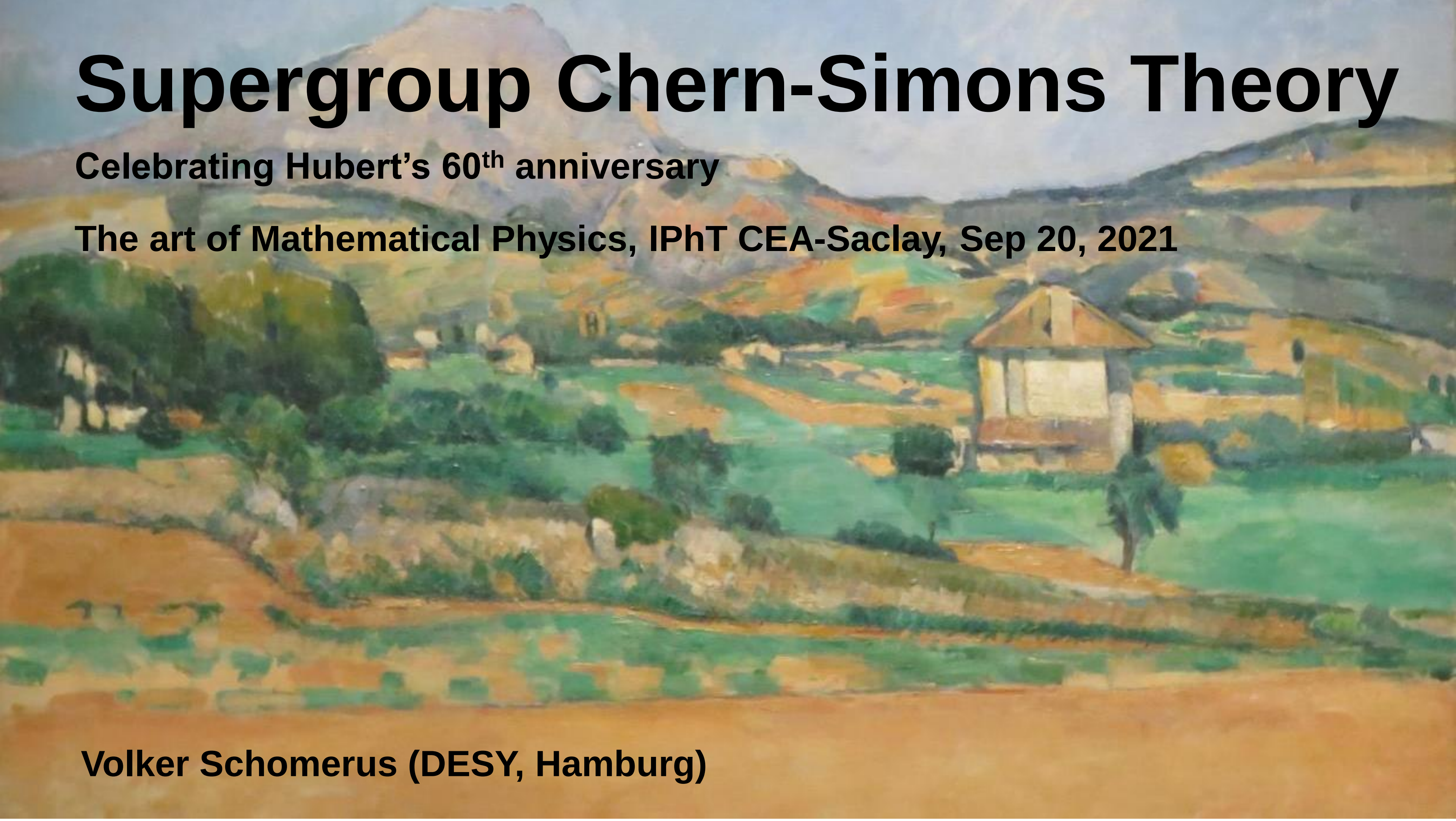

Supergroup Chern-Simons Theory

Celebrating Hubert's 60th anniversary

The art of Mathematical Physics, IPhT CEA-Saclay, Sep 20, 2021

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Based on work in progress with N. Aghaei, A. Gainutdinov, M. Pawelkiewicz

HELMHOLTZ RESEARCH FOR GRAND CHALLENGES

DESY.

