Thank you for the nice conference

XXZ at root of unity

COMMON STRUCTURES BETWEEN FINITE SYSTEMS AND CONFORMAL FIELD THEORIES THROUGH QUANTUM GROUPS

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Open XXZ Chain

$$\begin{aligned} H &= \sum_{j=1}^{L-1} \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + (\frac{q+q^{-1}}{2}) \sigma_j^z \sigma_{j+1}^z \\ &+ (\frac{q-q^{-1}}{2})(\sigma_1^z - \sigma_L^z) \end{aligned}$$

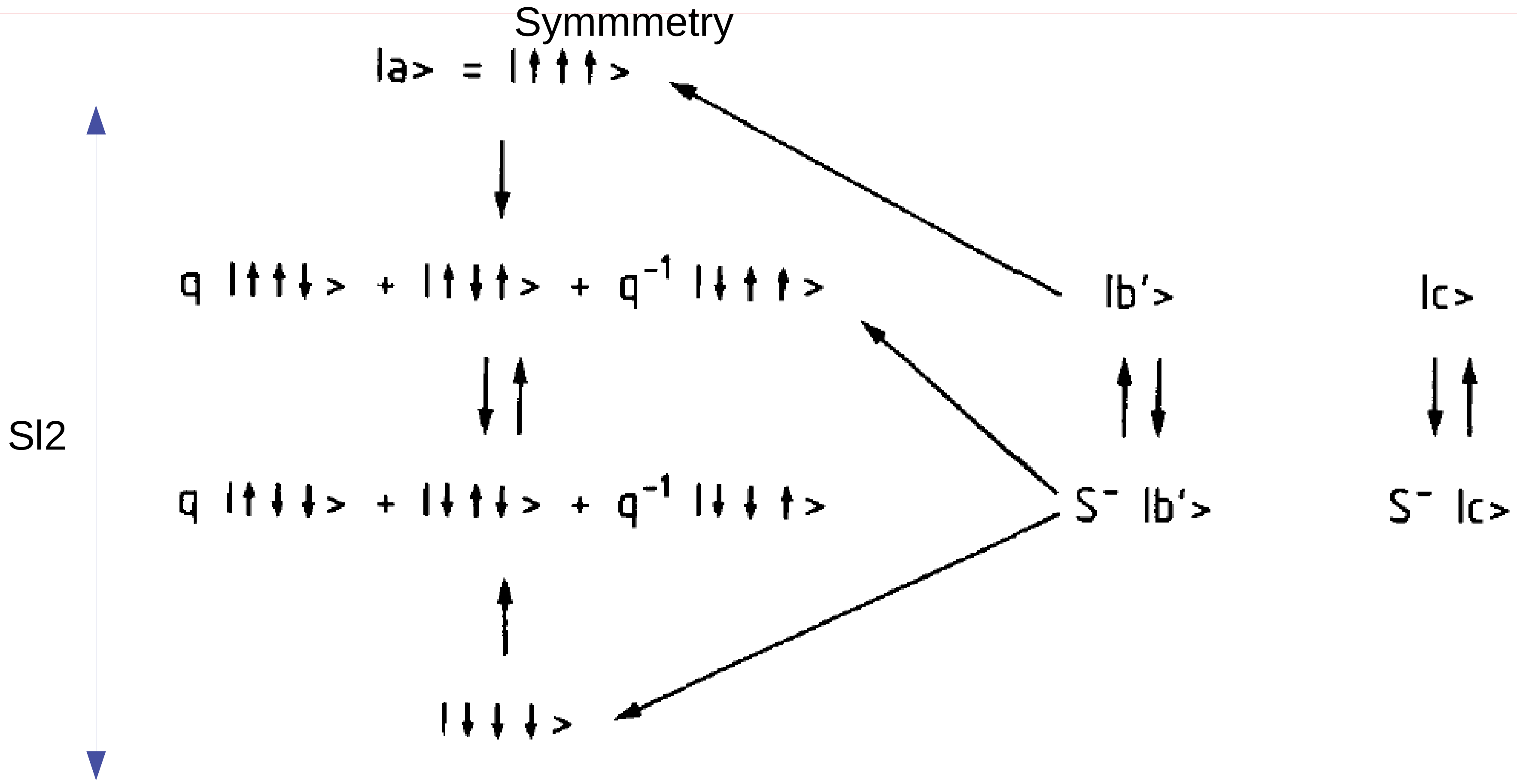


Fig. 2. Structure of $\mathbb{C}^6$ for $q = e^{i\pi/3}$.

