A master solution of the Yang-Baxter equation and classical discrete integrable equations.

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- Lattice models of statistical mechanics and field theory,
  - Quantum Yang-Baxter equation. Star-triangle relation.
  - low-temperature (quasi-classical) limit and its relation to classical mechanics.
- New “master” solution to the star-triangle relation (STR) contains
  - all previously known solutions to STR
    - Ising & Kashiwara-Miwa models
    - Fateev-Zamolodchikov & chiral Potts models
  - elliptic gamma-functions & Spiridonov’s elliptic beta integral
- Low-temperature (quasi-classical) limit of the “master solution”.
  - relation to the Adler-Bobenko-Suris classical non-linear integrable equations on quadrilateral graphs,
  - new integrable models of statistical mechanics where the Boltzmann weights are determined by classical integrable equations (Q_4).

Local "spins": $\sigma_i \in$ (set of values), $\quad \sigma_i \in \mathbb{R}$

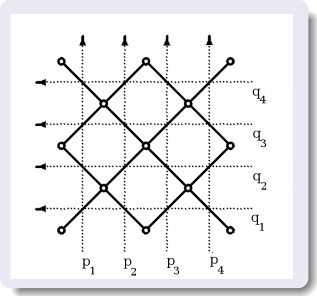
$$Z = \sum_{\{spins\}} e^{-E(\sigma)/T},$$

$$E(\{\sigma\}) = \sum_{(ij)\in edges} \epsilon(\sigma_i, \sigma_j),$$

Boltzmann weights

$$W(\sigma_i, \sigma_j) = e^{-\epsilon(\sigma_i,\sigma_j)/T}$$

$$Z = \sum_{\{spins\}} \prod_{(ij)\in edges} W(\sigma_i, \sigma_j).$$



The problem: calculate partition function when number of edges is infinite,

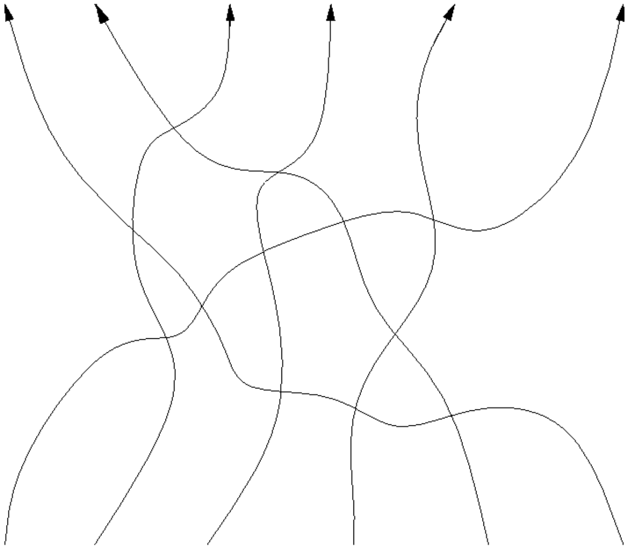
$$\log Z \ = \ -Nf/T + O(\sqrt{N}), \qquad N \to \infty$$

Solvable analytically if the Boltzmann weights satisfy the Yang-Baxter equation

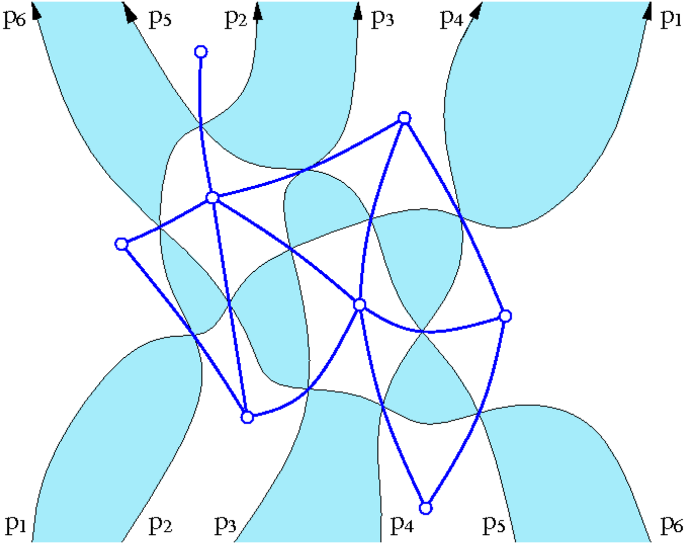
YBE is an overdetermined system of algebraic equations. Its general solution is unknown even in the simplest cases.

- Known solutions (various methods):McGuire, Yang, Baxter, Cherednik, Korepin, Izergin, Perk, Schultz, Fateev, Zamolodchikov(s), Kulish, Reshetikhin, Kirillov, Sklyanin, Smirnov, Belavin, McCoy, Au-Yang, Stroganov, Andrews, Forrester, Bazhanov, Jimbo, Kashiwara, Miwa, Date, Okado, Kuniba, Miki, Nakanishi, Hasegawa, Yamada, Pearce, Warnaar, Seaton, Nienhuis, Lukyanov, Faddeev, Volkov, Mangazeev, Kashaev, Akutsu, Deguchi, Wadati, Sergeev, Khoroshkin, Teschner, Lukowski, Frassek, Meneghelli, Staudacher, . . .
- Algorithmic recipes: Universal R-matrix for quantized affine Lie algebras (quantum groups) (Drinfeld-Jimbo)
- almost all known solutions have been included in the quantum group scheme (up to elliptic deformations, vertex-face transformations, etc.).
- 3D-generalization: tetrahedron equation, Zamolodchikov (1980) followed by Baxter, Bazhanov, Kashaev, Korepanov, Mangazeev, Maillet-Nijhoff, Sergeev, Stroganov,. . .

