In and around Koo-Saleur formulas

The art of mathematical physics

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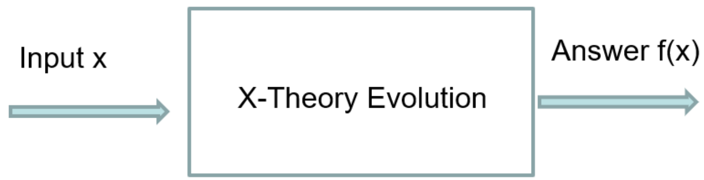
Motivation: Quantum Church-Turing Thesis

- **Church-Turing Thesis:**
  Any reasonable computation can be done by Turing machines. Algorithm=TM
  Turing machine can simulate any physical system, but slowly.
  (Assume any physical theory is computable.)

- **Extended** Church-Turing Thesis: **Efficiently**
  All physical theories can be simulated by TMs efficiently.
  Potential counterexample: Quantum computing

- **Quantum Church-Turing thesis:**
  There is a unique quantum model of computing, it implies that any physical quantum field theory can be **efficiently** simulated by quantum computers.
  Hence quantum computers can simulate conformal field theories. How?
  (All CFTs are two dimensional, unitary, and rational.)

Computational Power of Physical Theories



Each theory provides a computational model,
which selects class of efficiently solvable problems XP

Classical Physics	**Turing Machines P**
Quantum Mechanics	**Quantum Circuit Model BQP**
Quantum Field Theory	**? BQP**
String Theory	**???**

Rigorous classes of QFTs:
TQFTs, CFTs, …

True for TQFTs (Freedman, Kitaev, W. 02):
BQP=B-TQFT-P

How about CFTs?

What to Simulate

Two things: Evolutions and Partition functions.

For TQFTs, representations of mapping class groups (braid groups), and 3-manifold and link invariants.

Some key ingredients:

1) Finitely dimensional Hilbert spaces (TQFT=trivial QFT), but no tensor product
2) MCG and braid groups have local generators,
   complexity of MCG or braid for simulation is transparent
3) A form of locality of TQFTs as given by
   gluing formulas---quantum state can be reconstructed
   from local patches---leading to qudits

Simulating TQFTs I: Reps of braid groups (FKW)

Given some anyon statistics or rep of the braid groups

$$\rho: B_n \longrightarrow U(V_n), \qquad V_n = V(\quad)$$

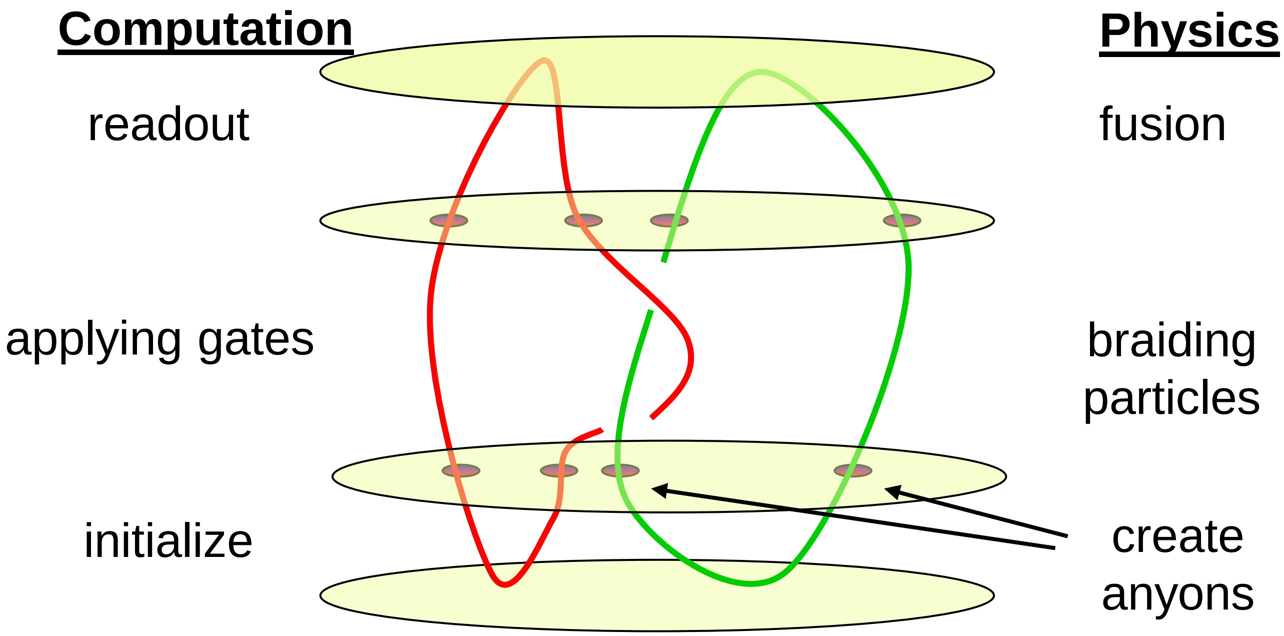
Find a quantum circuit on m(n) qudits W= $C^d = \bigoplus_{(a,b,c)\in\mathcal{L}^3} V_{abc}$