Closed XXZ Chain

- With q a root of unity :

$$H = \sum_{j=1}^{L} \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + (\frac{q+q^{-1}}{2}) \sigma_j^z \sigma_{j+1}^z$$

$$\Delta = \frac{q+q^{-1}}{2} = \cosh \eta, \qquad q = e^{\eta}.$$

J.Lamers, Y.Miao VP
2012.10224

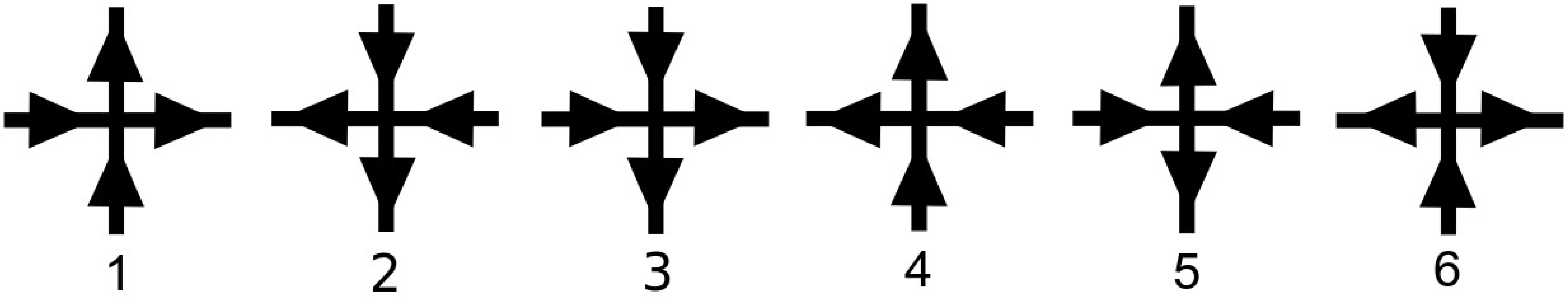
Fabricius McCoy

- Symmetry is now loop algebra
- Finite dimensional representations are tensor product of evaluation representations
- Tensor product of N spin 1/2 are Caracterized by Drinfeld polynomial :

