Integrability of anyonic chains with competing interactions

Paata Kakashvili
Eddy Ardonne

arXiv:1110.0719

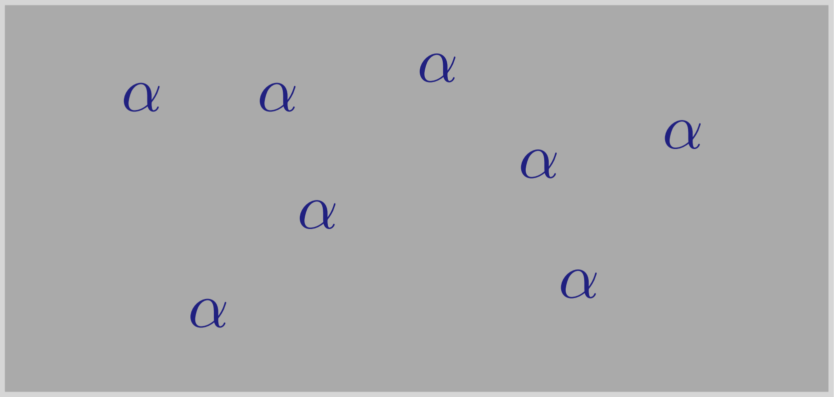
Nordita, IHP

CFT, T & I: 2011-11-02

Outline

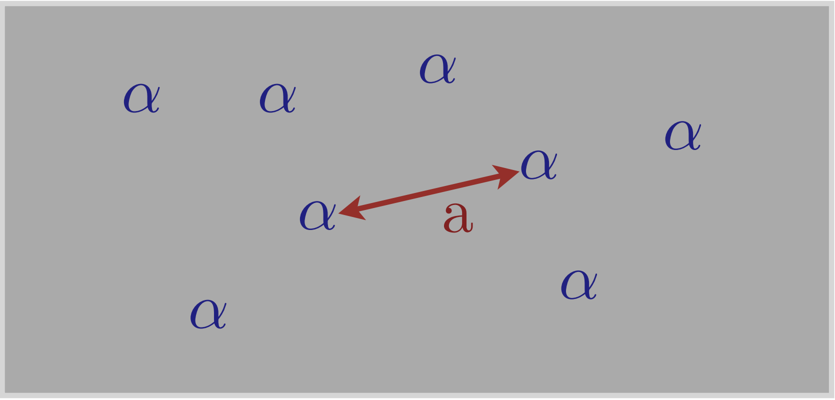
- Motivation
- Anyonic quantum chains
- Competing interactions
- Construction of a new, integrable 2-d model
- The corner transfer matrix method
- Analyzing the model

Anyons in quantum Hall systems



Abelian anyons: braiding gives phase

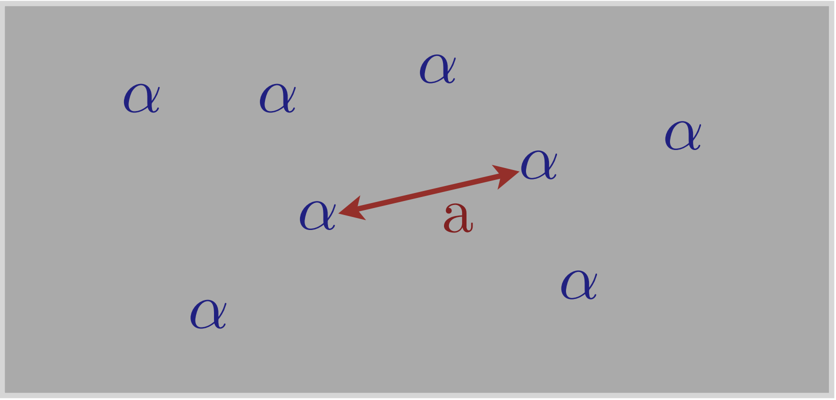
Anyons in quantum Hall systems



Abelian anyons: braiding gives phase

Non-abelian anyons give macroscopic degeneracy d_α^L in the ideal case: $a \gg \xi$

Anyons in quantum Hall systems

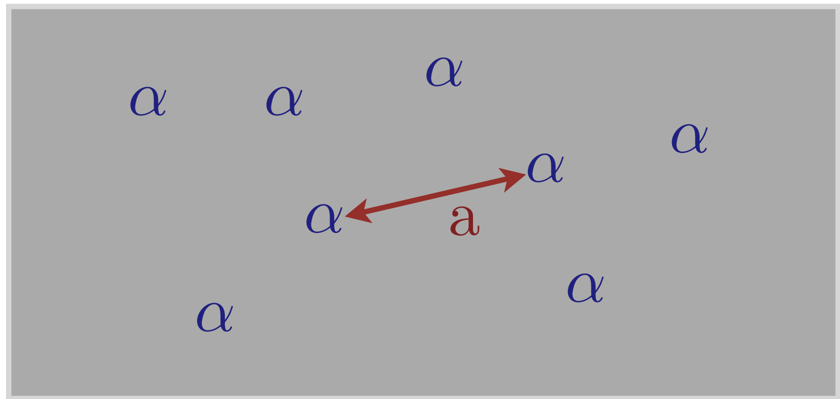


Abelian anyons: braiding gives phase

Non-abelian anyons give macroscopic degeneracy d_α^L in the ideal case: $a \gg \xi$

Degeneracy is lifted by anyon-anyon interactions when $a \ll \xi$

Anyons in quantum Hall systems



Abelian anyons: braiding gives phase

Non-abelian anyons give macroscopic degeneracy d_α^L
in the ideal case: $a \gg \xi$

Degeneracy is lifted by anyon-anyon interactions
when $a \ll \xi$

Study the collective anyonic behaviour in quantum chains!

The golden chain (Fibonacci anyons)

$$\mathbf{1} \times \mathbf{1} = \mathbf{1} \qquad \mathbf{1} \times \tau = \tau \qquad \tau \times \tau = \mathbf{1} + \tau$$

The golden chain (Fibonacci anyons)

$$\mathbf{1} \times \mathbf{1} = \mathbf{1} \qquad \mathbf{1} \times \tau = \tau \qquad \tau \times \tau = \mathbf{1} + \tau$$

Heisenberg hamiltonian:
penalize τ fusion channel of neighbouring anyons

The golden chain (Fibonacci anyons)

$$\mathbf{1} \times \mathbf{1} = \mathbf{1} \qquad \mathbf{1} \times \tau = \tau \qquad \tau \times \tau = \mathbf{1} + \tau$$

Heisenberg hamiltonian:
penalize τ fusion channel of neighbouring anyons

$$\tau \quad \tau \quad \tau \quad \tau \quad \tau \quad \tau$$

The golden chain (Fibonacci anyons)

$$\mathbf{1} \times \mathbf{1} = \mathbf{1} \qquad \mathbf{1} \times \tau = \tau \qquad \tau \times \tau = \mathbf{1} + \tau$$

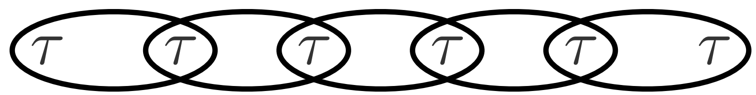
Heisenberg hamiltonian:
penalize τ fusion channel of neighbouring anyons

$$\tau \quad \tau \quad \tau \quad \tau \quad \tau \quad \tau$$

The golden chain (Fibonacci anyons)

$$\mathbf{1} \times \mathbf{1} = \mathbf{1} \qquad \mathbf{1} \times \tau = \tau \qquad \tau \times \tau = \mathbf{1} + \tau$$

Heisenberg hamiltonian:
penalize τ fusion channel of neighbouring anyons



The golden chain (Fibonacci anyons)

$$\mathbf{1} \times \mathbf{1} = \mathbf{1} \qquad \mathbf{1} \times \tau = \tau \qquad \tau \times \tau = \mathbf{1} + \tau$$

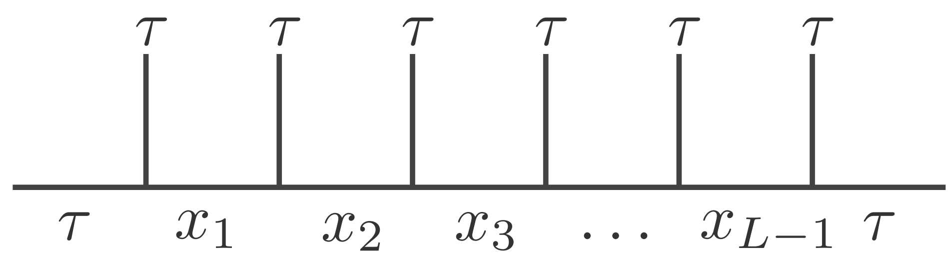
Heisenberg hamiltonian:
penalize τ fusion channel of neighbouring anyons

$$\tau \quad \tau \quad \tau \quad \tau \quad \tau \quad \tau$$

The golden chain (Fibonacci anyons)

$$\mathbf{1} \times \mathbf{1} = \mathbf{1} \qquad \mathbf{1} \times \tau = \tau \qquad \tau \times \tau = \mathbf{1} + \tau$$