Introduction: Chern Simons Theory

Hamiltonian Chern-Simons gauge the 3D manifold $M = \Sigma \times \mathbb{R}$

= theory of moduli spaces of flat connections on the surface Σ

- **Topological invariants of 3-manifold, knots and links**

  Heegaard splitting, knot surgery

- **2-dimensional CFT - WZNW models**

  CS states = conformal blocks

- **Host of integrable (quantum) mechanical systems**

  Gaudin, Hitchin → Calogero-Sutherland

Chern-Simons theory for gauge supergroups important extension

Pioneered by Hubert (with Lev Rozansky) in 92 ….

Introduction: Combinatorial Quantization

Idea: Quantization of Chern-Simons theory obtained from a lattice gauge theory with non-commutative (q-deformed) connections.

[Alekseev, Grosse, VS] $\leftarrow$ [Fock, Rosly]

$\rightarrow$ **Prime example for factorization homology of Lurie** [Ben-Zvi, Brochier, Jordan]

History and References (*incomplete*):

Compact gauge groups [Alekseev, Grosse, VS] [Buffenoir, Roche]

Non-compact groups [Buffenoir, Roche] [Meusburger, Schroers]

Non-semisimple case [Faitg]

Supergroups [Aghaei,Gainutdinov,Pawelkiewicz,VS]

Combinatorial Quantization: Building Blocks

Quasitriangular factorizable ribbon super Hopf algebra $\mathcal{G}$

$$R \in \mathcal{G} \otimes \mathcal{G} \qquad M = R'R = \Delta(v^{-1})\, v \otimes v$$ ribbon element

with right integral $\mu : \mathcal{G} \to \mathbb{C}$

The link algebra $\mathcal{U}$ [AGS 93] :

Close relative of quantum group

$$R' \overset{1}{U} \overset{2}{U} = \overset{2}{U} \overset{1}{U} R$$

admits two mutually commuting actions of $\mathcal{G}$

$$\xi_s(U) = (S(\xi) \otimes id)U, \qquad \xi_t(U) = U(\xi \otimes id)$$

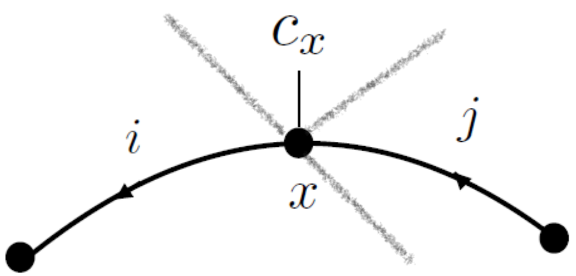
Deformed left and right regular action (of vector fields on group)

Combinatorial Quantization: CS Observables

$\Gamma = (V, E)$ graph on orientable 2-dimensional surface Σ of genus g

vertex set V, edge set E ***choose cilium at vertices, orientation of edges***

The graph algebra $\mathcal{U}(\Gamma_*)$



$$\overset{1}{U}(j) R \overset{2}{U}(i) = \overset{2}{U}(i) \overset{1}{U}(j)$$

$$\overset{1}{U}(j) \overset{2}{U}(k) = \overset{2}{U}(i) \overset{1}{U}(k) \qquad \partial k \cup \partial j = \emptyset$$