$$P(z) = \prod_{1}^{N} (z - a_i)$$

- Roots of P(z) are evaluation parameters

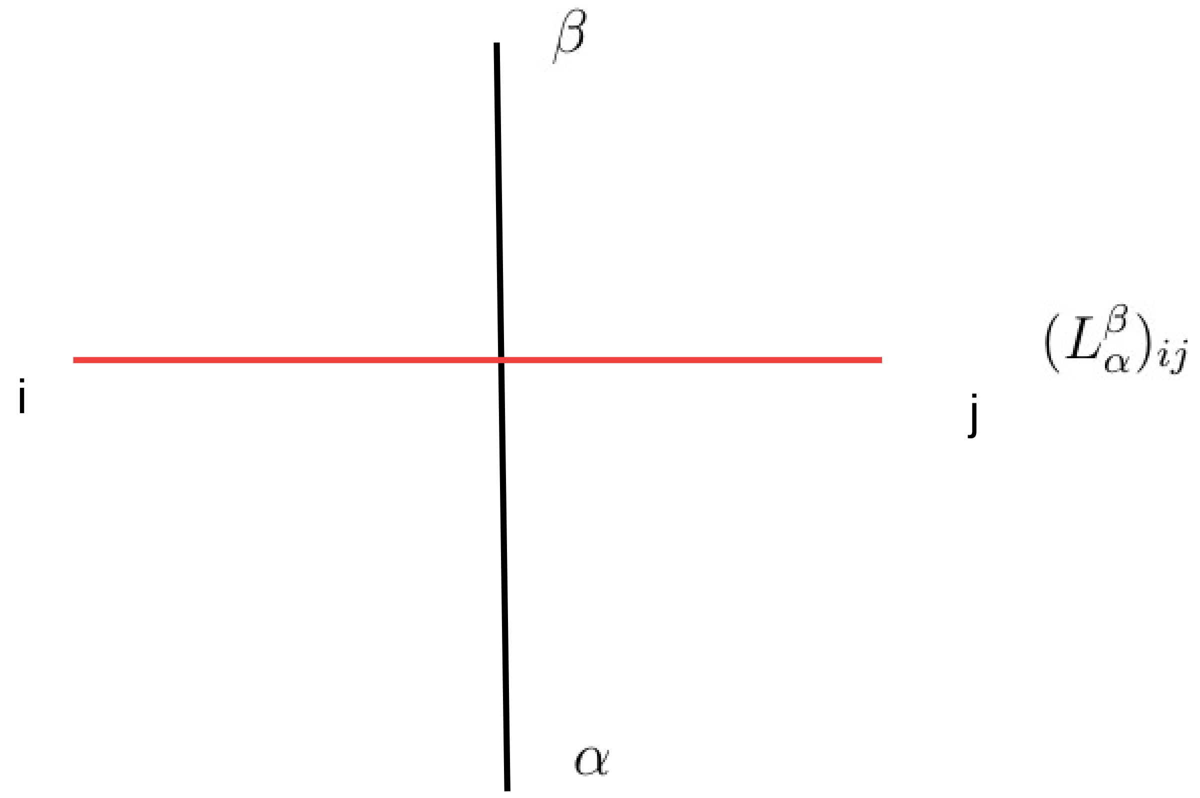
Lieb 6 Vertex Model



a a’ b b’ c c’

$$\frac{a^2 + b^2 - c^2}{2ab} = \Delta$$

Sklyanin 6 vertex model



L Matrix

$$L_s(u) = \begin{pmatrix} q^u K - q^{-u} K^{-1} & (q - q^{-1}) S^- \\ (q - q^{-1}) S^+ & q^u K^{-1} - q^{-u} K \end{pmatrix}$$

$$\begin{aligned} K S^{\pm} &= q^{\pm} S^{\pm} K \\ [S^+, S^-] &= (K^2 - K^{-2})/(q - q^{-1}) \end{aligned}$$

Quantum Sl2

Bethe equations