Heisenberg hamiltonian:
penalize τ fusion channel of neighbouring anyons

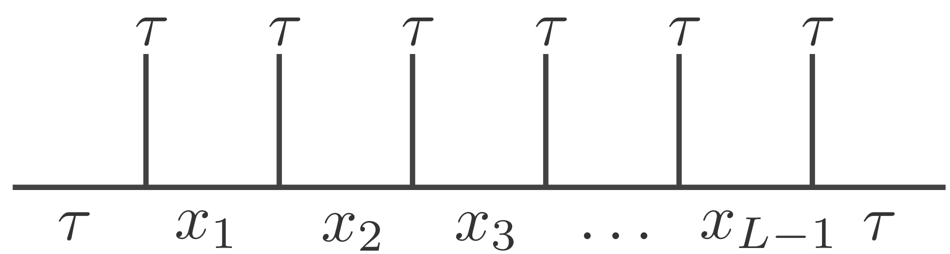


Hilbert space: consistent labelings of the ‘fusion tree’:

The golden chain (Fibonacci anyons)

$$\mathbf{1}\times\mathbf{1}=\mathbf{1} \qquad \mathbf{1}\times\tau=\tau \qquad \tau\times\tau=\mathbf{1}+\tau$$

Heisenberg hamiltonian:
penalize τ fusion channel of neighbouring anyons



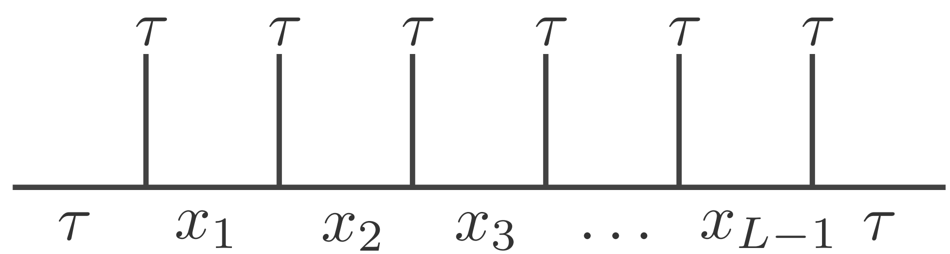
Hilbert space: consistent labelings of the ‘fusion tree’:

$$|x_1, x_2, \ldots, x_{L-1}\rangle$$

The golden chain (Fibonacci anyons)

$$\mathbf{1} \times \mathbf{1} = \mathbf{1} \qquad \mathbf{1} \times \tau = \tau \qquad \tau \times \tau = \mathbf{1} + \tau$$

Heisenberg hamiltonian:
penalize τ fusion channel of neighbouring anyons



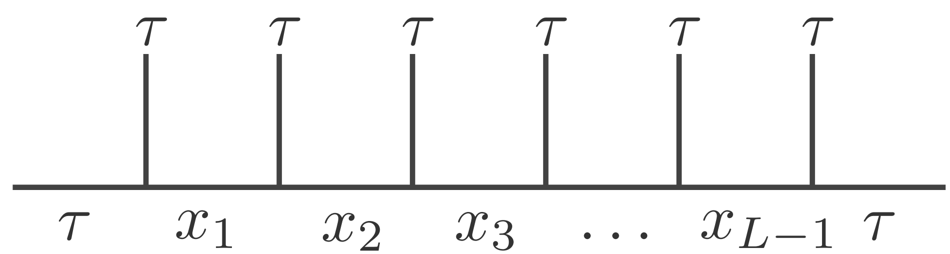
Hilbert space: consistent labelings of the ‘fusion tree’:

$|x_1, x_2, \ldots, x_{L-1}\rangle$ ~~$x_i = x_{i+1} = \mathbf{1}$~~

The golden chain (Fibonacci anyons)

$$\mathbf{1} \times \mathbf{1} = \mathbf{1} \qquad \mathbf{1} \times \tau = \tau \qquad \tau \times \tau = \mathbf{1} + \tau$$

Heisenberg hamiltonian:
penalize τ fusion channel of neighbouring anyons



Hilbert space: consistent labelings of the 'fusion tree':

$$|x_1, x_2, \ldots, x_{L-1}\rangle \qquad \text{~~}x_i = x_{i+1} = \mathbf{1}\text{~~}$$

No tensor product decomposition:

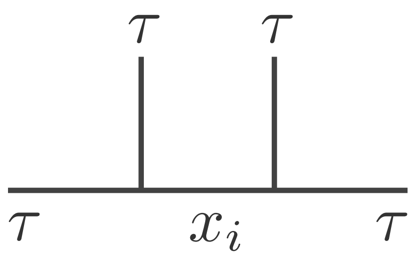
$$\dim \mathcal{H}_L = \mathrm{Fib}_{L+1} \propto \varphi^L \qquad \varphi = \frac{1+\sqrt{5}}{2}$$

The golden chain

Hamiltonian: sum of local projectors: $H = \sum_i P^{(\tau)}_{\mathrm{2-body,i}}$

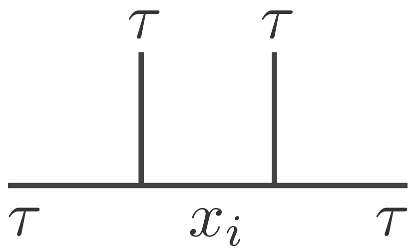
The golden chain

Hamiltonian: sum of local projectors: $H = \sum_i P^{(\tau)}_{\mathrm{2-body,i}}$

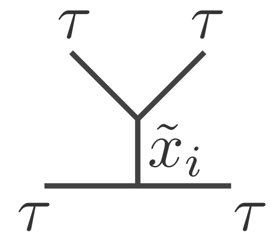


The golden chain

Hamiltonian: sum of local projectors: $H = \sum_i P^{(\tau)}_{2-\mathrm{body,i}}$

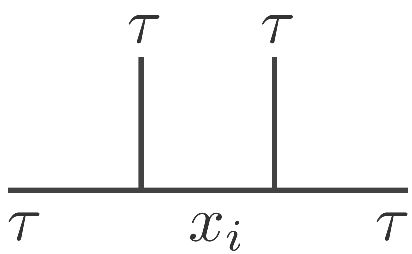


$$= \sum_{\tilde{x}_i} (F^{\tau\tau}_{\tau\tau})_{x_i,\tilde{x}_i}$$

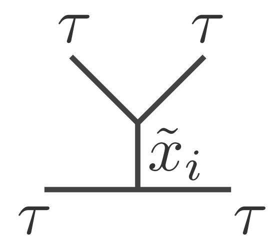


The golden chain

Hamiltonian: sum of local projectors: $H = \sum_i P^{(\tau)}_{2-\mathrm{body,i}}$

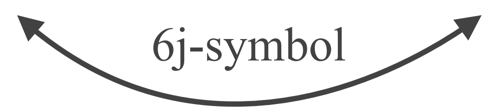


$$= \sum_{\tilde{x}_i} (F^{\tau\tau}_{\tau\tau})_{x_i,\tilde{x}_i}$$



$$F^{\tau\tau}_{\tau\tau} = \begin{pmatrix} \varphi^{-1} & \varphi^{-\frac{1}{2}} \\ \varphi^{-\frac{1}{2}} & -\varphi^{-1} \end{pmatrix}$$

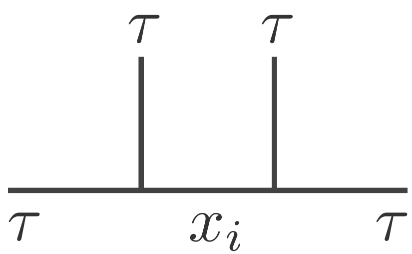
$$\left(\frac{1}{2} \otimes \frac{1}{2}\right) \otimes \frac{1}{2} \cong \frac{1}{2} \otimes \left(\frac{1}{2} \otimes \frac{1}{2}\right)$$



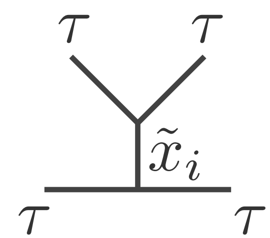
F-matrix is the anyon version of the Wigner 6j-symbol

The golden chain

Hamiltonian: sum of local projectors: $H = \sum_i P^{(\tau)}_{2-\mathrm{body,i}}$

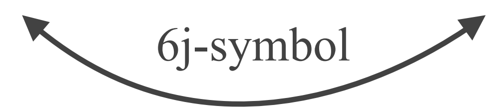


$= \sum_{\tilde{x}_i} (F^{\tau\tau}_{\tau\tau})_{x_i,\tilde{x}_i}$



$$F^{\tau\tau}_{\tau\tau} = \begin{pmatrix} \varphi^{-1} & \varphi^{-\frac{1}{2}} \\ \varphi^{-\frac{1}{2}} & -\varphi^{-1} \end{pmatrix}$$

$$\left(\frac{1}{2} \otimes \frac{1}{2}\right) \otimes \frac{1}{2} \cong \frac{1}{2} \otimes \left(\frac{1}{2} \otimes \frac{1}{2}\right)$$



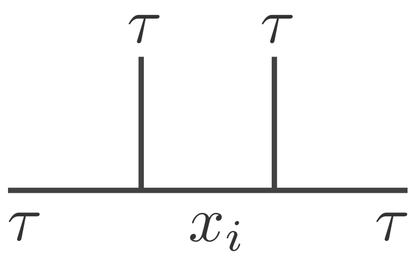
F-matrix is the anyon version of the Wigner 6j-symbol

The projectors can be written as: $P_i^{(\tau)} = F_i^{-1} \Pi_i^{(\tau)} F_i$

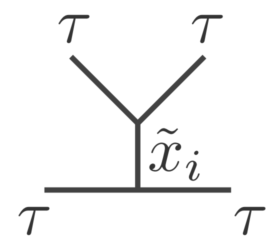
Feiguin et.al., PRL (2007)

The golden chain

Hamiltonian: sum of local projectors: $H = \sum_i P^{(\tau)}_{2-\text{body},\text{i}}$