Graph $\mathcal{L}$, an arrangement of pseudolines (rapidity lines)

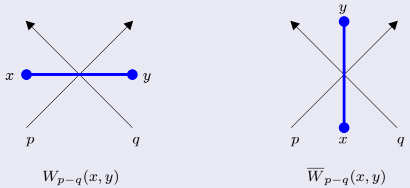


Planar graph G, where $\mathcal{L}$ is the medial graph

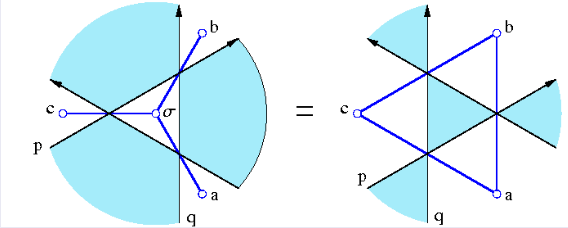


Two types of Boltzmann weights, depending on the arrangement of rapidity line wrt the edge

$W_{p-q}(x,y)$ and $\overline{W}_{p-q}(x,y)$.

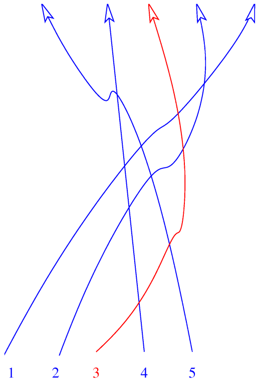


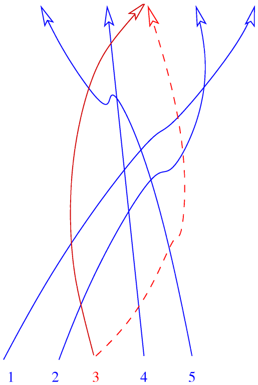
Simplest form of the Yang-Baxter equation: the *star-triangle relation*



$$\sum_{\sigma} \overline{W}_{p-q}(\sigma,b)\, W_{p-r}(c,\sigma)\, \overline{W}_{q-r}(a,\sigma) \;=\; W_{p-q}(c,a)\overline{W}_{p-r}(a,b)\, W_{q-r}(c,b)\,.$$

Partition function depends only on the boundary data (i.e., on values of boundary spins and values of rapidities) but not on details of the lattice inside.





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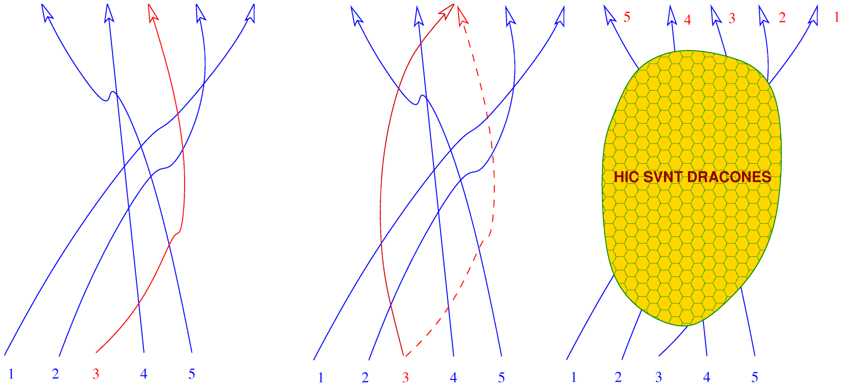
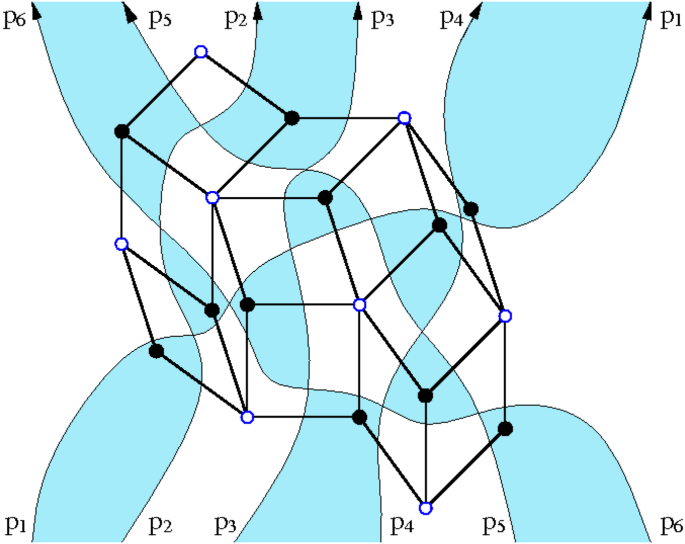
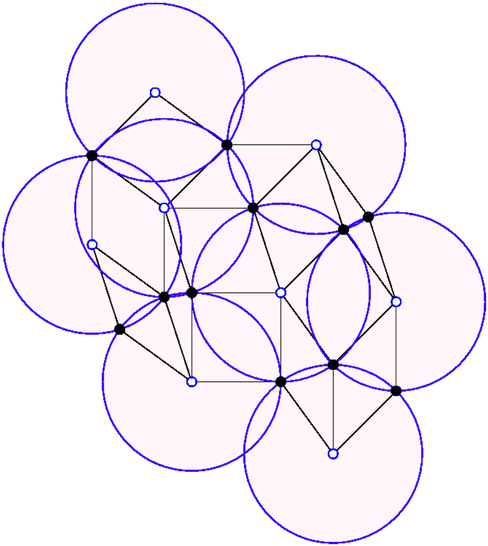