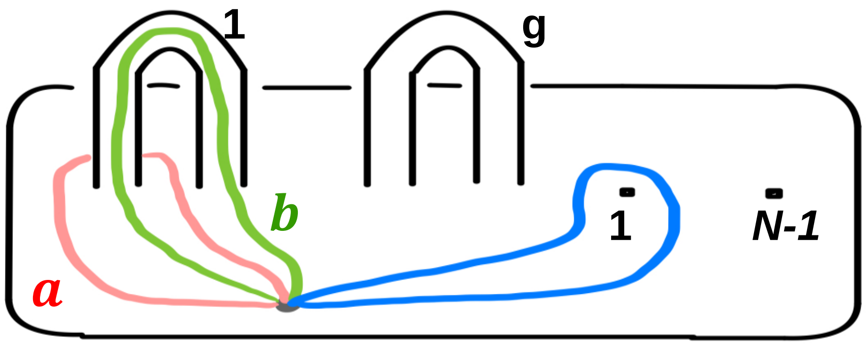
$$U(i) U(-i) = v \otimes id$$

Graph algebra admits gauge transformations $\xi \in \underline{\mathcal{G}} = \bigotimes_{x \in V} \mathcal{G}_x$

Chern-Simons <u>observables</u> $\mathcal{A}(\Gamma) = \mathcal{A}(\Gamma_*) = [\mathcal{U}(\Gamma_*)]^{\underline{\mathcal{G}}}$

Algebras $\mathcal{U}(\Gamma_*), \mathcal{A}(\Gamma)$ also defined for multigraphs with loops

The Anyonic Jordan Wigner Transformation



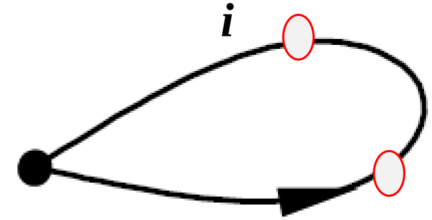
graph: $\Gamma_{g,N-1}$

$2g+N-1$ loops
single vertex

algebra: $\mathcal{U}(\Gamma_{g,N-1}) = \mathcal{U}_{g,N-1}$

$\partial\Sigma = S^1$

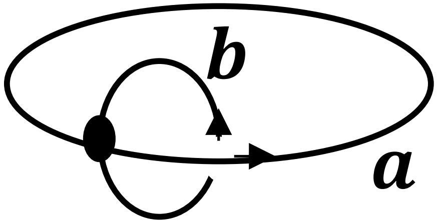
The Loop algebra $\mathcal{U}_{0,1} \cong \mathcal{G}$



$$R' \overset{1}{M}(i) R \overset{2}{M}(i) = \overset{2}{M}(i) R^{-1} \overset{1}{M}(i) R$$

The Handle algebra $\mathcal{U}_{1,0}$

Heisenberg double



$$R^{-1} \overset{1}{M}(a) R \overset{2}{M}(b) = \overset{2}{M}(b) R' \overset{1}{M}(a) R$$

Theorem: [Alekseev, VS]

$$\mathcal{G} \ltimes \mathcal{U}_{g,N-1} \cong \mathcal{G} \ltimes_{diag} \left(\mathcal{U}_{1,0}^{\otimes g} \otimes \mathcal{G}^{\otimes N-1} \right)$$

braided — mutually commuting

Representation theory of CS observables

The handle algebra $\mathcal{U}_{1,0}$ admits unique representation on space

$\mathscr{H}_{1,0} \cong \mathcal{G}$ generated by $M(a)$ out of ground state

$\rightarrow$ representations of $\mathcal{G} \ltimes \mathcal{U}_{g,N-1}$ for a choice of $\mathrm{N}-1$ reps of $\mathcal{G}$

$$\mathscr{H}_{g,N-1}^{\pi_1,\dots,\pi_{N-1}} = \mathscr{H}_{1,0}^{\otimes g} \otimes \bigotimes_{m=1}^{N-1} \mathscr{H}_{0,1}^{\pi_m}$$

We obtains representations of Chern-Simons observables $\mathcal{A}_{g,N-1}$ on multiplicity spaces $\mathscr{H}_{g,N-1}^{\pi_1,\dots,\pi_{N-1};\pi_N}$ of irreps π_N of $\mathcal{G}$ action.