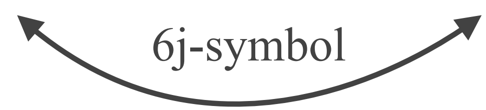


$$= \sum_{\tilde{x}_i} (F^{\tau\tau}_{\tau\tau})_{x_i,\tilde{x}_i}$$



$$F^{\tau\tau}_{\tau\tau} = \begin{pmatrix} \varphi^{-1} & \varphi^{-\frac{1}{2}} \\ \varphi^{-\frac{1}{2}} & -\varphi^{-1} \end{pmatrix}$$

$$\left(\frac{1}{2} \otimes \frac{1}{2}\right) \otimes \frac{1}{2} \cong \frac{1}{2} \otimes \left(\frac{1}{2} \otimes \frac{1}{2}\right)$$



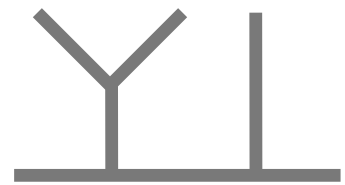
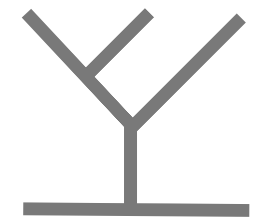
Explicit form of the local projectors:

$$P^{(\tau)}_{2-\text{body}} = \mathcal{P}_{1\tau\tau} + \mathcal{P}_{\tau\tau 1} + \varphi^{-2}\mathcal{P}_{\tau 1\tau} + \varphi^{-1}\mathcal{P}_{\tau\tau\tau}$$
$$+ \varphi^{-3/2}\left(|\tau 1\tau\rangle\langle\tau\tau\tau| + \text{h.c.}\right)$$

Feiguin et.al., PRL (2007)

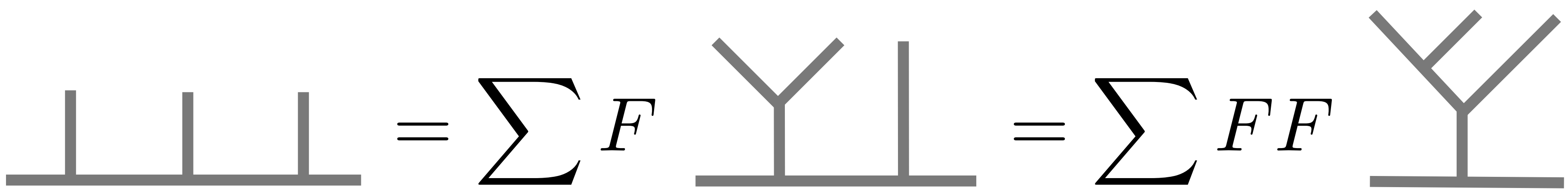
The 3-body interaction

We need to transform twice to find the fusion channel of three neighbouring anyons:

 $= \sum F$  $= \sum FF$ 

The 3-body interaction

We need to transform twice to find the fusion channel of three neighbouring anyons:



Explicitly, we find (please forget!):

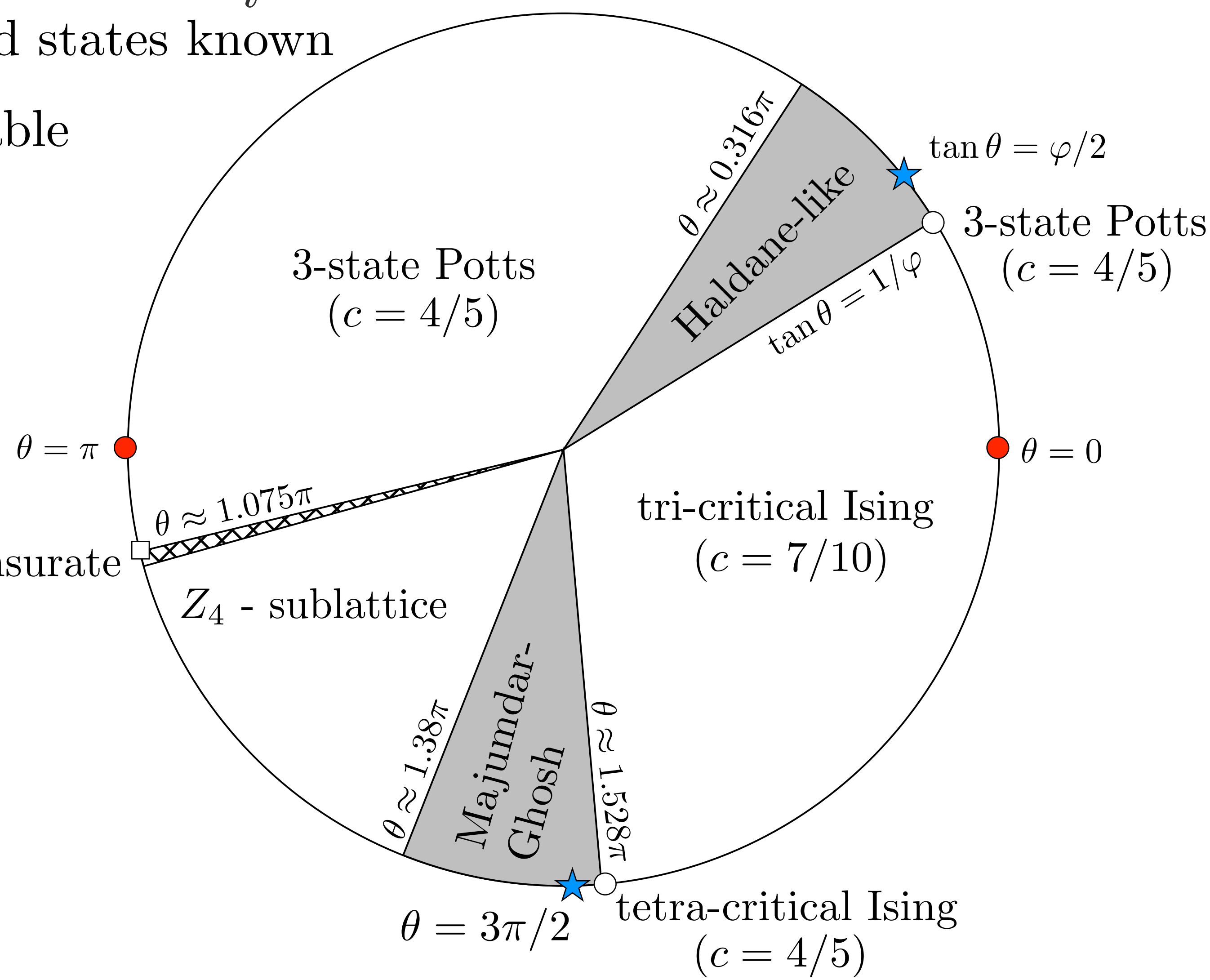
$$
\begin{aligned}
P^{(\tau)}_{3-\mathrm{body}} = {} & \mathcal{P}_{\tau 1 \tau 1} + \mathcal{P}_{1 \tau 1 \tau} + \mathcal{P}_{\tau\tau\tau 1} + \mathcal{P}_{1 \tau\tau\tau} + 2\varphi^{-2}\mathcal{P}_{\tau\tau\tau\tau} + \\
& \varphi^{-1}\left(\mathcal{P}_{\tau 1 \tau\tau} + \mathcal{P}_{\tau\tau 1 \tau}\right) - \varphi^{-2}\left(|\tau\tau 1 \tau\rangle\langle \tau 1 \tau\tau| + \mathrm{h.c.}\right) + \\
& \varphi^{-5/2}\left(|\tau 1 \tau\tau\rangle\langle\tau\tau\tau\tau| + |\tau\tau 1 \tau\rangle\langle\tau\tau\tau\tau| + \mathrm{h.c.}\right)
\end{aligned}
$$

Phase diagram of the model

$$H_{J_2-J_3} = \sum_i \cos\theta P^{(\tau)}_{2-\mathrm{body},i} + \sin\theta P^{(\tau)}_{3-\mathrm{body},i}$$

★ Ground states known

● Integrable



Trebst et.al., PRL (2008)

Integrability of the Golden chain

The operators $e_i = \varphi\left(1 - P^{(\tau)}_{\rm 2-body,i}\right)$ form a representation of the Temperly-Lieb algebra:

$$e_i^2 = de_i \qquad e_i e_{i\pm1} e_i = e_i$$

Pasquier (1987)

$$[e_i, e_j] = 0 \qquad \text{for } |i-j| \geq 2$$

$d = \varphi$ is the isotopy parameter

Integrability of the Golden chain

The operators $e_i = \varphi\big(1 - P^{(\tau)}_{\mathrm{2-body,i}}\big)$ form a representation of the Temperly-Lieb algebra:

$$e_i^2 = d e_i \qquad e_i e_{i\pm 1} e_i = e_i$$

Pasquier (1987)

$$[e_i, e_j] = 0 \quad \text{for } |i-j| \geq 2$$

$d = \varphi$ is the isotopy parameter

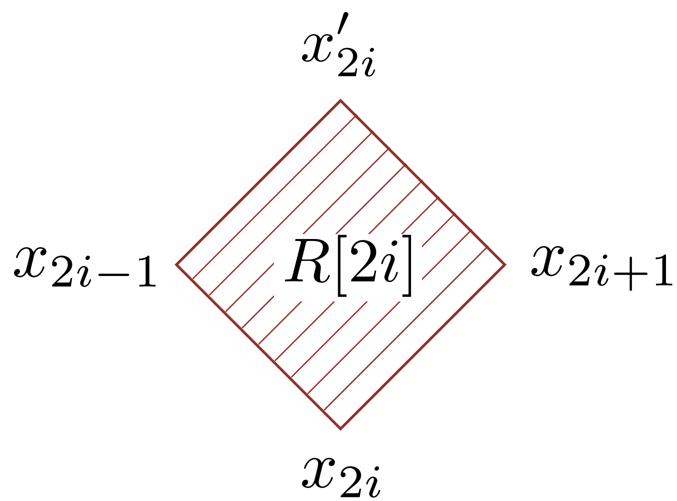
With the e's, we can construct an R-matrix (plaquette weights) which satisfies the Yang-Baxter equation!