Relation to quad-graphs and rhombic tilings





In general, weights $\overline{W}$ are related to W via

$$\overline{W}_{p-q}(x,y) \;=\; \sqrt{S(x)S(y)}W_{\eta-p+q}(x,y)\;,$$

where $S(x)$ are one-"spin" weights and η is the non-zero crossing parameter (value of straight angle).
In most cases the Boltzmann weights W are symmetric,

$$W_{p-q}(x,y) = W_{p-q}(y,x)\;.$$

Let for shortness

$$p-q=\alpha_1\;,\quad q-r=\alpha_3\;.$$

The star-triangle relation takes the form (assume continuous spins)

$$\int dx_0 S(x_0)W_{\eta-\alpha_1}(x_1,x_0)W_{\alpha_1+\alpha_3}(x_2,x_0)W_{\eta-\alpha_3}(x_3,x_0)$$

$$=\; W_{\alpha_1}(x_2,x_3)W_{\eta-\alpha_1-\alpha_3}(x_1,x_3)W_{\alpha_3}(x_1,x_2)$$

Partition function

$$Z = \int \prod_{(ij)} W_{\alpha_{ij}}(x_i, x_j) \ \prod_m S(x_m) \ dx_m$$

$$\alpha_{ij} = \begin{cases} p - q, & \text{for a first type edge} \\ \eta - p + q, & \text{for a second type edge} \end{cases}$$

Assume, there is a temperature-like parameter ε, such for $\varepsilon \to 0$

$$W_\alpha(x, y) \ = \ e^{-\Lambda_\alpha(x,y)/\varepsilon + \mathcal{O}(1)} \ , \quad S(x) \ = \ \varepsilon^{-1/2} e^{-C(x)/\varepsilon + \mathcal{O}(1)}$$

$$\log Z = -\frac{1}{\varepsilon}\, \mathcal{E}(x^{(cl)}) + O(1), \qquad \mathcal{E}(x) = \sum_{(ij)} \Lambda_{\alpha_{ij}}(x_i, x_j) + \sum_m C(x_m)$$

and the variables $x^{(cl)} = \{\boldsymbol{x}_1^{(cl)}, x_2^{(cl)}, \ldots\}$ solve the variational equations

$$\frac{\partial \mathcal{E}(x)}{\partial x_j}\Big|_{x = x^{(cl)}} = 0$$

Examples: Bobenko-Kutz-Pinkall ('93), Faddeev-Volkov ('94), Bazhanov-Bobenko-Reshetikhin ('96), Adler-Bobenko-Suris ('03).

$$\int \varepsilon^{-1/2} dx_0 \exp\left\{ -\frac{\mathcal{E}_\star(x_0)}{\varepsilon} + \mathcal{O}(1) \right\} = \exp\left\{ -\frac{\mathcal{E}_\triangle}{\varepsilon} + \mathcal{O}(1) \right\}$$

where

$$\mathcal{E}_\star = \Lambda_{\eta-\alpha_1}(x_0, x_1) + \Lambda_{\alpha_1+\alpha_3}(x_0, x_2) + \Lambda_{\eta-\alpha_3}(x_0, x_3) + C(x_0)\ ,$$

$$\mathcal{E}_\triangle = \Lambda_{\alpha_1}(x_2, x_3) + \Lambda_{\eta-\alpha_1-\alpha_3}(x_1, x_2) + \Lambda_{\alpha_3}(x_1, x_2)$$

the STR implies

$$\mathcal{E}_\star = \mathcal{E}_\triangle$$

at the stationary point

$$\frac{\partial \mathcal{E}_\star}{\partial x_0} = 0$$

Any solution of STR, admitting low-temperature expansion, leads to classical discrete integrable system, whose action is invariant under star-triangle moves

Elliptic gamma-function

$$\frac{\Gamma(x+1)}{\Gamma(x)} = x \,, \quad \frac{\Gamma_{\mathrm{trig}}(x+\delta)}{\Gamma_{\mathrm{trig}}(x)} \sim \sinh(x) \,, \quad \frac{\Gamma_{\mathrm{ell}}(x+\delta)}{\Gamma_{\mathrm{ell}}(x)} \sim \vartheta_1(x|\tau)$$

Let q,p be the temperature-like parameters (elliptic nomes)

$$\mathsf{q} \;=\; e^{\mathsf{i}\pi\tau'}\,,\quad \mathsf{p}=e^{\mathsf{i}\pi\tau}\quad \operatorname{Im}(\tau,\tau')>0\;.$$

The crossing parameter $\eta>0$ is given by

$$e^{-2\eta} \;=\; \mathsf{pq}\;,\quad \mathsf{i}\eta \;=\; \frac{1}{2}\pi(\tau+\tau')\;.$$