$$\left(\frac{\sinh(u_m+\eta/2)}{\sinh(u_m-\eta/2)}\right)^N \prod_{m'(\neq m)}^{M} \frac{\sinh(u_m-u_{m'}-\eta)}{\sinh(u_m-u_{m'}+\eta)} = e^{-\mathrm{i}\phi}, \qquad 1 \leq m \leq M.$$

If $l\eta$ is a multiple of π

$$\sinh(u-\alpha)\sinh(u-\alpha-\eta)\cdots\sinh(u-\alpha-(l-1)\eta)$$

For such polynomial, the numerator and denomintor cancel

Moreover it does not change the spectrum.

Questions ?

When such a factor occurs there are degenerate states with magnetization differing by l

Does the eigenstate depend on α no

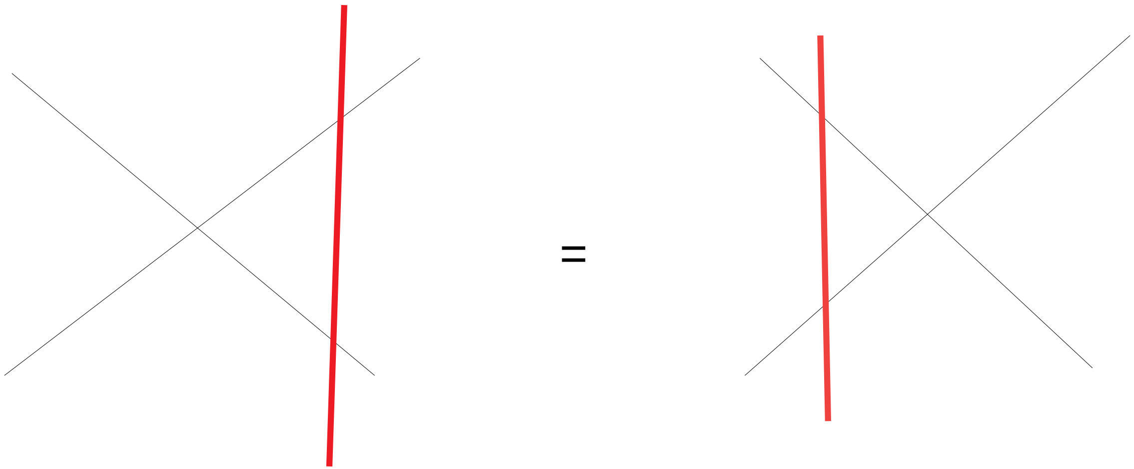
Does α has any physical meaning ? Center of the string