The R-matrix

$$R_i(u)^{\vec{x}'}_{\vec{x}} = \Big(\frac{\sin(\frac{\pi}{k+2} - u)}{\sin(\frac{\pi}{k+2})} \mathbf{1}^{\vec{x}'}_{\vec{x}} + \frac{\sin(u)}{\sin(\frac{\pi}{k+2})} e(i)^{\vec{x}'}_{\vec{x}} \Big)$$

$$e(i)^{\vec{x}'}_{\vec{x}} = \Big(\prod_{j \neq i} \delta_{x'_j, x_j} \Big) e(i)^{x_{i+1}}_{x_{i-1}}$$

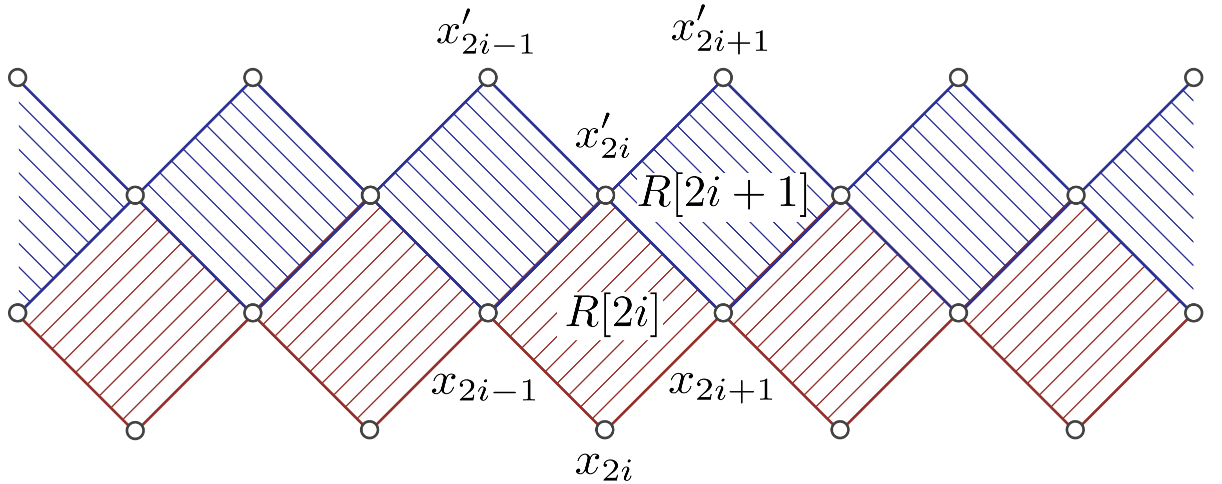
$$\mathbf{1}^{\vec{x}'}_{\vec{x}} = \prod_j \delta_{x'_j, x_j}$$



R-matrix satisfies the Yang-Baxter equation:

$$R_j(u) R_{j+1}(u+v) R_j(v) = R_{j+1}(v) R_j(u+v) R_{j+1}(u)$$

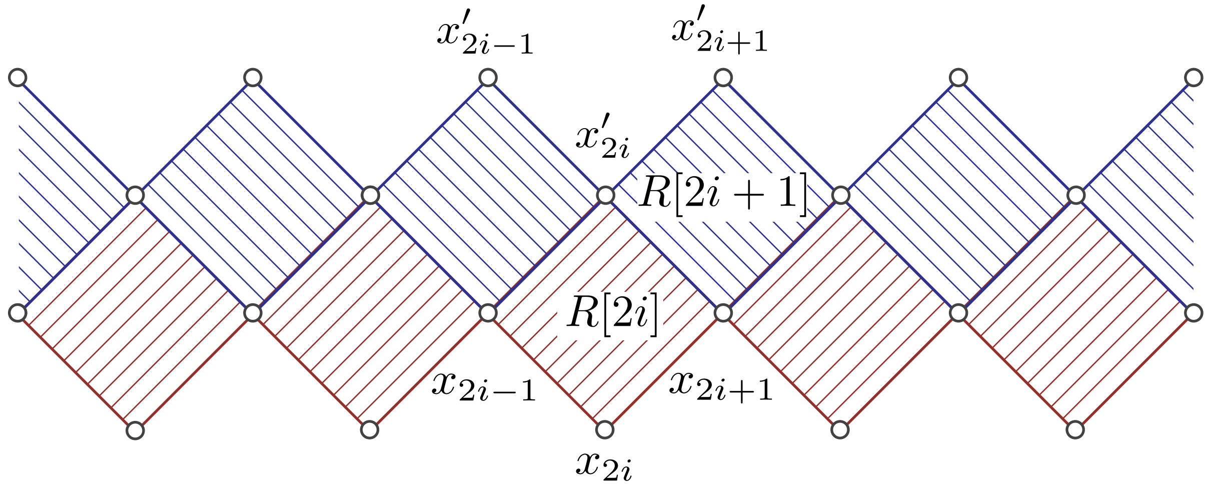
The transfer matrix



$$T(u) = \prod_i R[2i+1](u) R[2i](u)$$

The different transfer matrices commute, because R satisfies the Yang-Baxter equation.

The transfer matrix

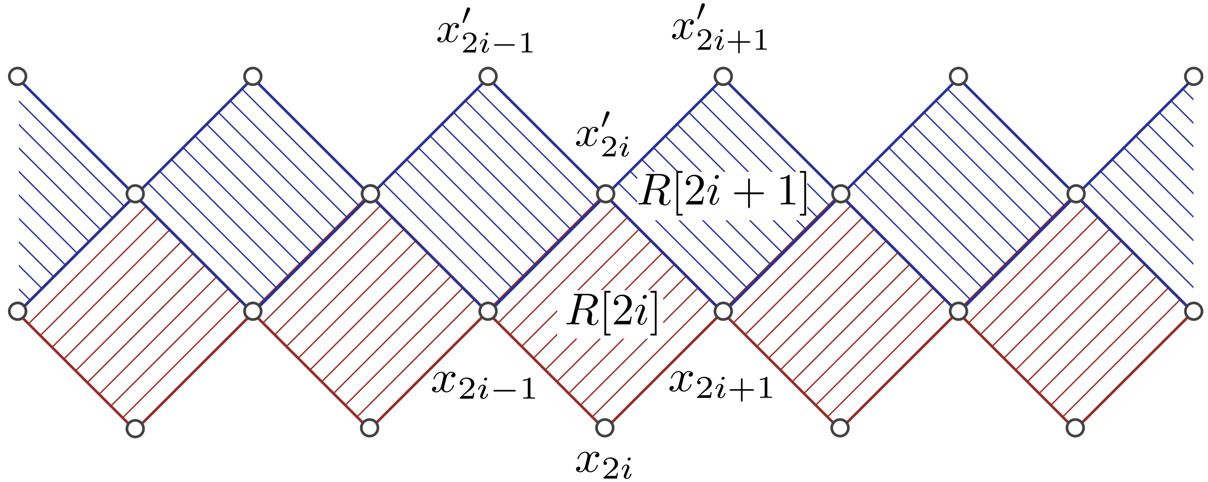


$$T(u) = \prod_i R[2i+1](u)R[2i](u)$$

The different transfer matrices commute, because R satisfies the Yang-Baxter equation.

Constructing a hamiltonian: $T(u) = e^{-uH+o(u^2)}$

The transfer matrix



$$T(u) = \prod_i R[2i+1](u) R[2i](u)$$

The different transfer matrices commute, because R satisfies the Yang-Baxter equation.

Constructing a hamiltonian: $T(u) = e^{-uH + o(u^2)}$

$$\frac{d \ln T(u)}{du}\Big|_{u=0} = c_1 H_{2-\mathrm{body}} + c_2$$

Composite R-matrix

The 3-body term requires a composite R-matrix:

$$\tilde{R}_j(u,\phi) = R_{2j+1}(u-\phi)R_{2j}(u)R_{2j+2}(u)R_{2j+1}(u+\phi)$$

$\tilde{R}_j(u,\phi)$ satisfies Yang-Baxter, because $R_j(u)$ does!

Composite R-matrix

The 3-body term requires a composite R-matrix:

$$\tilde{R}_j(u,\phi) = R_{2j+1}(u-\phi)R_{2j}(u)R_{2j+2}(u)R_{2j+1}(u+\phi)$$

$\tilde{R}_j(u,\phi)$ satisfies Yang-Baxter, because $R_j(u)$ does!