In what follows, we consider the primary physical regimes

$$\eta>0\;,\quad \mathsf{p},\mathsf{q}\in\mathbb{R}\quad \text{or}\quad \mathsf{p}^*=\mathsf{q}\;.$$

The elliptic gamma-function is defined by

$$\Phi(z) \;=\; \prod_{j,k=0}^{\infty}\frac{1-e^{2\mathsf{i}z}\mathsf{q}^{2j+1}\mathsf{p}^{2k+1}}{1-e^{-2\mathsf{i}z}\mathsf{q}^{2j+1}\mathsf{p}^{2k+1}} \;=\; \exp\left\{\sum_{n\neq 0}\frac{e^{-2\mathsf{i}zn}}{k(\mathsf{q}^n-\mathsf{q}^{-n})(\mathsf{p}^n-\mathsf{p}^{-n})}\right\}\;.$$

- $\Phi(z)$ is π-periodic,

$$\Phi(z+\pi) = \Phi(z)\ ,$$

- $\log\Phi$ is odd,

$$\Phi(z)\Phi(-z) = 1\ ,$$

- Zeros and poles:

$$\text{Zeros of } \Phi(z) = \{-\mathsf{i}\eta - j\pi\tau - k\pi\tau' \mod \pi\ , \quad j,k \geq 0\}\ ,$$

$$\text{Poles of } \Phi(z) = \{+\mathsf{i}\eta + j\pi\tau + k\pi\tau' \mod \pi\ , \quad j,k \geq 0\}\ ,$$

Exponential formula for $\Phi(z)$ is valid in the strip

$$-\eta < \mathrm{Im}(z) < \eta\ .$$

- Diference property:

$$\frac{\Phi(z-\frac{\pi\tau'}{2})}{\Phi(z+\frac{\pi\tau'}{2})} = \prod_{n=0}^{\infty}(1-e^{2\mathsf{i}z}\mathsf{p}^{2n+1})(1-e^{-2\mathsf{i}z}\mathsf{p}^{2n+1}) \sim \vartheta_4(z\,|\,\tau)\ ,$$

and similarly with $\tau \leftrightarrows \tau'$.

Boltzmann weights

Define the weights $\mathbb{W}$ and $\overline{\mathbb{W}}$ by

$$\mathbb{W}_\alpha(x,y) \;=\; \kappa(\alpha)^{-1}\,\frac{\Phi(x-y+\mathsf{i}\alpha)}{\Phi(x-y-\mathsf{i}\alpha)}\;\frac{\Phi(x+y+\mathsf{i}\alpha)}{\Phi(x+y-\mathsf{i}\alpha)}$$

and

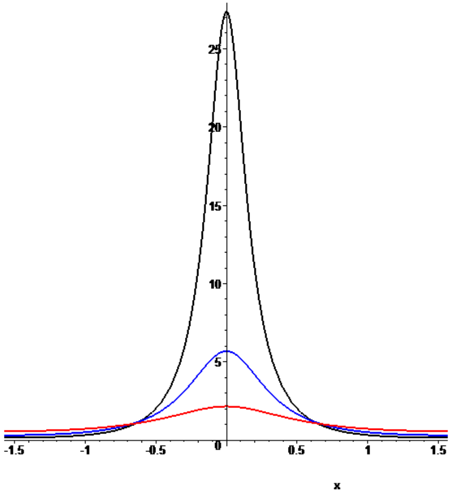
$$\overline{\mathbb{W}}_\alpha(x,y) = \sqrt{\mathbb{S}(x)\mathbb{S}(y)}\,\mathbb{W}_{\eta-\alpha}(x,y)\,, \quad \mathbb{S}(x) \;=\; \frac{e^{\eta/2}}{2\pi}\vartheta_1(2x\,|\,\tau)\vartheta_1(2x\,|\,\tau')\,.$$

Normalization factor (partition function per edge – exact solution) $\kappa(\alpha)$ is given by

$$\kappa(\alpha) \;=\; \exp\left\{\sum_{n\neq 0}\frac{e^{4\alpha n}}{n(\mathsf{p}^n-\mathsf{p}^{-n})(\mathsf{q}^n-\mathsf{q}^{-n})(\mathsf{p}^n\mathsf{q}^n+\mathsf{p}^{-n}\mathsf{q}^{-n})}\right\}\,.$$

It satisfies

$$\frac{\kappa(\eta-\alpha)}{\kappa(\alpha)} \;=\; \Phi(\mathsf{i}\eta-2\mathsf{i}\alpha)\,, \quad \kappa(\alpha)\kappa(-\alpha)=1\,.$$

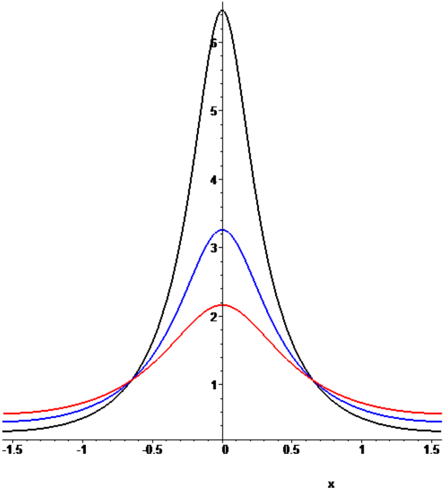


Plot of the real π-periodic function

$$R_\alpha(x) = \frac{\Phi(x + \mathsf{i}\alpha)}{\Phi(x - \mathsf{i}\alpha)}$$

for $\mathsf{p} = \mathsf{q} = \frac{1}{2}$ and

- red: $\alpha = \frac{\eta}{4}$
- blue: $\alpha = \frac{\eta}{2}$
- black: $\alpha = \frac{3\eta}{4}$



