpha(\lambda) Q_t(\lambda - 3\eta) - \beta(\lambda) t_1(\lambda - 2\eta) Q_t(\lambda - 2\eta)$$
$$+\gamma(\lambda) t_2(\lambda - \eta) Q_t(\lambda - \eta) - q\text{-det} M_a^{(K)}(\lambda - 2\eta) Q_t(\lambda) = 0$$

$$\begin{aligned} \alpha(\lambda) &= \gamma(\lambda)\gamma(\lambda - \eta)\gamma(\lambda - 2\eta) \\ \beta(\lambda) &= \gamma(\lambda)\gamma(\lambda - \eta) \\ \gamma(\lambda) &= \gamma_0 \prod_{a=1}^{N} (\lambda + \eta - \xi_a) \end{aligned}$$

with $\gamma_0^3 - \gamma_0^2 \operatorname{tr} K + \gamma_0 \frac{(\operatorname{tr} K)^2 - \operatorname{tr} K^2}{2} = \det K$. Moreover, up to a normalization the common transfer matrix eigenstate $|t\rangle$ admits the following separate representation:

$$\langle h_1, ..., h_N | t \rangle = \prod_{a=1}^{N} \gamma^{h_a}(\xi_a) Q_t^{h_a}(\xi_a - \eta) Q_t^{2-h_a}(\xi_a)$$

Scalar products - I

Choice of left and right SoV bases $(\{h_1, ..., h_{\mathsf{N}}\} \in \{0, 1, 2\}^{\mathsf{N}})$

$$\langle \underline{\mathbf{h}}| = \langle \underline{\mathbf{1}}| \prod_{n=1}^{\mathsf{N}} T_2^{(K)\delta_{h_n,0}}(\xi_n^{(1)}) T_1^{(K)\delta_{h_n,2}}(\xi_n) \quad \text{and} \quad |\underline{\mathbf{h}}\rangle \equiv \prod_{n=1}^{\mathsf{N}} T_2^{(K)\delta_{h_n,1}}(\xi_n) T_1^{(K)\delta_{h_n,2}}(\xi_n)|\underline{\mathbf{0}}\rangle$$

chosen (in tensor product form for Jordan form of K) such that $\langle \underline{\mathbf{k}}|\underline{\mathbf{0}}\rangle = \prod_{a=1}^{\mathsf{N}} \delta_{0,k_a}$

Scalar products

$$\mathcal{N}_{\underline{\mathbf{h}},\underline{\mathbf{k}}} = \langle \underline{\mathbf{h}}|\underline{\mathbf{k}}\rangle = \langle \underline{\mathbf{k}}|\underline{\mathbf{k}}\rangle \left(\delta_{\underline{\mathbf{h}},\underline{\mathbf{k}}} + C_{\underline{\mathbf{h}}}^{\underline{\mathbf{k}}} \sum_{r=1}^{n_{\underline{\mathbf{k}}}} (\det K)^r \sum_{\substack{\alpha\cup\beta\cup\gamma=\mathbf{1}_{\underline{\mathbf{k}}}, \\ \alpha,\beta,\gamma \text{ disjoint}, \#\alpha=\#\beta=r}} \delta_{\underline{\mathbf{h}},\underline{\mathbf{k}}_{\alpha,\beta}^{(\underline{\mathbf{0}},\underline{\mathbf{2}})}} \right),$$

$C_{\underline{\mathbf{h}}}^{\underline{\mathbf{k}}}$ non-zero constants defined recursively and independent w.r.t. det K, $n_{\underline{\mathbf{k}}} = [(\sum_{a=1}^{\mathsf{N}} \delta_{k_a,1})/2]$.

$$\mathrm{N}_{\underline{\mathbf{h}}} = \langle \underline{\mathbf{h}}|\underline{\mathbf{h}}\rangle = \left(\prod_{a=1}^{\mathsf{N}} \frac{d\left(\xi_a^{(1)}\right)}{d\left(\xi_a^{(1+\delta_{h_a,1}+\delta_{h_a,2})}\right)} \right) \frac{V^2(\xi_1, ..., \xi_{\mathsf{N}})}{V\left(\xi_1^{(\delta_{h_1,2}+\delta_{h_1,1})}, ..., \xi_{\mathsf{N}}^{(\delta_{h_{\mathsf{N}},1}+\delta_{h_{\mathsf{N}},2})}\right) V\left(\xi_1^{(\delta_{h_1,2})}, ..., \xi_{\mathsf{N}}^{(\delta_{h_{\mathsf{N}},2})}\right)}$$

See also related results by A. Cavaglia, N. Gromov, F. Levkovich-Maslyuk, P. Ryan, D. Volin.

Non-orthogonal SoV bases.....

Scalar products - II

⇒ Changing the set of conserved charges to get orthogonal bases

Let $\{|t_a^{(K)}\rangle,\ a \in \{1,..,3^{\mathsf{N}}\}\}$ be the eigenvector basis and let $\{\langle t_a^{(K)}|,\ a \in \{1,..,3^{\mathsf{N}}\}\}$ be the eigenco-vector basis associated to the transfer matrix $T_1^{(K)}(\lambda)$ (that has simple spectrum).