Symmetries Zcharges

commute

Arbitrary representation
Conserved charges

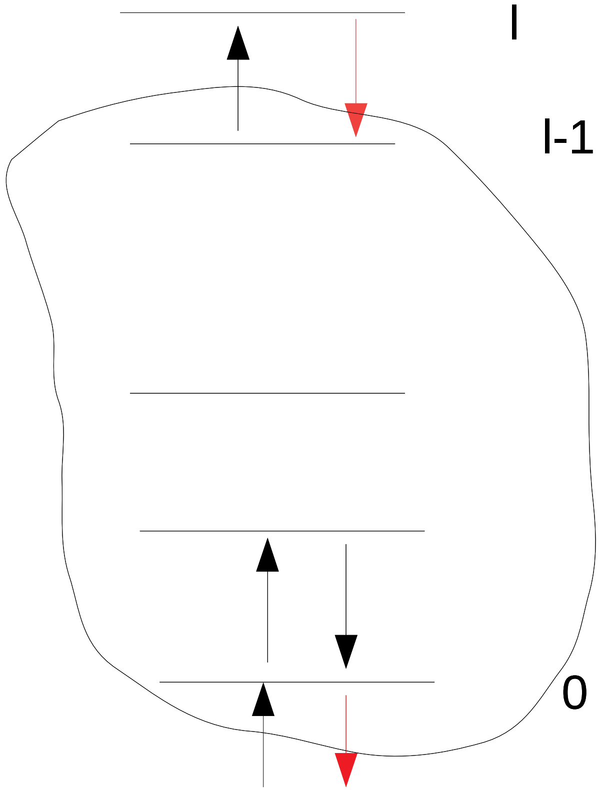
YB



$$RLL' = L'LR$$

2 black and one red lines, R 4x4 6-Vertex matrix

Root of unity reps



$$\mathbf{WX} = \mathbf{qXW}$$

$$\mathbf{S}^{-} = (\mathbf{W} - \mathbf{W}^{-1})\mathbf{X}$$

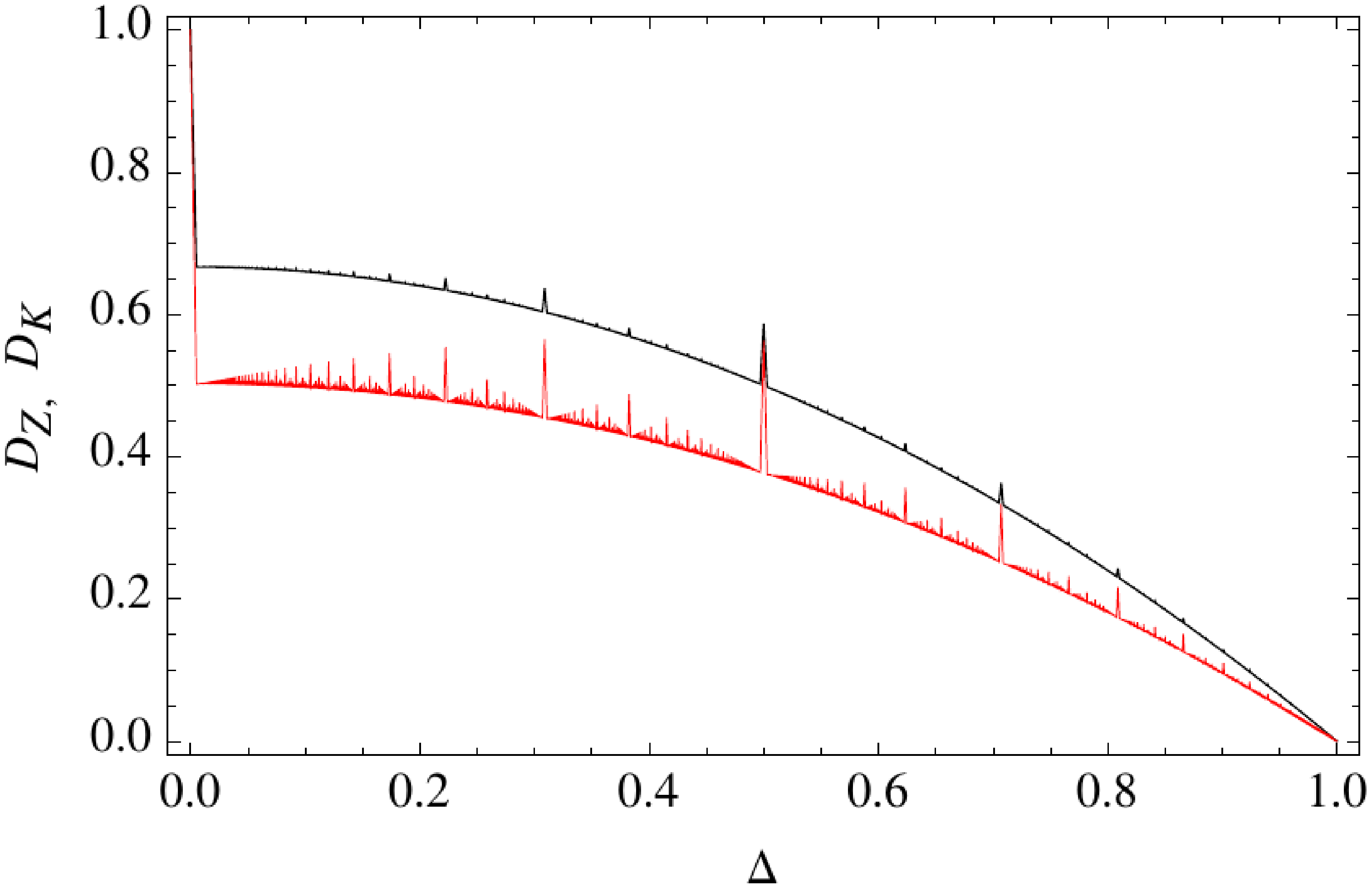
Physical motivation

Prosen, Pereira Sirker Affleck P

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{d}t' \langle A(0)A(t')\rangle_\beta \geq \sum_k \frac{\langle AQ_k\rangle_\beta^2}{\langle Q_k^2\rangle_\beta}.$$