Note: algebras $\mathcal{A}_{0,N-1}$ are relevant for decomposition of tensor products of representations of $\mathcal{G} \overset{q\to 1}{\longrightarrow}$ Gaudin integrable systems

$\mathcal{A}_{0,3}$ for $U_q(sl_2)$ is spherical DAHA of type $(C_1^\vee, C_1)$ [Cooke]

Mapping Class Group & 3-manifold Invariants

Mapping class group of $\Sigma_{g,N}$ is generated by Dehn twists $v(p)$ along simple closed curves p.

p is unique product of edges $c_\nu \in E_{g,N-1}$

A projective representation of the mapping class group through Chern-Simons observables is given by: **[Alekseev,VS],[Aghaei et al.]**

$$v(p) \mapsto \hat{v}(p) = (\mu \times id)\Big((v^{-1} \otimes 1) \cdot M(p)\Big) \in \mathcal{A}_{g,N-1}$$

right integral **ribbon element** $M(p) = \prod_\nu M(c_\nu)$

→ projective representation of MCG on spaces $\mathscr{H}_{g,N-1}^{\pi_1,\dots,\pi_{N-1};\pi_N}$

graded extension of [Turaev, Reshetikhin], [Lyubashenko,Majid]

→ construction of 3-manifold invariants through Heegaard spitting

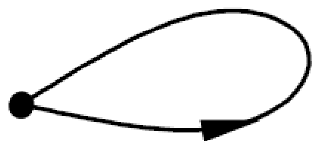
and link invariants **[Kohno][Lyubashenko]**

GL(1|1) Chern-Simons: Building Blocks

$\mathcal{G} = U_q(gl(1|1))$

generators $k_\alpha, k_\beta, e_\pm$

$\approx$ loop algebra $\mathcal{U}_{0,1}$



$$[k_\alpha, k_\beta] = 0 \qquad k_\alpha^p = k_\beta^p = 1$$

$$[k_\alpha, e_\pm] = 0 \qquad k_\beta e_\pm = q^{\pm 1} e_\pm k_\beta$$

$$\{e_\pm, e_\pm\} = 0 \qquad \{e_+, e_-\} = \frac{k_\alpha - k_\alpha^{-1}}{q - q^{-1}}$$

Link algebra $\mathcal{U}$

A q-deformation of

$Fun(GL(1|1))$

but not quite $GL_q(1|1)$

$$\ell_\alpha \ell_\beta = \ell_\beta \ell_\alpha \qquad \ell_\alpha^p = 0 = \ell_\beta^p$$

$$\ell_\alpha \xi_\pm = \xi_\pm \ell_\alpha \qquad \ell_\beta \xi_\pm = q^{\mp 1} \xi_\pm \ell_\beta$$

$$\{\xi_\pm, \xi_\pm\} = 0 \qquad \{\xi_+, \xi_-\} = q - q^{-1}$$

GL(1|1) Chern-Simons: $U_q(gl(1|1))$ basics

$\mathcal{G}$ is a quasitriangular factorizable ribbon super Hopf algebra with:

$$R = \frac{1}{p^2}\left(1 \otimes 1 - (q - q^{-1}) e_+ \otimes e_-\right) \sum_{n,m=0}^{p-1} \sum_{s,t=0}^{p-1} q^{nt+ms} k_\alpha^n k_\beta^m \otimes k_\alpha^{-s} k_\beta^{-t}$$

$$v = \frac{1}{p} k_\alpha (1 \mp (q - q^{-1}) k_\alpha e_- e_+) \sum_{n,m=0}^{p-1} q^{2nm} k_\alpha^{2n} k_\beta^{2m}$$

$$\mu(k_\alpha^n k_\beta^m e_+^r e_-^s) = \mathcal{N} \delta_{n,-1} \delta_{m,0} \delta_{r,1} \delta_{s,1}$$

Representations:

(1) 2-dimensional typical irreps $\pi_{e,n}$ with $\begin{cases} e = 1, \ldots, p-1 \\ n = 0, \ldots, p-1 \end{cases}$