$$U_L: W^{\otimes m} \longrightarrow W^{\otimes m}$$

so that the following commutes ideally:

$$\begin{array}{ccc} V_n & \rightarrow & W^{\otimes m} \\ \downarrow \rho & & \downarrow U_L \\ V_n & \rightarrow & W^{\otimes m} \end{array}$$

Simulation of TQFT II: Approximation Link Invariants (FKW)



Can we and how to extend these to CFTs?

Bulk-edge correspondence

Edge physics of fractional quantum Hall liquids:

∂Witten-Chern-Simons theories

$\sim$Wen’s chiral Luttinger liquids

∂TQFTs/UMTCs $\sim$ χCFTs/VOA

Conjecturally, every anyon model is a representation category of some vertex operator algebra (VOA).

Efficient simulation of TQFTs hints at efficient simulation of CFTs

Two analogues to simulate

- Representations of Diff(S^1)
- Approximate correlation functions $|(1, \mathcal{Y}_n(a_n, z_n) \dots \mathcal{Y}_1(a_1, z_1) 1)|.$

Some issues:

- Infinite dimensional Hilbert spaces
- No obvious local generating set of Diff(S^1)
- No obvious gluing formulas, what does locality imply for simulating CFTs?
- …

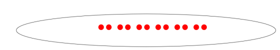
Gapped vs gapless

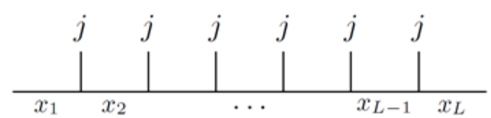
Finite approximation of CFTs: TQFT Hilbert spaces

$\mathrm{Diff}(S^1)$ $\qquad\qquad$ $B_\infty = \lim_n B_n$

VOA $V = \bigoplus_i V_{[i]}$ $\qquad\qquad$ $V_\infty = \lim_n V_n$

? $\qquad\qquad$ V_n--Hilbert space of n anyons

$V_n =$ V()

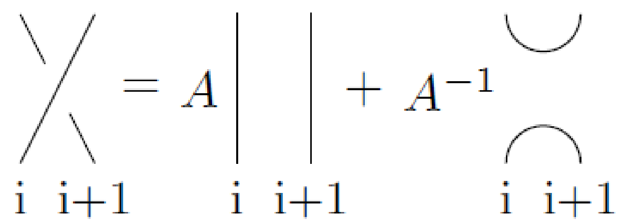


? $\qquad\qquad$ braid σ_i

($\bigoplus_{i \le n} V_{[i]}$ not good as $L_m = 0$ if m large enough)

Temperley-Lieb algebras and braid groups

Kauffman bracket/Jones representation



$$\sigma_i = A\, Id + A^{-1} d\, e_i$$

Compare with $e^{iPx} = cosx\, Id + i \sin x\, P$

" Lie algebra" vs "Lie group"

VOA and LCN

Two definitions of χCFT:

VOA	LCN (local conformal net)
"Lie algebra"	"Lie group"
$\{L_n\}$	$\{ e^{iL(f)} \}$"

$L(f) = \sum f\text{^}_n L_n,\ \ f \in C^\infty(S^1)$, $f\text{^}_n$ Fourier coefficient

smeared conformal field $Y(\omega, f)$

Admissible unitaries:

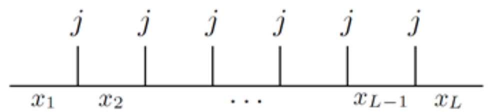
$e^{iL(f)}$ is admissible if L(f) is a finite sum.

Thm: (Goodman-Wallach):

Every infinitesimally unitary projective representation of the subalgebra of real vector fields on S^1 with finite Fourier series can be integrated to a continuous projective unitary representation of Diff(S^1).