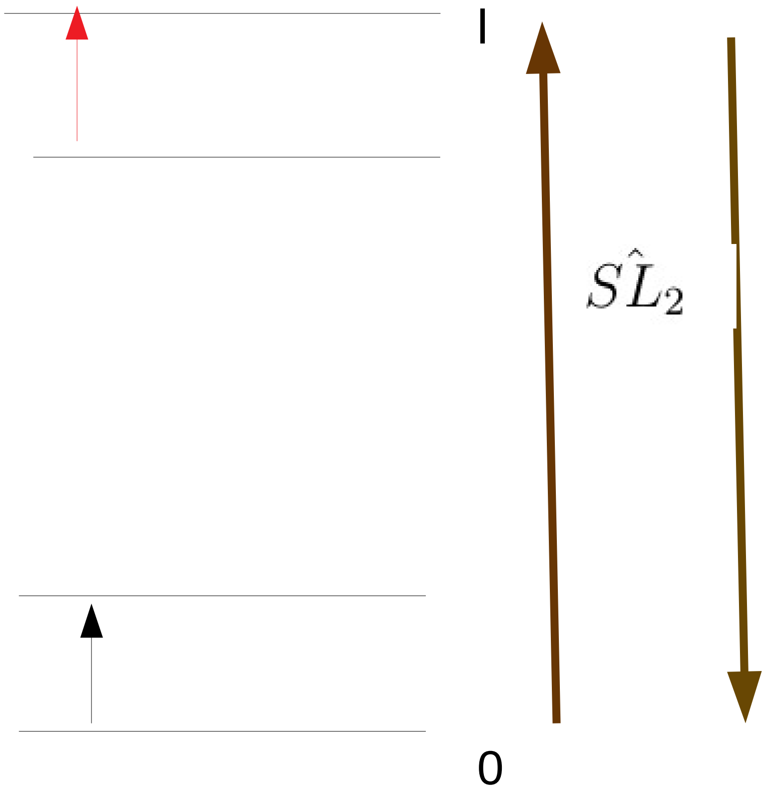
Mazur bound   A the integrated current

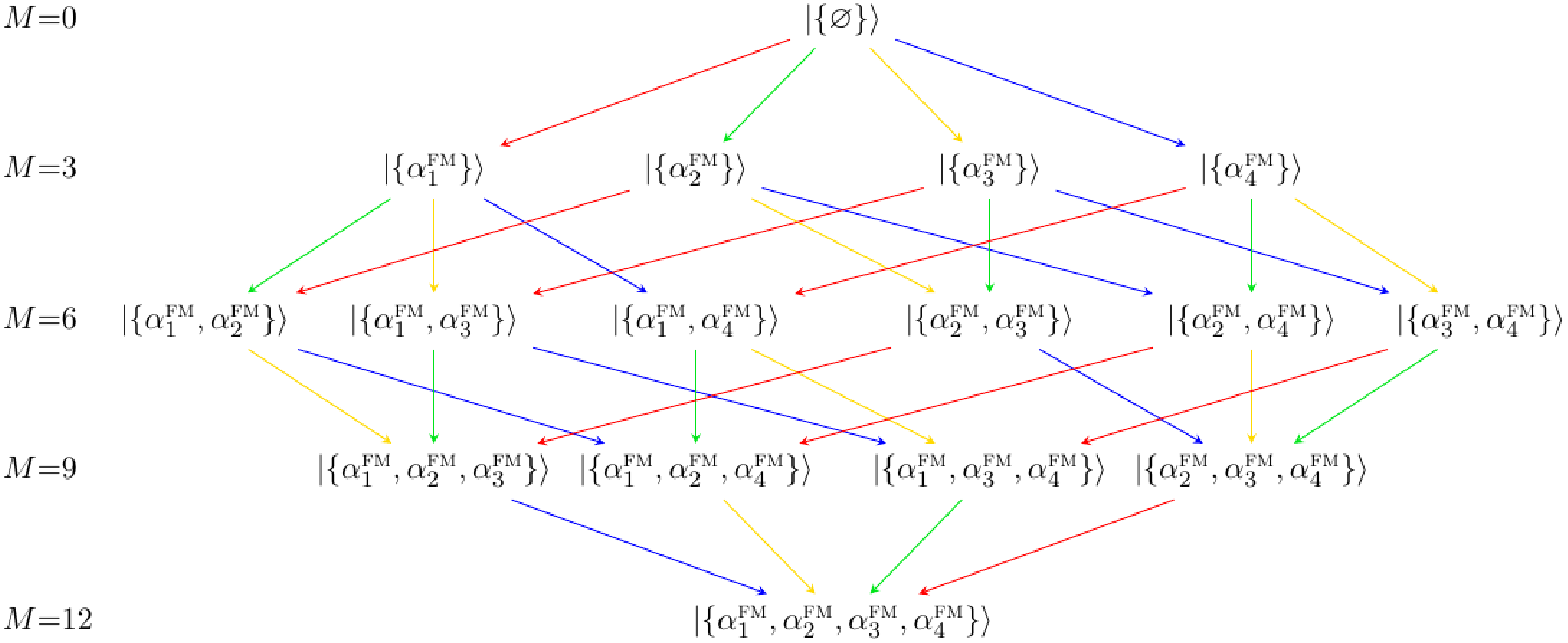
Q= Z charge associated to auxiliary transfer matrix



Fractal Drude Weight

Mathematical motivation





$$V_{1/2}(z_i)^N$$

Interpretation of z_i?

Takahashi

- $$\frac{13}{16} = \frac{1}{1 + \frac{1}{4+\frac{1}{3}}}$$

8 strings

- (11,+) (5,-)

2 last strings

Takahashi strings= FM strings ?

$$\eta = i\pi\frac{p}{l}$$

$$\alpha_k = \alpha_{FM} + \frac{i\pi}{l}(\frac{l+1}{2} - k)$$

$$\begin{aligned}\alpha_{FM} &= i\frac{\pi p}{l} \;\; p\; odd \\ &= 0 \;\; p\; even\end{aligned}$$

Physical=Mathematical ?

Degeneracy only for commensurate twist

$$q^{lN}\mu^{l}=1$$

Changing the twist away from commensurate, FM strings split into 2 last strings which carry the charge (conjecture)