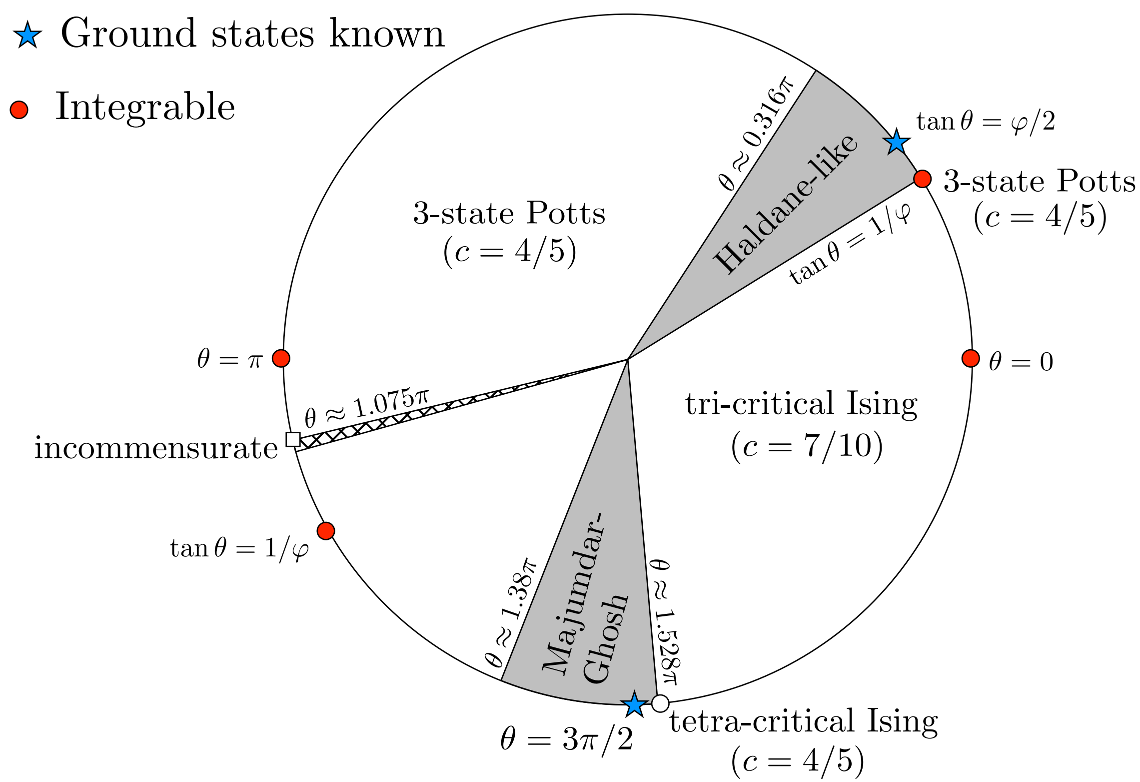
The Hamiltonian indeed contains 2 and 3-body terms:

$$\begin{aligned} H &= c_4 + \sum_i c_1(e_i + e_{i+1})/2 + c_2(e_i e_{i+1} + e_{i+1} e_i) \\ &\quad + c_3(-1)^i (e_i e_{i+1} - e_{i+1} e_i) \end{aligned}$$

See Iklhef et.al., JPA (2009)

Composite R-matrix

$$\phi = 0 \quad \rightarrow \quad \tan\theta = 0$$
$$\phi = \pi/2 \quad \rightarrow \quad \tan\theta = 1/\varphi$$

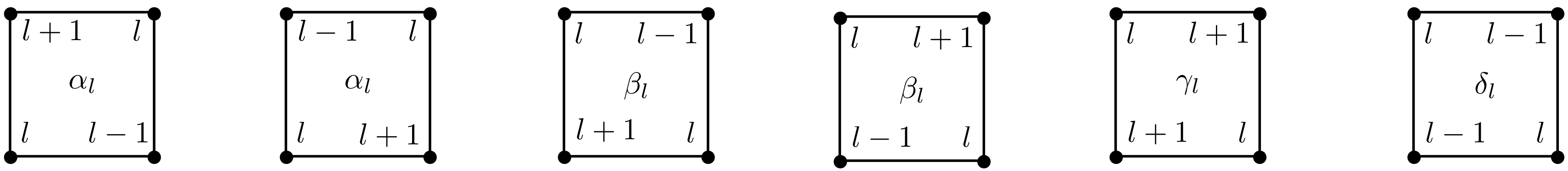


$$H_{J_2-J_3} = \sum_i \cos\theta P^{(\tau)}_{2-\mathrm{body},i} + \sin\theta P^{(\tau)}_{3-\mathrm{body},i}$$

Andrews-Baxter-Forrester model

Solvable height model on square lattice.

Heights take the values $l = 1, 2, \ldots, r-1$
Neighbouring heights differ by one!

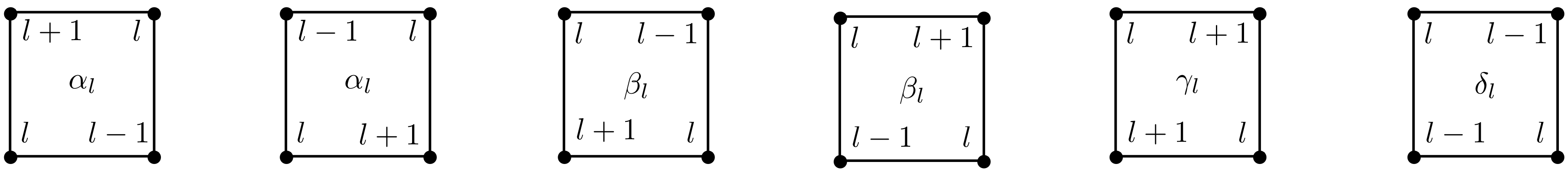


ABF, Nucl.Phys B (1984)

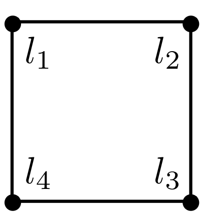
Andrews-Baxter-Forrester model

Solvable height model on square lattice.

Heights take the values $l = 1, 2, \ldots, r-1$
Neighbouring heights differ by one!



$$Z = \sum_{\text{configurations}} \prod_{\text{plaquettes}} W(l_{j_1}, l_{j_2}, l_{j_3}, l_{j_4})$$



$$\leftrightarrow W(l_1, l_2, l_3, l_4)$$

ABF, Nucl.Phys B (1984)

Parameters & phase diagram

Two important parameters in the model: ABF, Huse (1984)

$-1 \leq p \leq 1$ drives a phase transition at $p = 0$

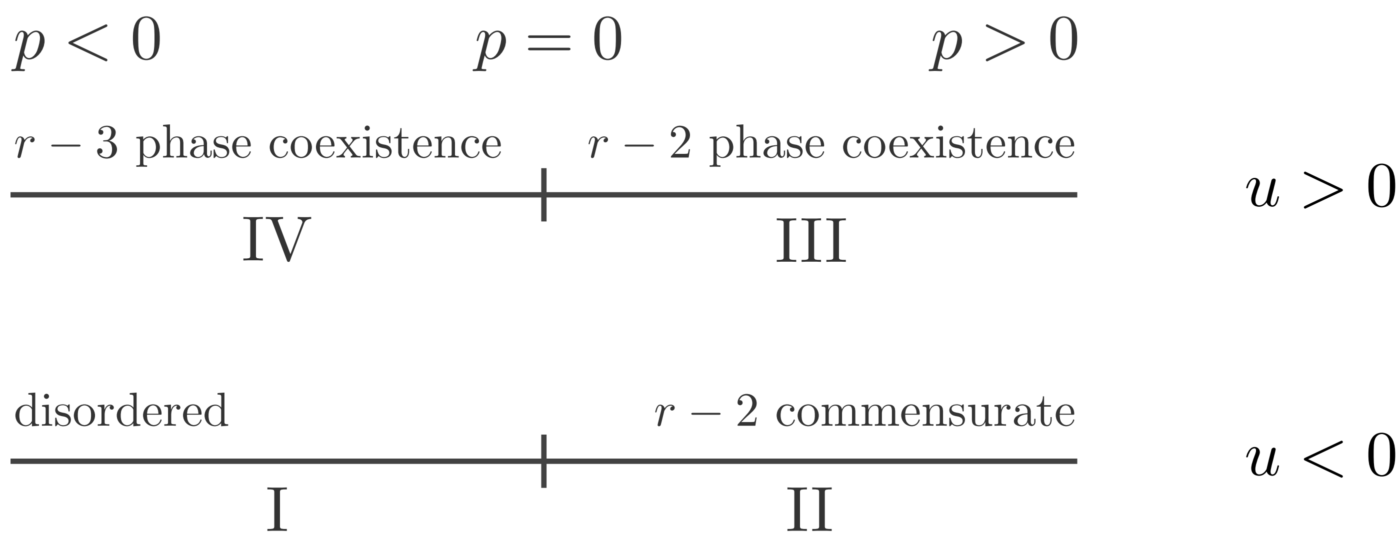
u is related to the anisotropy of the lattice

Parameters & phase diagram

Two important parameters in the model: ABF, Huse (1984)

$-1 \leq p \leq 1$ drives a phase transition at $p = 0$

u is related to the anisotropy of the lattice



The critical behaviour of this model describes the 2-body golden chain, for both signs of the interaction

Form of the weights

The weights are given in terms of elliptic functions:

$$
\begin{aligned}
h(u) &= \theta_1(\frac{u\pi}{2K}, p)\theta_4(\frac{u\pi}{2K}, p) \\
&= 2p^{1/4}\sin(\frac{\pi u}{2K})\prod_{n=1}^{\infty}(1-2p^n\cos(\frac{\pi u}{K})+p^{2n})(1-p^{2n})^2
\end{aligned}
$$