Plot of the real π-periodic function

$$R_\alpha(x) = \frac{\Phi(x + \mathsf{i}\alpha)}{\Phi(x - \mathsf{i}\alpha)}$$

for $\alpha = \eta/4$ and

- red: $\mathsf{p} = \mathsf{q} = 0.5$
- blue: $\mathsf{p} = \mathsf{q} = 0.6$
- black: $\mathsf{p} = \mathsf{q} = 0.7$

- The weights $\mathbb{W}_\alpha(x,y)$ and $\overline{\mathbb{W}}_\alpha(x,y)$ are real positive for

$$x, y \in \mathbb{R} \quad \text{and} \quad 0 < \alpha < \eta$$

- The weights are symmetric and π-periodic,

$$\mathbb{W}_\alpha(x,y) = \mathbb{W}_\alpha(y,x) = \mathbb{W}_\alpha(-x,y) = \mathbb{W}_\alpha(x+\pi,y) = \ldots \ .$$

- Difference properties of the weights:

$$\frac{\mathbb{W}_\alpha(x - \frac{\pi\tau'}{2}, y)}{\mathbb{W}_\alpha(x + \frac{\pi\tau'}{2}, y)} = \frac{\vartheta_4(x - y + \mathsf{i}\alpha \,|\, \tau)}{\vartheta_4(x - y - \mathsf{i}\alpha \,|\, \tau)} \frac{\vartheta_4(x + y + \mathsf{i}\alpha \,|\, \tau)}{\vartheta_4(x + y - \mathsf{i}\alpha \,|\, \tau)}$$

and similarly with $\tau \leftrightarrows \tau'$.

As a mathematical identity the star-triangle realtion for this solution is equivalent to Spiridonov's selebrated elliptic beta integral (2001).

This identity lies in the basis of the theory of elliptic hypergeometric functions.

Its connection with the Yang-Baxter equation (star-triangle relation) was not hitherto known

"Trigonometric" limit.

$$\tau = \mathsf{ib}/R\,, \quad \tau' = \mathsf{ib}^{-1}/R\,, \quad R \to \infty$$

Gamma-function with small argument

$$\Phi(\frac{\pi}{R}\sigma) \;\to\; \varphi(\sigma) = \;\exp\left\{\frac{1}{4}\int_{\mathrm{pv}} \frac{dw}{w}\,\frac{e^{-2\mathsf{i}\sigma w}}{\sinh(\mathsf{b}w)\sinh(w/\mathsf{b})}\right\}$$

Gamma-function with big argument

$$\Phi(\frac{\pi}{R}\sigma + \mathrm{const}) \;\to\; 1\,, \quad \mathrm{const} = \mathcal{O}(R^0)\,.$$

Two regimes of the star-triangle equation:

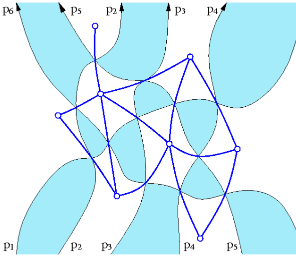
$$x_j = \mathrm{const} + \frac{\pi}{R}\sigma_j \quad \text{and} \quad x_j = \frac{\pi}{R}\sigma_j\,.$$

Faddeev-Volkov solution with continuous spins $s_i \in \mathbb{R}$

$$Z = \sum_{\{spins\}} \prod_{(ij)\in edges} W_{\theta_{ij}}(s_i - s_j), \quad \theta_{ij} = \begin{cases} p_i - p_j, \\ \pi - (p_i - p_j) \end{cases}$$

$$W_\theta(s) = \frac{e^{2\eta\theta s}}{\varkappa(\theta)} \frac{\varphi(s + \mathsf{i}\eta\theta/\pi)}{\varphi(s - \mathsf{i}\eta\theta/\pi)}, \qquad \eta = (b + b^{-1})/2$$

$$\varphi(z) = \exp\left(\frac{1}{4}\int_{\mathbb{R}+\mathsf{i}0} \frac{e^{-2\mathsf{i}zw}}{\sinh(wb)\sinh(w/b)} \frac{dw}{w}\right),$$



- ["continuation" of Fateev-Zamolodchikov Z_N-model to negative N.]
- Boltzmann weights $W_\theta(s)$ are strictly positive.
- Modular duality (Faddeev 1994,2000; Ponsot-Teschner 2001)

$$U_q(sl_2) \otimes U_{\tilde{q}}(sl_2), \qquad q = e^{i\pi b^2}, \quad \tilde{q} = e^{-i\pi/b^2}, \quad c_L = 1 + 6(b + b^{-1})^2.$$

- Describes "quantum circle patterns"(Bazhanov-Mangazeev-Sergeev)

Quasi-classical limit, $b \to 0$, $c_L \to +\infty$

Parameter b^2 = Planck constant $\hbar$.