Two last strings

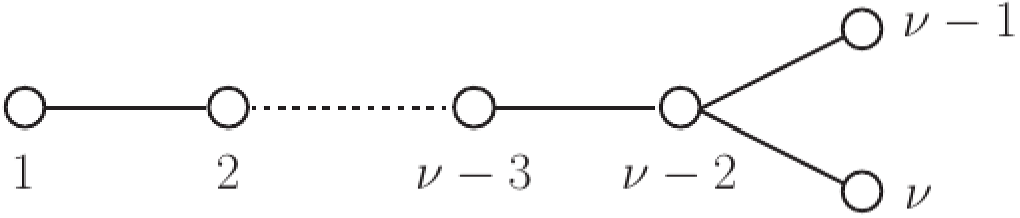
- Kuniba Sakai Suzuki

$$Y_j(v+\mathrm{i})Y_j(v-\mathrm{i}) = (1+Y_{j-1}(v))(1+Y_{j+1}(v)) \quad \text{for } 1 \le j \le \nu-3,$$

$$Y_{\nu-2}(v+\mathrm{i})Y_{\nu-2}(v-\mathrm{i}) = (1+Y_{\nu-3}(v))(1+Y_{\nu-1}(v))(1+Y_{\nu}(v)) \quad \text{for } \nu \ge 3,$$

$$Y_{\nu-1}(v+\mathrm{i})Y_{\nu-1}(v-\mathrm{i}) = \mathrm{e}^{-\mathrm{i}\nu\phi}(1+Y_{\nu-2}(v)),$$

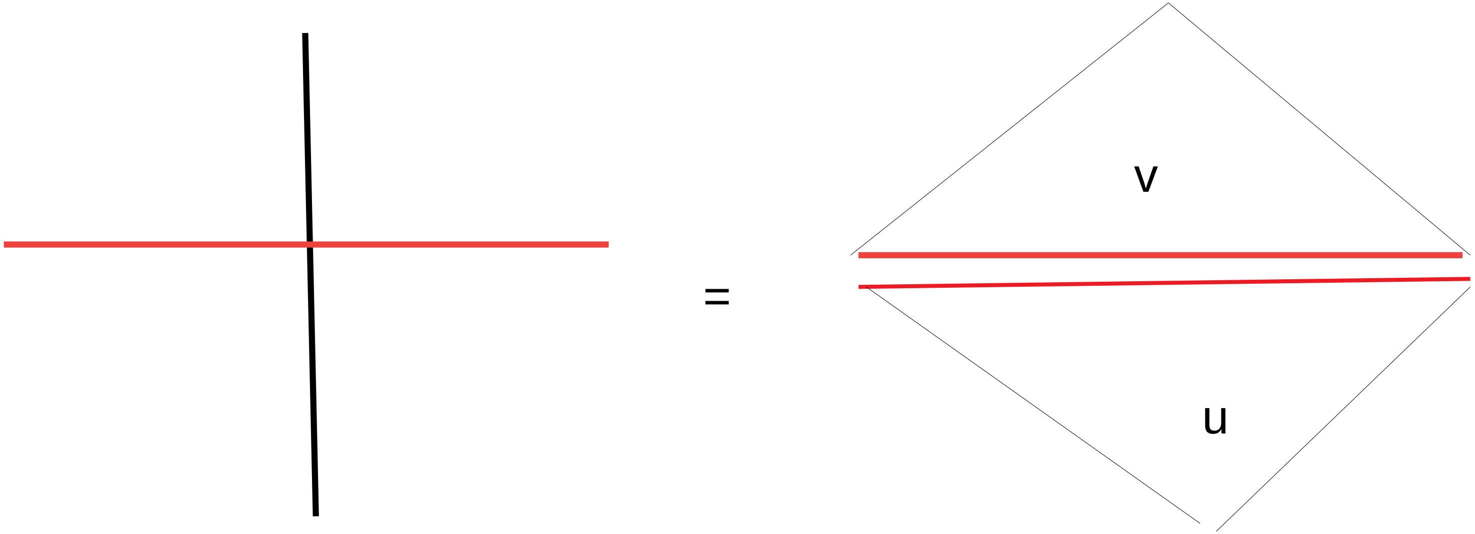
$$Y_{\nu}(v+\mathrm{i})Y_{\nu}(v-\mathrm{i}) = \mathrm{e}^{\mathrm{i}\nu\phi}(1+Y_{\nu-2}(v)).$$