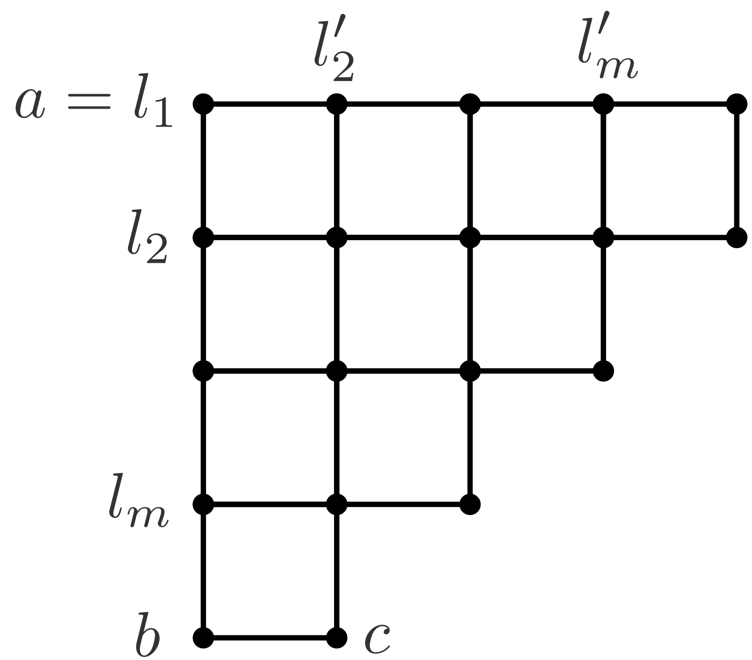
$$
\begin{aligned}
\alpha_l &= \frac{h(2\eta-u)}{h(2\eta)}, \\
\beta_l &= \frac{h(u)}{h(2\eta)}\frac{h(w_{l-1})^{1/2}h(w_{l+1})^{1/2}}{h(w_l)}, \\
\gamma_l &= \frac{h(w_l+u)}{h(w_l)}, \\
\delta_l &= \frac{h(w_l-u)}{h(w_l)}.
\end{aligned}
$$

$$\eta = \frac{K}{r}$$

$$w_l = 2\pi\eta l$$

Corner transfer matrices

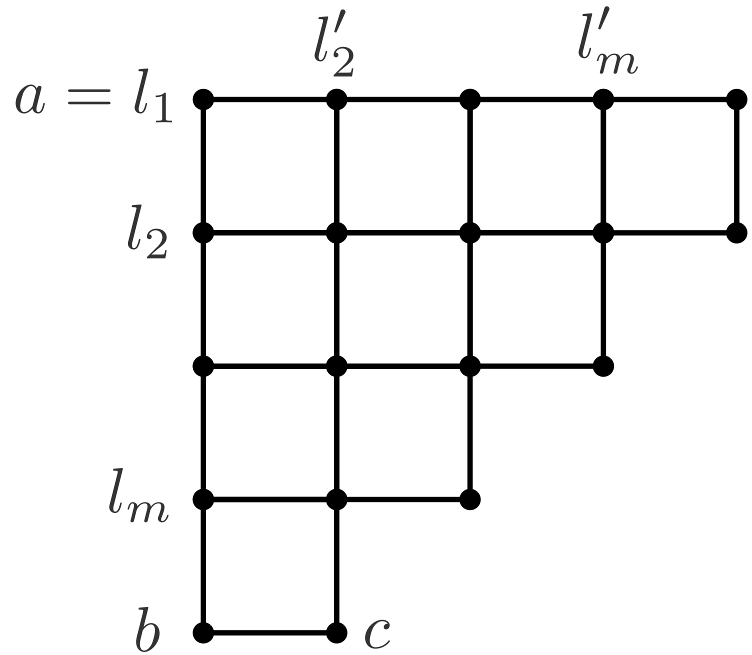
$$A_{l,l'} =$$



$$= \prod_{\text{plaquettes}} W(l_{j1}, l_{j2}, l_{j3}, l_{j4})$$

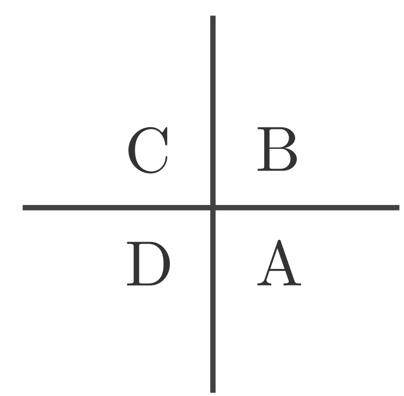
Corner transfer matrices

$A_{l,l'} =$



$$= \prod_{\text{plaquettes}} W(l_{j1}, l_{j2}, l_{j3}, l_{j4})$$

B, C and D are defined analogously, by rotating successively over 90 degrees



Corner transfer matrix method

One can ‘solve’ the model by calculating the probability for the height of the center vertex.

In terms of corner transfermatrices, Z reads $Z = Tr(ABCD)$

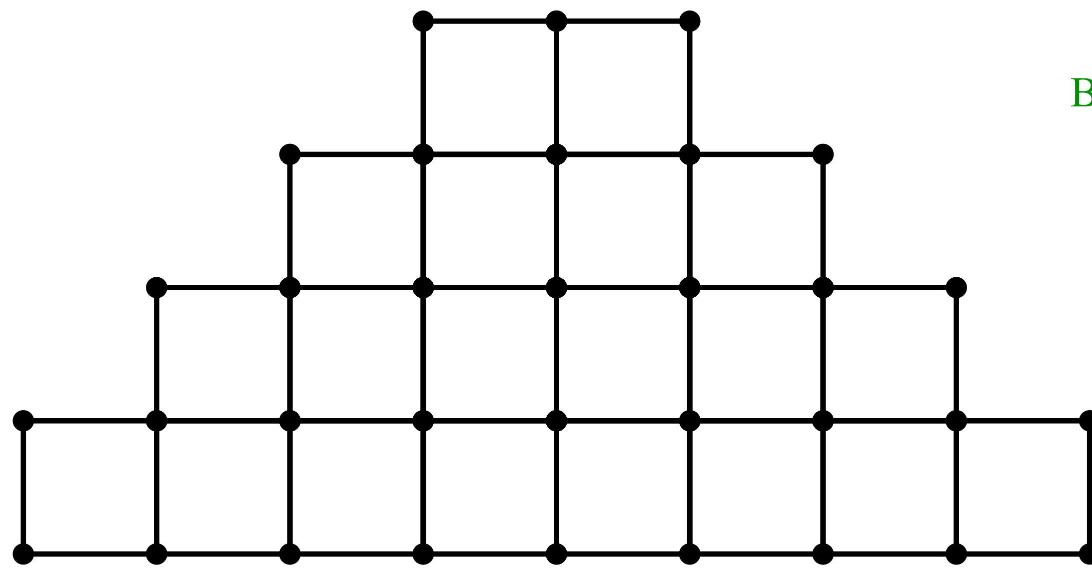
The height probabilities are $P_a = \dfrac{Tr(S_a ABCD)}{Tr(ABCD)}$

S_a is a diagonal matrix, with 1’s on the diagonal for the block with $l_1 = a$

How the method works

Equate, in the large lattice limit, the following:

Baxter's book (1984)

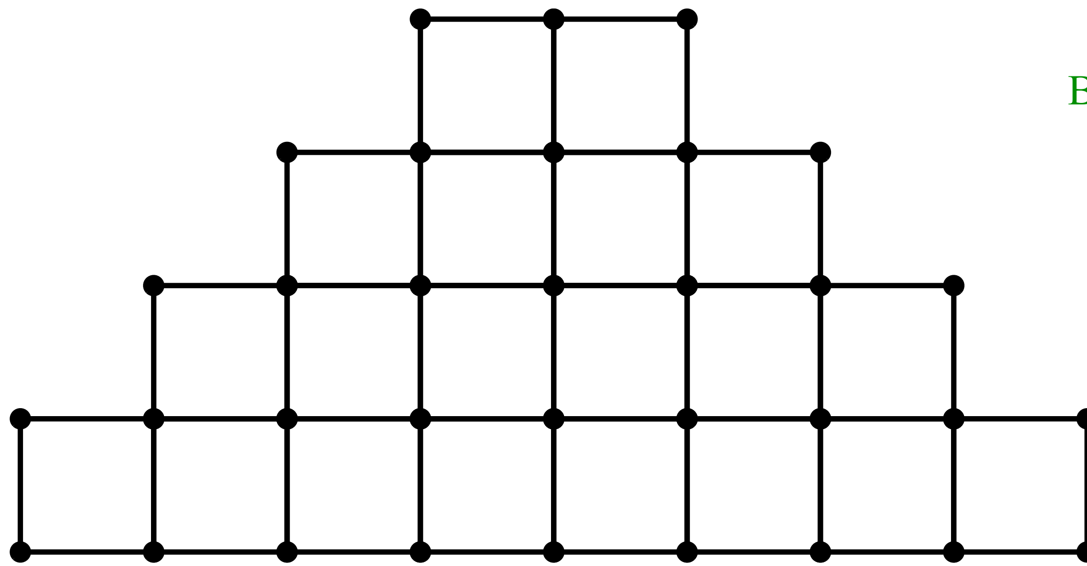
$BC =$ 

How the method works

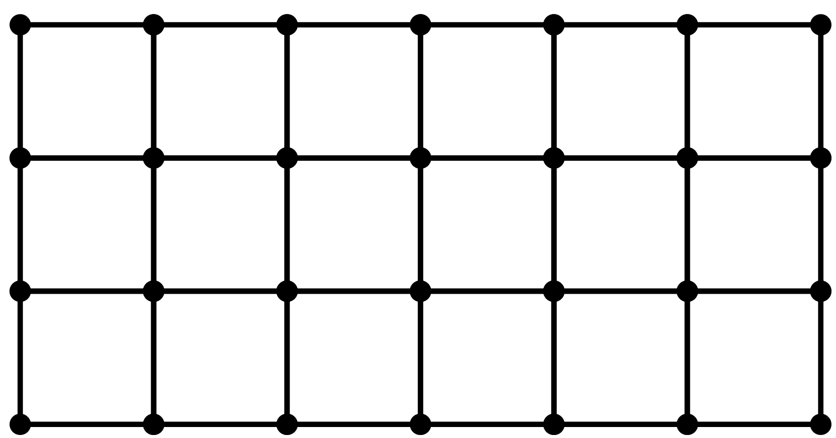
Equate, in the large lattice limit, the following:

Baxter's book (1984)

$BC =$



$T^n =$



Diagonal form of the CTM's

It follows that one can write A,B,C,D in diagonal form

$$\begin{aligned} A(u) &= Q_1 M_1 e^{-u\mathcal{H}} Q_2^{-1} , \\ B(u) &= Q_2 M_2 e^{u\mathcal{H}} Q_3^{-1} , \\ C(u) &= Q_3 M_3 e^{-u\mathcal{H}} Q_4^{-1} , \\ D(u) &= Q_4 M_4 e^{u\mathcal{H}} Q_1^{-1} , \end{aligned}$$

Baxter's book (1984)

$\mathcal{H}$ and the M_i are diagonal, dependence on anisotropy u is explicit

Diagonal form of the CTM's

It follows that one can write A,B,C,D in diagonal form

$$
\begin{aligned}
A(u) &= Q_1 M_1 e^{-u\mathcal{H}} Q_2^{-1} \; , \\
B(u) &= Q_2 M_2 e^{u\mathcal{H}} Q_3^{-1} \; , \\
C(u) &= Q_3 M_3 e^{-u\mathcal{H}} Q_4^{-1} \; , \\
D(u) &= Q_4 M_4 e^{u\mathcal{H}} Q_1^{-1} \; ,
\end{aligned}
$$

Baxter's book (1984)

$\mathcal{H}$ and the M_i are diagonal, dependence on anisotropy u is explicit

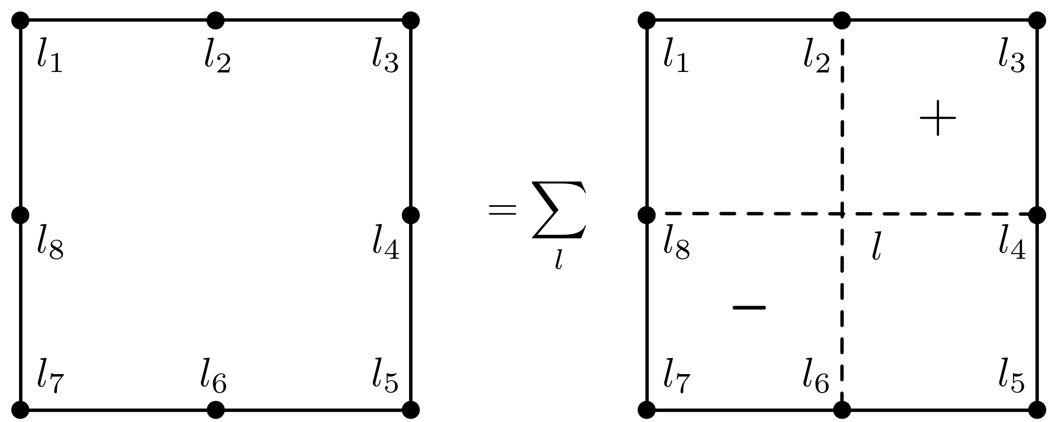
The height probabilities take the following form:

$$P_a = Tr(S_a M_1 M_2 M_3 M_4)/Tr(M_1 M_2 M_3 M_4)$$

The final result follows from considering various limits, such as $u = 0$, $u = (2 \pm r)\eta$ and $p = 1$

New lattice model

We introduce the following new lattice model with the following plaquettes:



$$\tilde{W}(l_1,\ldots,l_8) = \sum_l W(l_1,l_2,l,l_8;u)W(l_2,l_3,l_4,l;u+K)W(l,l_4,l_5,l_6;u)W(l_8,l,l_6,l_7;u-K)$$

Not 6, but 66 different types of plaquettes!

Height probability

When the dust settles, the height probability is given by:

$$P_a = \frac{1}{\mathcal{N}} v_a X_m(a; b, c, d, e; x^t) \qquad x = e^{-4\pi\eta/K'}$$

$$X_m(a; b, c, d, e; q) = \sum_{l_2, \ldots, l_m} q^{\phi(\mathbf{l})} \qquad \phi(\mathbf{l}) \text{: next slide}$$

$\mathcal{N}$ normalization

v_a depends only on the central height

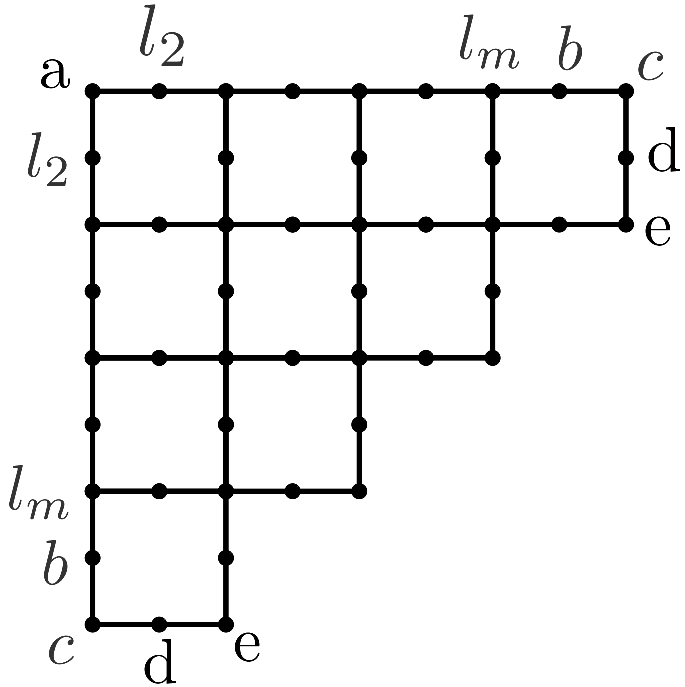
t depends on the region:

$$t = \begin{cases} 2 + r & \text{for } u > 0 \\ 2 - r & \text{for } u < 0 \end{cases}$$

Height probability

$$\phi(\mathbf{l}) = \sum_{j=1}^{(m+1)/2} j\Big(\frac{|l_{2j+3} - l_{2j-1}|}{4} + \delta_{l_{2j-1},l_{2j+1}}\delta_{l_{2j+1},l_{2j+3}}\delta_{l_{2j},l_{2j+2}}\Big)$$

Calculated from the limit $p = 1$, in which only 'diagonal' plaquettes contribute



Ordered phases

The ordered phases at $p > 0$ are obtained by: minimizing $\phi(\mathbf{l})$ for $u > 0$

$$
\begin{array}{ccccccccc}
1 & 2 & 3 & 2 & 1 & 2 & 3 & 2 & 1 \\
2 & & 2 & & 2 & & 2 & & 2 \\
3 & 2 & 1 & 2 & 3 & 2 & 1 & 2 & 3 \\
2 & & 2 & & 2 & & 2 & & 2 \\
1 & 2 & 3 & 2 & 1 & 2 & 3 & 2 & 1
\end{array}
$$

Ordered phases

The ordered phases at $p > 0$ are obtained by:
minimizing $\phi(\mathbf{l})$ for $u > 0$

1	2	3	2	1	2	3	2	1
2		2		2		2		2
3	2	1	2	3	2	1	2	3
2		2		2		2		2
1	2	3	2	1	2	3	2	1

maximizing $\phi(\mathbf{l})$ for $u < 0$

1	2	1	2	1	2	1	2	1
2		2		2		2		2
1	2	1	2	1	2	1	2	1
2		2		2		2		2
1	2	1	2	1	2	1	2	1

Intermezzo:

Intermezzo:

Intermezzo:

meets

Intermezzo:

meets

Ising meets Fibonacci:
Relation between characters of theories with Ising and Fibonacci particles

Grosfeld & Schoutens, PRL (2008)

Intermezzo: Fibonacci meets Leonardo Pisano

Intermezzo: Fibonacci meets Leonardo Pisano

meets

Intermezzo:
Fibonacci meets Leonardo Pisano

meets

Critical behaviour

For the critical behaviour at $p = 0$, we need information about the whole function X_m

$$X_m(a;b,c,d,e;q) = \sum_{l_2,\ldots,l_m} q^{\phi(\{l\})}$$

Critical behaviour

For the critical behaviour at $p = 0$, we need information about the whole function X_m

$$X_m(a; b, c, d, e; q) = \sum_{l_2, \ldots, l_m} q^{\phi(\{l\})}$$

Relevant for $u > 0$

$$X_{43}(1; 2, 1, 2, 3; q) =$$
$$1 + 3q^2 + 4q^3 + 9q^4 + 12q^5 + 22q^6 + 30q^7$$
$$+ \ldots + 5875310q^{121} + \ldots +$$
$$+8q^{235} + 7q^{236} + 4q^{237} + 3q^{238} + 2q^{239} + q^{240} + q^{242}$$

Relevant for $u < 0$

Critical behaviour

$X_{43}(1;2,1,2,3;q) =$

$1 + 3q^2 + 4q^3 + 9q^4 + 12q^5 + 22q^6 + 30q^7$ $\qquad u > 0$

$+ \ldots + 5875310q^{121} + \ldots +$

$+8q^{235} + 7q^{236} + 4q^{237} + 3q^{238} + 2q^{239} + q^{240} + q^{242}$ $\qquad u < 0$