$$Z = \int e^{-\frac{1}{2\pi b^2}\mathcal{A}[\rho]} \prod_i \frac{d\rho_i}{2\pi b}, \qquad \sigma_i \to \frac{\rho_i}{2\pi b}$$

$$\mathcal{A}[\rho] = \sum_{\text{edges } (ij)} \mathcal{L}_{\theta_{(ij)}}(\rho_i - \rho_j), \qquad \mathcal{L}_\theta(\rho) = \frac{1}{\mathsf{i}} \int_0^\rho \log\left(\frac{1 + e^{\xi + \mathsf{i}\theta}}{e^\xi + e^{\mathsf{i}\theta}}\right) d\xi$$

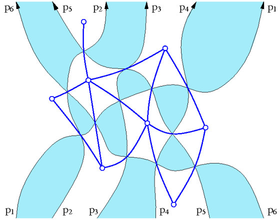
Stationary point

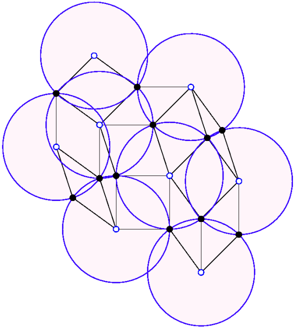
$$\frac{\partial \mathcal{A}[\rho]}{\partial \rho_i}\Big|_{\rho = \rho^{(cl)}} = 0, \quad \Rightarrow \sum_{(ij) \in star(i)} \phi(\theta_{ij} | \rho_i - \rho_j) = \sum_{(ij) \in star(i)} \theta_{ij} \quad \ldots = 2\pi,$$

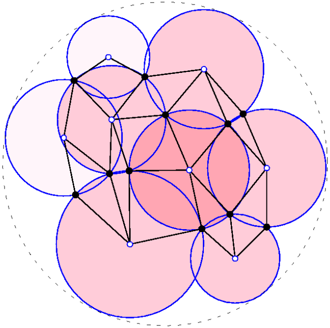
$$\phi(\theta|\rho) = \frac{1}{i} \log\left(\frac{1 + e^{\rho + \mathsf{i}\theta}}{1 + e^{\rho - \mathsf{i}\theta}}\right)$$

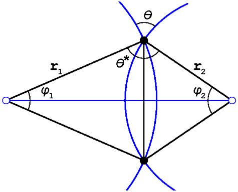
Hirota equations (second order difference eqns. for e^{ρ_i}). Admit a trivial solution

$$\rho_i = const, \qquad \phi_{ij} = \theta_{ij}, \qquad \rho_i = \log r_i$$









"Circle flower" equations (Hirota equations):

$$\sum_{(ij)\in star(i)} \varphi_{(ij)} = 2\pi, \qquad i \in V_{int}(\mathcal{G}), \qquad\qquad \varphi_1 = \frac{1}{i}\log\frac{r_1 + r_2 e^{i\theta}}{r_1 + r_2 e^{-i\theta}},$$

Integrable circle patterns (admit iso-radial solution),

$$\sum_{(ij)\in star(i)} \theta_{ij} = 2\pi \ ,$$

We consider the low-temperature limit outside the primary physical regime:

$$\mathsf{p}^2 = \boldsymbol{e}^{2\mathsf{i}\pi\tau} \quad \text{and} \quad \mathsf{q}^2 = e^{-T/N^2}\omega\,, \quad \omega = e^{2\pi\mathsf{i}/N}\,, \quad T \to 0\,.$$

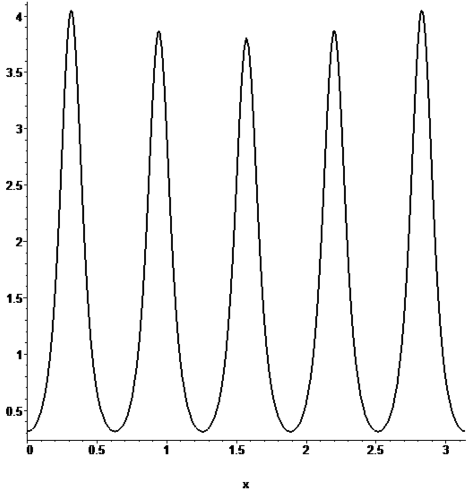
Asymptotic of $\mathbb{W}$: the low-T expansion

$$\mathbb{W}_\alpha(x,y) = \exp\left\{-\frac{\Lambda_\alpha(x,y)}{T}\right\} \cdot W_\alpha(x,y) \cdot (1 + \mathcal{O}(T))$$

where the Lagrangian density $\Lambda_\alpha(x,y)$ is $\frac{\pi}{N}$ periodic in x and y while the finite part $W_\alpha(x,y)$ is π-periodic.
Asymptotic of the partition function:

$$\mathcal{Z} = \underbrace{\int \ldots \int}_{0\le x_m \le \pi} \exp\left\{-\frac{\mathcal{E}(\{x\})}{T} + \mathcal{O}(1) + \mathcal{O}(T)\right\} \prod_m \frac{dx_m}{\sqrt{T}}\,, \quad T \to 0\,,$$

where $\mathcal{E}(\{x\})$ is an action for a classical discrete integrable system. The ground state of the system is highly degenerate due to π/N periodicity.