Lattice version of Virasoro generators L_m

VOAs V approximated by

{  }

$$e_\theta \xrightarrow{SL} Y(\omega, e^{i\theta}) \qquad Y(\omega, z) = \sum L_m z^{-m-2}$$

Temperley-Lieb generators e_i are finite lattice version of Fourier transform of the Virasoro generators L_m. How are they related precisely, i.e., how conformal symmetry emerges from lattice?

Elementary unitary (energy local) should come from $e^{iL_m[L]}$, $L_m[L]$=some lattice version of L_m such as Koo-Saleur formulas.

Spin chains: Koo-Saleur formulas

$$\mathscr{H} = \frac{\gamma}{2\pi \sin\gamma} \sum_{j=1}^{N} \left[\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta(\sigma_j^z \sigma_{j+1}^z - 1) + 2e_\infty \right]$$

$$\mathscr{L}_n[N] = \frac{N}{4\pi} \left[-\frac{\gamma}{\pi \sin\gamma} \sum_{j=1}^{N} e^{inj2\pi/N} \left(e_j - e_\infty + \frac{i\gamma}{\pi \sin\gamma} [e_j, e_{j+1}] \right) \right] + \frac{c}{24} \delta_{n,0} \, ,$$

$$\bar{\mathscr{L}}_n[N] = \frac{N}{4\pi} \left[-\frac{\gamma}{\pi \sin\gamma} \sum_{j=1}^{N} e^{-inj2\pi/N} \left(e_j - e_\infty - \frac{i\gamma}{\pi \sin\gamma} [e_j, e_{j+1}] \right) \right] + \frac{c}{24} \delta_{n,0}$$

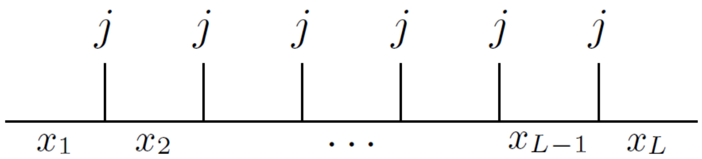
$$[\mathscr{L}_m, \mathscr{L}_n] = (m-n)\mathscr{L}_{m+n} + \delta_{m+n,0} \frac{1}{12} (m^3 c^* - mc) \, .$$

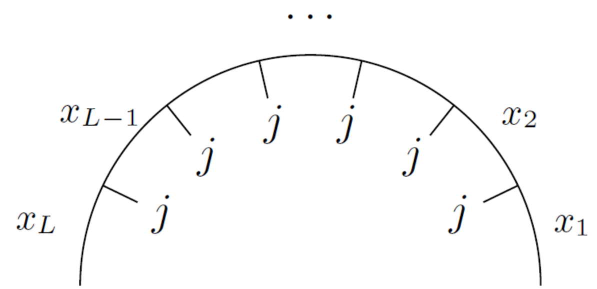
W. M. Koo and H. Saleur, Nucl. Phys. B **426**, 459 (1994).

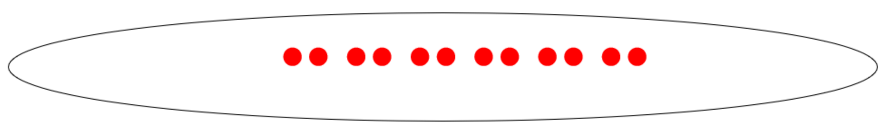
Linnea Grans-Samuelsson, Jesper Lykke Jacobsen & Hubert Saleur

Journal of High Energy Physics **2021**, Article number: 130 (2021)

Anyonic Chains







$$H = J \sum_i P_i^0,$$

$$H = -\frac{1}{d} \sum_{i=1}^{L-1} e_i.$$

Anyonic Chain Approach

- **Jones:** a general spin chain = Hilbert spaces tensored together by Connes fusion.
- Explicit study of golden chain (Phys. Rev. Lett. **98**, 160409 ·)

1) Exactly solvable related to RSOS models, but not known rigorously solvable.
2) Numerical simulations show they converge to CFTs.
3) Protected by topological symmetries.