Critical behaviour

$$X_{43}(1;2,1,2,3;q) =$$

$$1 + 3q^2 + 4q^3 + 9q^4 + 12q^5 + 22q^6 + 30q^7 \qquad u > 0$$

$$+ \ldots + 5875310q^{121} + \ldots +$$

$$+8q^{235} + 7q^{236} + 4q^{237} + 3q^{238} + 2q^{239} + q^{240} + q^{242} \qquad u < 0$$

$$1 + 3q^2 + 4q^3 + 9q^4 + 12q^5 + 22q^6 + 30q^7 + \ldots$$

Critical behaviour

$X_{43}(1;2,1,2,3;q) =$

$1 + 3q^2 + 4q^3 + 9q^4 + 12q^5 + 22q^6 + 30q^7$ $\qquad u > 0$

$+ \ldots + 5875310q^{121} + \ldots +$

$+8q^{235} + 7q^{236} + 4q^{237} + 3q^{238} + 2q^{239} + q^{240} + q^{242}$ $\qquad u < 0$

$1 + 3q^2 + 4q^3 + 9q^4 + 12q^5 + 22q^6 + 30q^7 + \ldots$

is a character of (Gepner) $su(3)_2$ parafermions

Critical behaviour

$X_{43}(1;2,1,2,3;q) =$

$1 + 3q^2 + 4q^3 + 9q^4 + 12q^5 + 22q^6 + 30q^7$ $\qquad u > 0$

$+ \ldots + 5875310q^{121} + \ldots +$

$+8q^{235} + 7q^{236} + 4q^{237} + 3q^{238} + 2q^{239} + q^{240} + q^{242}$ $\qquad u < 0$

$1 + 3q^2 + 4q^3 + 9q^4 + 12q^5 + 22q^6 + 30q^7 + \ldots$

is a character of (Gepner) $su(3)_2$ parafermions

$1 + q^2 + 2q^3 + 3q^4 + 4q^5 + 7q^6 + 8q^7 + \ldots$

Critical behaviour

$$X_{43}(1;2,1,2,3;q) =$$
$$1 + 3q^2 + 4q^3 + 9q^4 + 12q^5 + 22q^6 + 30q^7 \qquad u > 0$$
$$+ \ldots + 5875310q^{121} + \ldots +$$
$$+8q^{235} + 7q^{236} + 4q^{237} + 3q^{238} + 2q^{239} + q^{240} + q^{242} \qquad u < 0$$

$$1 + 3q^2 + 4q^3 + 9q^4 + 12q^5 + 22q^6 + 30q^7 + \ldots$$
is a character of (Gepner) $su(3)_2$ parafermions

$$1 + q^2 + 2q^3 + 3q^4 + 4q^5 + 7q^6 + 8q^7 + \ldots$$
is a character of Z_3 parafermions

Critical behaviour

$X_{43}(1;2,1,2,3;q) =$

$1+3q^2+4q^3+9q^4+12q^5+22q^6+30q^7$ $\qquad u>0$

$+\ldots+5875310q^{121}+\ldots+$

$+8q^{235}+7q^{236}+4q^{237}+3q^{238}+2q^{239}+q^{240}+q^{242}$ $\qquad u<0$

$1+3q^2+4q^3+9q^4+12q^5+22q^6+30q^7+\ldots$

is a character of (Gepner) $su(3)_2$ parafermions

$1+q^2+2q^3+3q^4+4q^5+7q^6+8q^7+\ldots$

is a character of Z_3 parafermions

So, Fibonacci meets Fibonacci!

Connection with CFT

For $r = 5$ $(k = 3)$, we have explicit formulas for the functions X_m

These reproduce all the characters of the Z_3 and $su(3)_2$ parafermions.

Connection with CFT

For $r = 5$ ($k = 3$), we have explicit formulas for the functions X_m

These reproduce all the characters of the Z_3 and $su(3)_2$ parafermions.

The critical behaviour for arbitrary k is given by:

$$\frac{su(2)_1 \times su(2)_1 \times su(2)_{k-2}}{su(2)_k} \quad \text{for} \quad u > 0$$

Z_k parafermions for $u < 0$

Similar results on related models :
Date, Jimbo...
Saleur, Bauer
...

Connection with CFT

The scaling dimensions of the primary fields $\Phi^{t_1,s_1}_{s_2}$ of

$$\frac{su(2)_1 \times su(2)_1 \times su(2)_{k-2}}{su(2)_k}$$

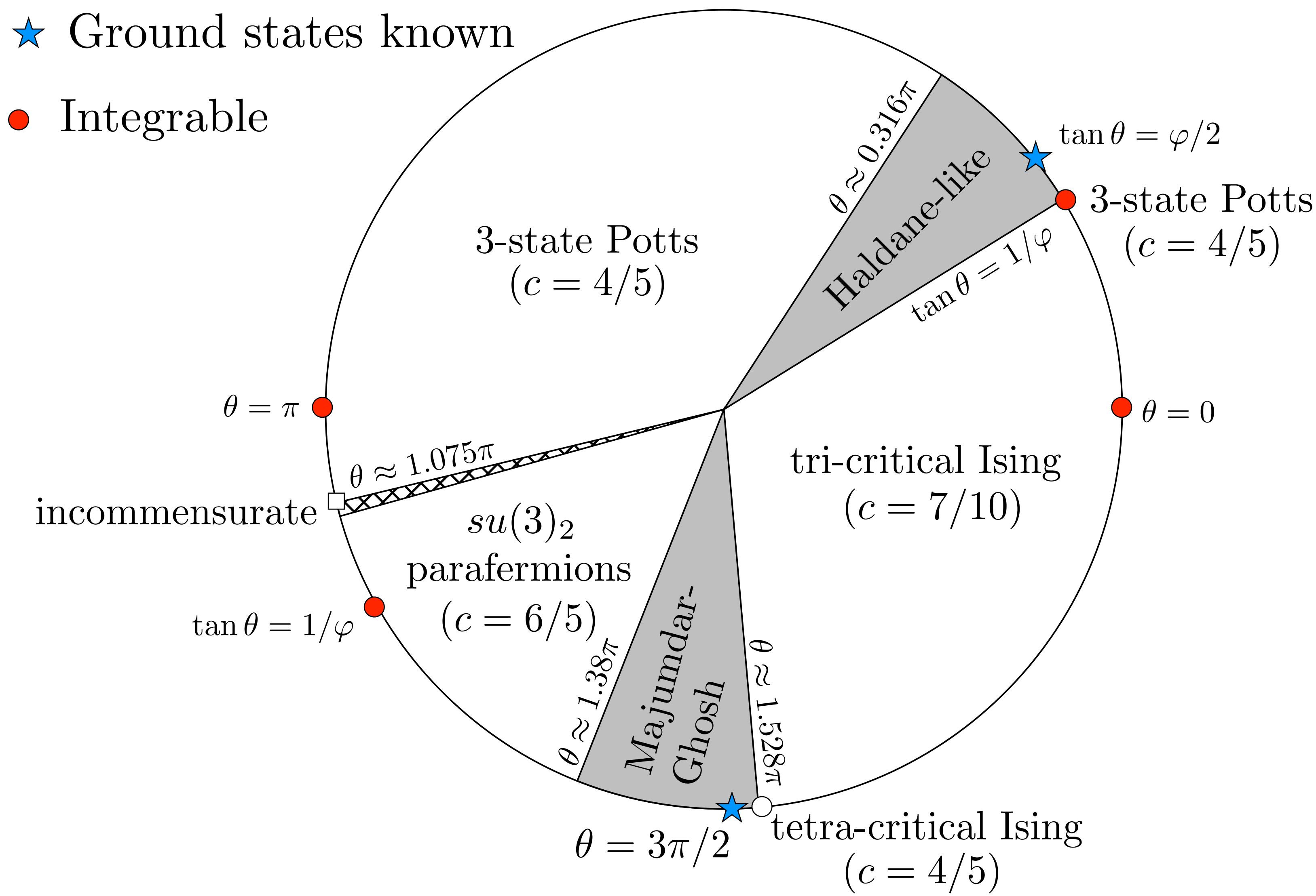
can be obtained from the Coulomb gas results by Ikhlef et al.:

Ikhlef et.al., JPA (2010)

$$h(t,s_1,s_2) = \begin{cases} \frac{(s_1(k+2)-s_2k)^2-4}{8k(k+2)} + \frac{1}{2} - \frac{(s_1-s_2+2t) \bmod 4}{4} & s_1+s_2 \bmod 2 = 0 \\ \frac{(s_1(k+2)-s_2k)^2-4}{8k(k+2)} + \frac{1}{8} & s_1+s_2 \bmod 2 = 1 \end{cases}$$

Explicit character formulas for $k > 3$ have not yet been found...

Updated phase diagram



Conclusions

- Studied an exactly solvable point in an anyonic chain with competing interactions.
- Introduced a new 2-d, solvable height model
- Obtained the critical behaviour, explaining an extended critical region in the chain.
- Connection with CFT was made

Outlook

- Connection with SU(2) Heisenberg chains
- Understanding of (topological) phase transitions
- Connection with related loop models
- Other types of anyonic chains
- Relation with Rogers-Ramanujan identities?
- Finitization of characters might have other (qHe) applications

Nordita program

July 30 - August 25, 2012:

Topological States of Matter:
Insulators, Superconductors, and Hall Liquids

www.nordita.org/~ardonne/workshops.html