- The SoV measure becomes diagonal if det $K = 0$ while keeping simple spectrum for K
- Define $\mathbb{T}_j^{(K)}(\lambda) = \sum_{a=1}^{3^{\mathsf{N}}} t_{j,a}^{(\hat{K})}(\lambda) \frac{|t_a^{(K)}\rangle\langle t_a^{(K)}|}{\langle t_a^{(K)}|t_a^{(K)}\rangle}$, $\quad j \in \{1,2\}$, with simple spectrum $\hat{K}$ and det $\hat{K} = 0$

$$\left[\mathbb{T}_l^{(K)}(\lambda), \mathbb{T}_m^{(K)}(\lambda)\right] = \left[T_l^{(K)}(\lambda), \mathbb{T}_m^{(K)}(\lambda)\right] = 0 \quad l, m \in \{1,2\}$$

$$\mathbb{T}_2^{(K)}(\xi_a^{(1)})\mathbb{T}_1^{(K)}(\xi_a) = \mathbb{T}_2^{(K)}(\xi_a^{(1)})\mathbb{T}_2^{(K)}(\xi_a) = 0$$
$$\mathbb{T}_1^{(K)}(\xi_a^{(1)})\mathbb{T}_1^{(K)}(\xi_a) = \mathbb{T}_2^{(K)}(\xi_a)$$

Orthogonal SoV bases

$$\langle \widehat{\underline{\mathbf{h}}}| \equiv \langle \underline{\mathbf{1}}| \prod_{n=1}^{\mathsf{N}} \mathbb{T}_2^{(K)\delta_{h_n,0}}(\xi_n^{(1)})\mathbb{T}_1^{(K)\delta_{h_n,2}}(\xi_n), \quad \forall\ h_n \in \{0,1,2\},$$

$$|\widehat{\underline{\mathbf{h}}}\rangle \equiv \prod_{n=1}^{\mathsf{N}} \mathbb{T}_2^{(K)\delta_{h_n,1}}(\xi_n)\mathbb{T}_1^{(K)\delta_{h_n,2}}(\xi_n)|\underline{\mathbf{0}}\rangle, \quad \forall\ h_n \in \{0,1,2\}.$$

$$\langle \widehat{\underline{\mathbf{k}}}|\widehat{\underline{\mathbf{h}}}\rangle = \mathsf{N}_{\underline{\mathbf{h}}} \prod_{a=1}^{\mathsf{N}} \delta_{h_a,k_a}$$

Scalar products - III

⇒ Determinant representations of the scalar products of separate states

We have the following representation of the vector of the original transfer matrix $T_1^{(K)}(\lambda)$:

$$|t_a^{(K)}\rangle = \sum_{\underline{\mathbf{h}}} \prod_{n=1}^{\mathsf{N}} t_{2,a}^{(\hat{K})\delta_{h_n,0}}(\xi_n^{(1)}) t_{1,a}^{(\hat{K})\delta_{h_n,2}}(\xi_n) \, \frac{|\widehat{\mathbf{h}}\rangle}{\mathsf{N}_{\underline{\mathbf{h}}}}$$

Separate states have the form

$$\langle\alpha| = \sum_{\underline{\mathbf{h}}} \prod_{a=1}^{\mathsf{N}} \alpha_a^{(h_a)} \, \frac{\langle\widehat{\mathbf{h}}|}{\mathsf{N}_{\underline{\mathbf{h}}}},$$

there exists a permutation π_n of the set $\{1, ..., \mathsf{N}\}$ such that:

$$t_{1,n}^{(\hat{K})}(\xi_{\pi_n(b)}) = t_{2,n}^{(\hat{K})}(\xi_{\pi_n(a)} - \eta) = 0, \quad \forall (a,b) \in A \times B,$$

$$t_{1,n}^{(\hat{K})}(\xi_{\pi_n(a)}) \neq 0, \quad t_{2,n}^{(\hat{K})}(\xi_{\pi_n(b)} - \eta) \neq 0, \quad \forall (a,b) \in A \times B,$$

where we have defined:

$$A \equiv \{1, ..., \mathsf{M}_n\}, \quad B \equiv \{\mathsf{M}_n + 1, ..., \mathsf{N}\}.$$

$$\langle\alpha|t_n^{(K)}\rangle = \prod_{a=1}^{\mathsf{N}} \frac{d(\xi_a^{(2)})}{d(\xi_a^{(1)})} \frac{V\left(\xi_{\pi n(1)}^{(1)}, ..., \xi_{\pi n(\mathsf{M}_n)}^{(1)}\right)}{V\left(\xi_{\pi n(1)}, ..., \xi_{\pi n(\mathsf{M}_n)}\right)} \times \frac{\det_{\mathsf{N}-\mathsf{M}_n} \mathcal{M}_{+,\mathsf{N}-\mathsf{M}_n}^{(\alpha|x_A t_{2,n})}}{V\left(\xi_{\pi n(\mathsf{M}_n+1)}, ..., \xi_{\pi n(\mathsf{N})}\right)} \frac{\det_{\mathsf{M}_n} \mathcal{M}_{-,\mathsf{M}_n}^{(\alpha|x_B t_{1,n})}}{V\left(\xi_{\pi n(1)}, ..., \xi_{\pi n(\mathsf{M}_n)}\right)}$$

Other models

- Fundamental quasi-periodic $Y(gl_n)$ models
- Fundamental quasi-periodic $U_q(gl_n)$ models
- Higher-spin $Y(gl_2)$ models
- $Y(gl_n)$ models with integrable boundaries
- $Y(gl_{(m,n)})$ and Hubbard models (SoV bases, conjectures for the closure relations)

Some general features in these models:

- The transfer matrix indeed provides SoV bases
- There are several choices for the SoV bases
- The full spectrum can be characterized (needs closure relations)
- Quantum spectral curve equations from fusion relations
- Equivalent SoV basis from the Q-operator
- A construction of the Q-operator

Results

- ✔ SoV bases from complete sets of commuting conserved charges
- ✔ Complete spectrum with eigenvectors determined from eigenvalues
- ✔ SoV for higher rank models

Questions

- ⊛ Algebraic construction of "optimal" SoV bases having also orthogonality property
- ⊛ Extension of the notion of Q-operators in general models
- ⊛ Form factors and correlation function in this new SoV scheme

HAPPY BIRTHDAY HUBERT!