Koo-Saleur Modified

M. Shokrian-Zini and Z. Wang, Comm. Math. Phys. **363**, 877–953 (2018)

Our modified formulas:

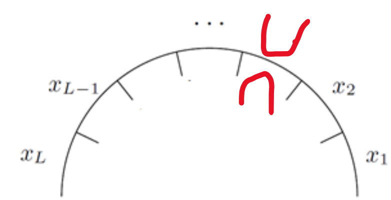
$$O_n^c = -\sum_{j=1}^{2n-1} \cos\left(\frac{m(j+\frac{1}{2})\pi}{2n+1}\right)e_j, \quad O_n^s = i\sum_{j=1}^{2n-2} \sin\left(\frac{m(j+1)\pi}{2n+1}\right)[e_j, e_{j+1}],$$

we have operators $\widetilde{L}^c_{\pm m}, \widetilde{L}^s_{\pm m} \xrightarrow{SL} L_{\pm m}$ *satisfying the properties in* $\boxed{\mathbf{4.3}}$ *and,*

$$\frac{\widetilde{L}^c_m + \widetilde{L}^c_{-m}}{2} = \alpha_n^c O_n^c + \beta_n^{m,c}\mathbf{1} \xrightarrow{SL} \frac{L_m + L_{-m}}{2},$$

$$\frac{i(\widetilde{L}^s_m - \widetilde{L}^s_{-m})}{2} = \alpha_n^s O_n^c + \beta_n^{m,s}\mathbf{1} \xrightarrow{SL} \frac{i(L_m - L_{-m})}{2},$$

where $\alpha_n^c, \alpha_n^s, \beta_n^{m,c}$*, and* $\beta_n^{m,s}$ *are suitable scaling factors.*



Comparison with Koo-Saleur formulas: $j + 1 \ vs \ j + \frac{1}{2}$

In what sense, the finite approximations converge to CFTs?

Mathematical Scaling Limit

M. Shokrian-Zini and Z. Wang, Comm. Math. Phys. **363**, 877–953 (2018)

Denote by $\mathcal{W}_n^M$ the Hilbert space $\mathcal{W}_n$ restricted to energies at most M, i.e. $\mathcal{W}_n^M = \bigoplus_{\lambda_i^{(n)} \leq M} E_{\lambda_i^{(n)}}$. Assume the following set of properties (**P**)

- $\lambda_i = \lim_{n\to\infty} \lambda_i^{(n)}$ exists for all $i \in \mathbb{N}$ with the convention $\lambda_i^{(n)} = 0$ for $i > d(n)$,
- (*connecting maps*) for all $M > \lambda_1$, there exist *connecting* unitary maps $\phi_n^M : \mathcal{W}_n^M \to \mathcal{W}_{n+1}^M$ for large enough n,
- (*extension*) ϕ_n^M is an extension of $\phi_n^{M'}$ when $M \geq M'$.

Definition 3. Given a sequence of quantum theories $(\mathcal{W}_n, H_n)$ with given connecting maps ϕ_n^M satisfying properties (**P**), the *scaling limit* $(\mathcal{V}, H)$ is the result of the double colimit construction. This limit will be written as $(\mathcal{W}_n, H_n) \xrightarrow{SL} (\mathcal{V}, H)$.

The definition is inspired by earlier works of Gainutdinov-Read-Saleur and numerical simulation of anyonic chains.

Algorithmic. Scaling limit of observables as algebras.
Rate of convergence (scaling dimension of irrelevant operators).
…

Scaling Limit of Ising ACs

M. Shokrian-Zini and Z. Wang, Comm. Math. Phys. **363**, 877–953 (2018)

Hamiltonian

$$H = i \sum_{a=1}^{2n-1} t_a \psi_{a+1} \psi_a.$$

(a) $\mathcal{W}_n = (\frac{1}{2}, \frac{1}{2})$, $H_n = -\sum_{j=1}^{2n-1} e_j$. *Then* $(\mathcal{W}_n, H_n) \xrightarrow{SL} (\chi_0 + \chi_{\frac{1}{2}}, L_0)$.

(b) $\mathcal{W}_n = (0,0)$ *or* $(1,1)$, $H_n = -\sum_{j=2}^{2n-2} e_j$. *Then* $(\mathcal{W}_n, H_n) \xrightarrow{SL} (\chi_0, L_0)$.

(c) $\mathcal{W}_n = (0,1)$ *or* $(1,0)$, $H_n = -\sum_{j=2}^{2n-2} e_j$. *Then* $(\mathcal{W}_n, H_n) \xrightarrow{SL} (\chi_{\frac{1}{2}}, L_0)$.

(d) $\mathcal{W}_n = (\frac{1}{2}, 1)$ *or* $(\frac{1}{2}, 0)$, $H_n = -\sum_{j=1}^{2n-2} e_j$. *Then* $(\mathcal{W}_n, H_n) \xrightarrow{SL} (\chi_{\frac{1}{16}}, L_0)$.