Plot of

$$\mathrm{abs}\left(\frac{\Phi(x+\mathsf{i}\alpha)}{\Phi(x-\mathsf{i}\alpha)}\right)$$

with $\mathsf{p}=\frac{1}{2}$ and

$$\mathsf{q}=0.99\cdot e^{\mathsf{i}\pi/5}\ .$$

The peaks are at

$$x=\frac{\pi}{N}(n+\tfrac{1}{2}),$$

Expression for the Lagrangian density:

$$\Lambda_\alpha(x,y) = 2\mathrm{i}N\int_0^{x-y} d\xi \log\frac{\vartheta_3(N(\xi-\mathrm{i}\alpha)\,|\,N\tau)}{\vartheta_3(N(\xi+\mathrm{i}\alpha)\,|\,N\tau)} + 2\mathrm{i}N\int_{\pi/2N}^{x+y} d\xi \log\frac{\vartheta_3(N(\xi-\mathrm{i}\alpha)\,|\,N\tau)}{\vartheta_3(N(\xi+\mathrm{i}\alpha)\,|\,N\tau)}$$

$$\begin{aligned}\Lambda_{\eta-\alpha}(x,y) &= \frac{\pi^2}{2} - (Nx)^2 - (Ny)^2 \\ &+2\mathrm{i}N\int_0^{x-y} d\xi \log\frac{\vartheta_1\big(N(\mathrm{i}\alpha+\xi)\,|\,N\tau\big)}{\vartheta_1\big(N(\mathrm{i}\alpha-\xi)\,|\,N\tau\big)} + 2\mathrm{i}N\int_{\pi/2N}^{x+y} d\xi \log\frac{\vartheta_1\big(N(\mathrm{i}\alpha+\xi)\,|\,N\tau\big)}{\vartheta_1\big(N(\mathrm{i}\alpha-\xi)\,|\,N\tau\big)}\ .\end{aligned} \tag{1}$$

$$\mathcal{C}(x) \;=\; 2\Big(x-\frac{\pi}{2}\Big)^2\ . \qquad 0<x<\frac{\pi}{N} \tag{2}$$

Energy for the regular square lattice

$$\mathcal{E}(X) \;=\; \sum_{(ij)}\Lambda(\alpha\,|\,x_i,x_j) + \sum_{(kl)}\Lambda(\eta-\alpha\,|\,x_k,x_l) + \sum_m \mathcal{C}(x_m)\ , \tag{3}$$

Variational equations (Adler-Bobenko-Suris Q_4 eqns.)

$$\frac{\partial\mathcal{E}(X)}{\partial x_i} = 0, \quad\Rightarrow\quad \Psi_3\big(x,x_r\big)\Psi_3(x,x_\ell) = \Psi_1\big(x,x_u\big)\Psi_1(x,x_d)\ ,$$

$$\Psi_j(x,y) \;=\; \frac{\vartheta_j\big(N(x-y+\mathrm{i}\alpha)\,|\,N\tau\big)\ \vartheta_j\big(N(x+y+\mathrm{i}\alpha)\,|\,N\tau\big)}{\vartheta_j\big(N(x-y-\mathrm{i}\alpha)\,|\,N\tau\big)\ \vartheta_j\big(N(x+y-\mathrm{i}\alpha)\,|\,N\tau\big)}\ , \qquad j=1,2,3,4.$$

Due to $\frac{\pi}{N}$-periodicity of the leading term, we introduce the discrete spin variables n_j,

$$x_j = \xi_j + \frac{\pi}{N} n_j \ , \quad 0 < \mathrm{Re}(\xi_j) < \frac{\pi}{2N} \ , \quad n_j \in \mathbb{Z}_N$$

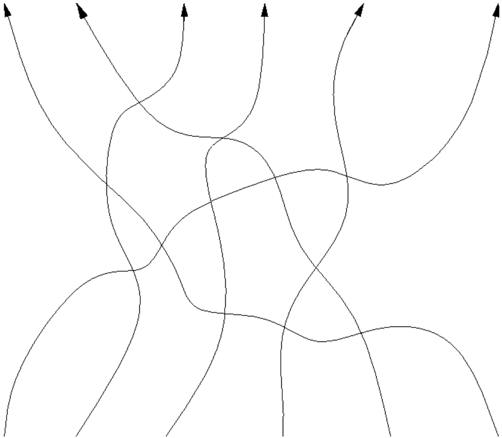
where parameter ξ_0 is the solution of the variational equation (in general: parameters ξ_j are solution of classical integrable equations). Canceling then the $\mathcal{T}^{-1}$ term, we come to the most general discrete-spin star-triangle equation:

$$\sum_{n_0 \in \mathbb{Z}_N} \overline{W}_{pq}(x_0, x_1) W_{pr}(x_0, x_2) \overline{W}_{qr}(x_0, x_3) \\ = \mathcal{R}_{pqr} W_{pq}(x_2, x_3) \overline{W}_{pr}(x_1, x_3) W_{qr}(x_1, x_2)$$

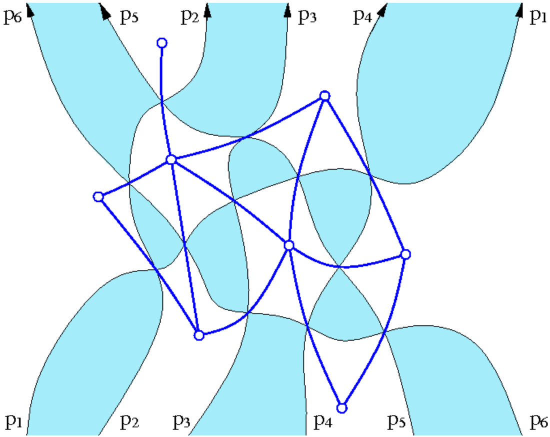
Note: we consider the star-triangle equation in the orders $\mathcal{T}^{-1}$ and $\mathcal{T}^{0}$, however it is satisfied in all orders of $\mathcal{T}$-expansion.