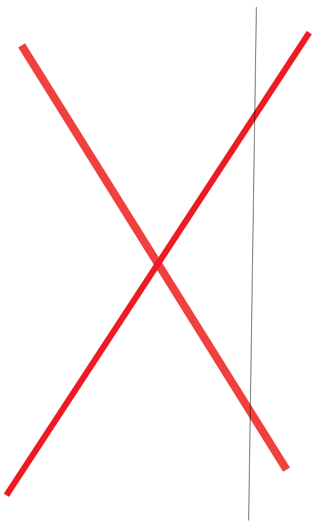
Factorization

$$\mathbf{L}_{aj}(u,s) = \frac{1}{2}\begin{pmatrix}\mathbf{X}_a^{\dagger} & 0 \\ 0 & 1\end{pmatrix}_j \mathbf{u}_j(x)\begin{pmatrix}\mathbf{W}_a & 0 \\ 0 & \mathbf{W}_a^{-1}\end{pmatrix}_j \mathbf{v}_j(y)^{\mathrm{T}}\begin{pmatrix}\mathbf{X}_a & 0 \\ 0 & 1\end{pmatrix}_j$$

$$\mathbf{WX} = \mathbf{qXW}$$



YB



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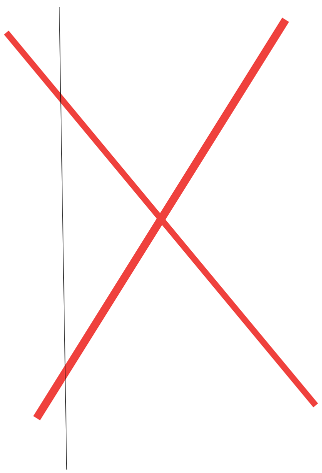

Exchange u_1, u_2

$$u_1 v_1 u_2 v_2 \xrightarrow{F_1} v_1 u_1 u_2 v_2 \xrightarrow{H_{12}} v_1 u_2 u_1 v_2 \xrightarrow{F_1^{-1}} u_2 v_1 u_1 v_2$$

$$R_u = F_1^{-1}(u_2, v_1) H_{12}(u_1, u_2) F_1(u_1, v_1)$$

P and Q matrices

$$T(\mathbf{u},\mathbf{v})T(\mathbf{u_0},\mathbf{v_0}) = T(\mathbf{u},\mathbf{v_0})T(\mathbf{u_0},\mathbf{v}) = Q(u)P(v)$$

P and Q commute

$$T(u)Q(u) = T_0(u-\eta)Q(u+\eta) + \mu T_0(u+\eta)Q(u-\eta)$$
$$T(u)P(u) = \mu T_0(u-\eta)P(u+\eta) + T_0(u+\eta)P(u-\eta)$$

From which Bethe equations derive

Decomposition

$(\mathbf{u}, \mathbf{v}) \equiv (u, s)$ u spectral parameter s spin

When s is half integer, one has

$$T(u,s) = \begin{pmatrix} \mathbf{T}(u,s) & * \\ 0 & T(u,-s-1) \end{pmatrix} \equiv \mathbf{T}(u,s) + T(u,-s-1)$$

$$(1-q^{Nl}\mu^{l})\mathbf{T_0}(u) = Q(u+1/2)P(u-1/2) - \mu^{1}Q(u-1/2)P(u+1/2)$$

Wronskian vanishes for commensurate twist

Complete strings

Vanishing of Wronskian means P and Q differ by complete strings

$$P(u) = Q_r(u)P_s(u)$$
$$Q(u) = Q_r(u)Q_s(u)$$

Drinfeld Polynomial

Specialize at s=0 using periodicity $$\frac{P}{Q}(u+l\eta) = q^{Nl}\frac{P}{Q}(u)$$

$$\frac{P}{Q}(u) = \sum_{0}^{l-1} \mu^k \frac{\mathbf{T}_0(u+k-1/2)}{Q(u+k\eta)Q(u+(k+1)\eta)}$$

Consequence for the spectrum

Since Wronskian vanishes at commensurate twists, P and Q differ only by the part Invariant under $u \to u + \eta$ polynomial in e^{lu}

$$Y(u) = P_s Q_s(u) = \sum_0^{l-1} \mu^k \frac{\mathbf{T}_0(u + k - 1/2)}{Q_r(u + k\eta) Q_r(u + (k+1)\eta)}$$

Equal to the polynomial Y(u) introduced by Fabricius and McCoy. Explains degeneracy
P 13 Center of FM strings are meaningless for 6Vertex but meaningfull for Q

Difference with Fabricius McCoy

- This Q matrix can only be constructed for $e^{2l\eta} = 1$ as a consequence center of FM strings are not the limit of root $e^{2l\eta} \to 1$ Center of FM strings coincide with roots of Y(u).
- The Q matrix itself is an interesting object related to so called τ_2 matrix of Bazanov Stroganov

Happy birthday

Hubert

Happy birthday Jesper !