(e) $\mathcal{W}_n$ *be the periodic chain of size* $2n$, *and* $H_n = -\sum_{j=1}^{2n} e_j$. *Then*

$$(\mathcal{W}_n, H_n) \xrightarrow{SL} (\chi_0\overline{\chi}_0 + \chi_{\frac{1}{2}}\overline{\chi}_{\frac{1}{2}} + \chi_{\frac{1}{16}}\overline{\chi}_{\frac{1}{16}}, L_0 + \overline{L}_0)$$

if n *is even.*

Furthermore, the rate of convergence of each scaling limit is $O(\frac{1}{n})$ *while we have restriction of energies up to* $O(\sqrt[3]{n})$.

2- For the corresponding higher Virasoro generators action, with the same rate of convergence as above, given a fixed $m \neq 0$, *we have (up to some scalings)*

(a) $-\sum_{j=1}^{2n-1} \cos(\frac{m(j+\frac{1}{2})\pi}{2n+1}) e_j \xrightarrow{SL} L_m + L_{-m}$,
$i\sum_{j=1}^{2n-2} \sin(\frac{m(j+1)\pi}{2n+1})[e_j, e_{j+1}] \xrightarrow{SL} i(L_m - L_{-m})$

Some partial simulation results

Theorem 3.5.1 *Given $f \in C^\infty_{\mathbb{R}}(S^1)$ with finitely many Fourier coefficients, a quantum computer can approximate the following up to an error ϵ in polynomially many steps in $\frac{1}{\epsilon}$*

$$|(\Omega, e^{iL(f)}\Omega)|.$$

Theorem 3.5.3 *Consider the same settings as in Conjecture $\boxed{3.3.1}$, with the exception of a prepared homogeneous state ξ of energy E instead of Ω, and the promise that*

$$\sum_{j=1}^{k} ||(f^{(j)})'||_{r_\omega+2}$$

is of order $O(\log(n_1, \ldots, n_k, k))$. Then one can efficiently approximate the state

$$\prod_{j=1}^{k} e^{iL(f^{(j)})}\xi$$

up to error ϵ in polynomially many steps in $\{\frac{1}{\epsilon}, n_1, \ldots, n_k, k, E\} \cup \{(\hat{f^{(j)}})_m\}_{j,m}$ on a quantum computer.

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Scaling Limit of Anyonic Chains and Quantum Simulation of Conformal Field Theory

A dissertation submitted in partial satisfaction
of the requirements for the degree

Doctor of Philosophy
in
Mathematics

by

Modjtaba Shokrian Zini

Committee in charge:

Professor Zhenghan Wang, Chair
Professor Michael Freedman
Professor David Morrison

September 2020

Dream: Golden chain

- The Fibonacci chain or golden chain converges to the tri-critical minimal model M(5,4) c=7/10.
- In general, scaling limit of ACs of spin=1/2 in $SU(2)_k$ converges to minimal models M(k+2,k+1):

UMMs with central charge $c = 1 - \frac{6}{(k+1)(k+2)}$.

Conjecture 4.3. *For any UMM VOA $\mathcal{V} = \mathcal{V}_{c,0}$ and chiral representation $\mathcal{V}_{c,h}$, there is a sequence of quantum theories with* **strong** *scaling limit $(\mathcal{V}_{c,h}, L_0)$ such that for each L_m, we have a sequence $\widetilde{L}_m \in \mathcal{A}_n$ with the following properties:*

- *It is a space local observable with hermitian operators $a\widetilde{L}_m + \overline{a}\widetilde{L}_{-m} \in \mathcal{A}_n^H$.*
- *It shifts the energy no more than $|m|$.*
- *Restricted to energy at most n^{d_ω} it has the following approximation by $L_m|_{n^{d_\omega}}$ with the rest being R_n^m:*

$$\widetilde{L}_m = L_m|_{n^{d_\omega}} + O(\frac{1}{n^{g_\omega}}) + R_n^m,$$

where d_ω, g_ω are positive constants.
- *Its norm is bounded by $O(n^{e_\omega})$ for some constant e_ω.*

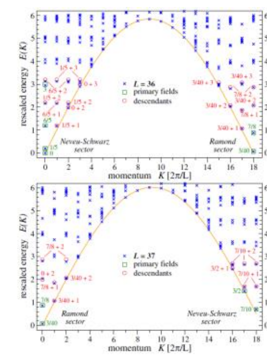

FIG. 3 (color online). Energy spectra for periodic chains of size L. Energies are rescaled and shifted such that the two lowest eigenvalues match the CFT assignments. Open boxes indicate positions of primary fields of the $c = \frac{7}{10}$ CFT. Open circles give positions of descendant fields as indicated. As a guide to the eye the solid line is a cosine-fit of the dispersion.

Happy Birthday!