(2) 1-dimensional atypical irreps π_n with $n = 0, \ldots, p-1$

(3) 4-dimensional projective covers $\pi_{\mathcal{P}_n}$ $\pi_n \to \pi_{n+1} \oplus \pi_{n-1} \to \pi_n$

GL(1|1) Chern-Simons: The state space

Under adjoint $\mathcal{G}$ action state space of handle algebra decomposes as

$$\mathscr{H}_{1,0} \cong \mathcal{G} \cong (p^2-1)\mathscr{P}_0 \oplus 2\pi_0 \oplus \pi_{\pm 1}$$

→ decomposition of $\mathscr{H}_{g,N-1}^{\pi_1,\dots,\pi_{N-1}}$ by evaluating tensor products.

From decompositions we can read off dimension of multicity spaces

e.g. $dim \mathscr{H}_{g,0}^{;\pi_0} = (p^{2g}-1)\binom{2g-2}{g-1} + \binom{2g}{g}$ $\quad dim \mathscr{H}_{1,0}^{;\pi_0} = p^2+1$

For g = 0 the Chern-Simons state space is isomorphic to the space of conformal blocks in the GL(1|1) WZW model on sphere [VS,Saleur]

$$dim \mathscr{H}_{0,N-1}^{(e_1,n_1)\dots(e_{N-1},n_{N-1});(e_N,n_M)} = \delta_{\sum e_i,0}\binom{N-2}{n_1 \dots + n_{N-1} - n_N}$$

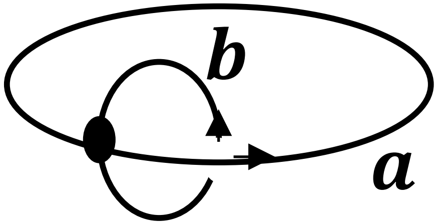
GL(1|1) Chern-Simons: MCG & 3D invariants

The general formula for Dehn twists generators $\hat{v}(c) \in \mathcal{A}_{g,N-1}$ gives

$$\hat{v}(c) = -\frac{i}{p}\sum_{n,m=0}^{p-1} q^{m(2n+1)}(k_\alpha^{(c)})^{2n}(k_\beta^{(c)})^{2m}(1+(q-q^{-1})k_\alpha^{(c)}e_+^{(c)}e_-^{(c)})$$

Specializing to the torus $\Sigma_{1,0}$ we obtain a (p^2+1)-dimensional representation of the modular group $SL(2,\mathbb{Z})$ on $\mathscr{H}_{1,0}^{;\pi_0}$

$$S = \hat{v}(b)\hat{v}(a)\hat{v}(b) \quad T = (\hat{v}(a))^{-1}$$



↔ [Lyubashenko, Majid]

… [Mikhaylov] for gl(1|1)

3-manifold invariants e.g. of Lens spaces $L(u,v)$ through Heegaard splitting as matrix elements of $T^{r_m} S T^{r_{m-1}} \ldots T^{r_1} S$

$$\frac{u}{v} = r_m - \frac{1}{r_{m-1}\cdots - \frac{1}{r_1}}$$

= Alexander-Conway invariant of [Rozansky,Saleur]

Conclusions and Outlook

Combinatorial quantization provides very universal access to observables and states of supergroup Chern-Simons theory.

Interesting examples include $GL(1|1)$ ✓, $SL(1|2), PSL(2|2), \ldots$

Chern-Simons theory is host for super² - integrable systems

Dehn twists along maximal set of non-intersecting cycles degenerate $_{q\to 1}$ to

(sub)set of Hamiltonians in certain limit of Gaudin/Hitchin integrable system

↑ **if $\mathcal{A}_{0,2}$ is non-abelian** ↑ **OPE limit of [Mann, Lacroix, Quintavalle, VS]**

The algebra $\mathcal{A}_{0,2}$ is of particular interest for future studies


Happy 60th
WAKASAYA
NIKE