Hybrid model

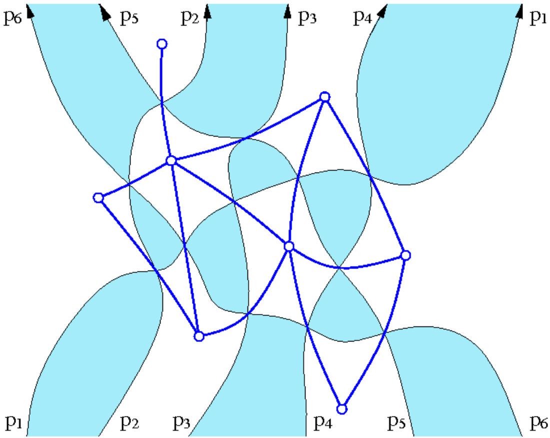
$$\mathcal{Z} = \int_{0\le x_m \le \pi} \!\!\dots\!\! \int \exp\left\{ -\frac{\mathcal{E}(\{x\})}{T} + \mathcal{O}(1) + \mathcal{O}(T) \right\} \prod_m \frac{dx_m}{\sqrt{T}}\,, \quad T \to 0\,,$$



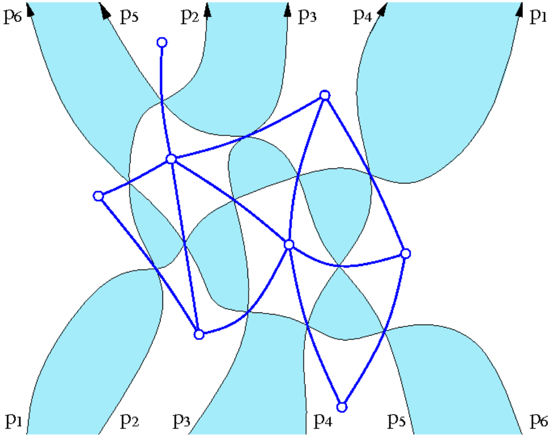
I. Rapidity lattice



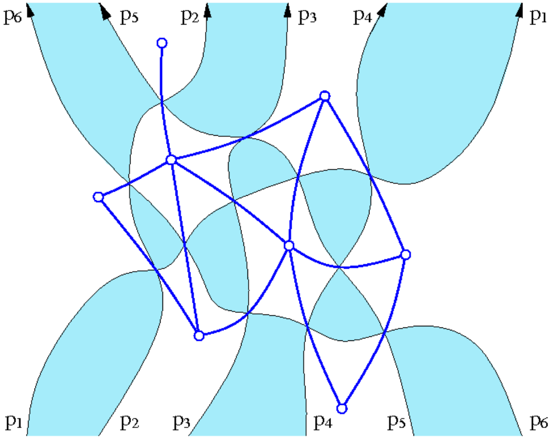
II. Bipartite graph, to each site assign a pair (ξ_j, n_j), where ξ_j are continuous and $n_j \in \mathbb{Z}_N$.



III. Fix all boundary variables (ξ_i, n_i).



IV. Solve classical integrable variational equations for the parameters ξ_j in the bulk (Dirichlet problem for the *Adler-Bobenko-Suris* system)



V. All discrete-spin Boltzmann weights W and $\overline{W}$ entering the partition function are now defined, the lattice statistical mechanics begins.

Asymptotics of the partition function:

$$\log \mathcal{Z} = -\frac{\mathcal{E}(\{\xi^{(cl)}\})}{T} + \log \mathcal{Z}_0 + \mathcal{O}(T) ,$$

where $\{\xi^{(cl)}\}$ denote the stationary point of the classical action,

$$\frac{\partial \mathcal{E}(\{\xi\})}{\partial \xi_m}\Big|_{\{\xi\}=\{\xi^{(cl)}\}} \ = \ 0 ,$$

and $\mathcal{Z}_0 = \mathcal{Z}_0(\{\xi^{(cl)}\})$ is the partition function for the discrete-spin system.

- We presented a new solution to the star-triangle equation expressed in terms of elliptic Gamma-functions
- This solution involves two temperature-like parameters (elliptic nomes p and q)
- This solution contains as specials cases all previously known solutions of the star-triangle equation both with discrete and continuous spin variables
- When one elliptic nome tends to a root of unity, $\mathsf{q}^2 \to e^{2\pi \mathsf{i}/N}$, we obtain a hybrid of a classical non-linear integrable system and a solvable model of statistical mechanics. In particular, it contains the chiral Potts and Kashiwara-Miwa models. This is analogous to the background field quantization in Quantum Field Theory.
- Connection to superconformal indices and electric-magnetic dualities (Dolan-Osborn, Spiridonov-Vartanov)

THANK YOU

Few references

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