Quantum geometry of 3D lattices: Existence as Integrability

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Theory of integrable (exactly solvable) quantum systems

- Algebraic structures: Yang-Baxter equation & quantum groups
- 3D-generalization: tetrahedron equation (Zamolodchikov)
- Integrability: Zero-curvature representation (discovered in soliton theory)

Discrete differential geometry (combines ideas from geometry, topology, combinatorics, ...)

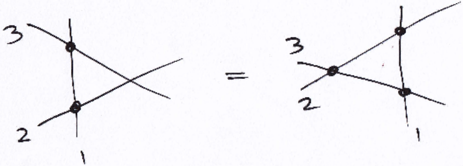
- Discretization principle: preserve as many features of the continuous theory as possible, including *transformation groups*.
- "Consistency as integrability" (Adler-Bobenko-Suris). A discrete analog of the zero-curvature representation for *classical* evolution equations on a lattice
- "Existence as integrability". Zero curvature representation $\Leftrightarrow$ Incidence theorems of elementary geometry.

Quantum geometry

$$Z = \sum_{geometries} e^{-\frac{S(\text{geometry})}{\hbar}}$$

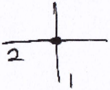
Classical geometry arises at $\hbar \to 0$ as a stationary configuration minimizing the action $\mathcal{S}$.

- Quantization of discrete orthogonal coordinate systems (3D circular lattices). Discrete analog of triply-orthogonal co-ordinate systems (Lamé & Darboux)
- Quantum Yang-Baxter equation $\Leftrightarrow$ quantization of incidence theorem in geometry

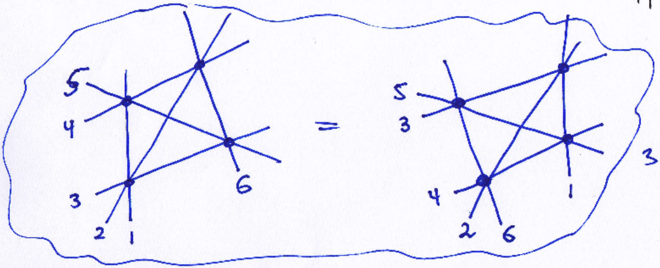


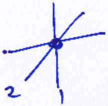
$$S_{12} S_{13} S_{23} = S_{23} S_{13} S_{12}$$

$$S : \quad V \otimes V \longrightarrow V \otimes V ;$$



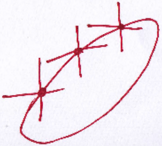
$= S_{12}$





R_{123}

$$R_{123} R_{145} R_{246} R_{356} = R_{356} R_{246} R_{145} R_{123}$$

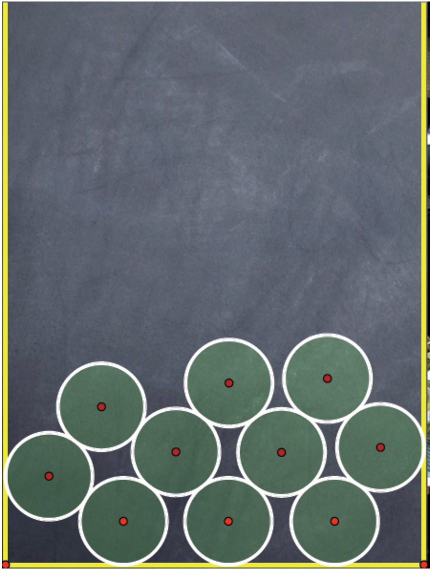


$$R : \quad V \otimes V \otimes V \longrightarrow V \otimes V \otimes V$$

YBE is an overdetermined system of algebraic equations. Its general solution is unknown even in the simplest cases.

- Known solutions (various methods):Onsager, McGuire, Yang, Baxter, ...(over 50 different authours)
- Algorithmic recipes: Universal R-matrix for quantized affine Lie algebras (quantum groups) (Drinfeld-Jimbo)
- almost all known solutions have been included in the quantum group scheme (up to elliptic deformations, vertex-face transformations, etc.).
- 3D-generalization: tetrahedron equation, Zamolodchikov (1980) followed by Baxter, Bazhanov, Kashaev, Korepanov, Mangazeev, Maillet-Nijhoff, Sergeev, Stroganov,...
- Question: Could one obtain *all* solutions of the YBE from solutions of the tetrahedron equation? Plausibly the answer is affirmative. Confirmed for $U_q(\widehat{sl}(n))$ and $U_q(\widehat{sl}(m|n))$.
- Strategy: obtain solutions of the tetrahedron equations from incidence theorems. Then obtain solutions of the YBE by a projection from 3D

What is an incidence theorem? (J.Richter-Gebert, 2007)



http://www.math.tu-berlin.de/geometrie/ps/ddg07/slides/Richter-Gebert.pdf

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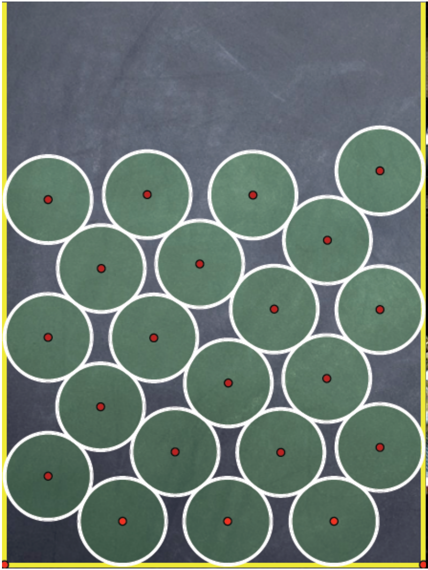

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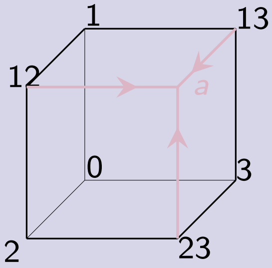
- tessellations of the flat 3-space with planar quad faces
- quadrilateral lattices are integrable (Doliwa-Santini)

Elementary geometry Theorem

Consider four points $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ in general position in $\mathbb{R}^N$, $N \geq 3$. On each of the three planes $(\mathbf{x}_0, \mathbf{x}_i, \mathbf{x}_j)$, $0 \leq i < j \leq 3$ chose an extra point $\mathbf{x}_{ij}$ not lying on the lines $(\mathbf{x}_0, \mathbf{x}_i)$, $(\mathbf{x}_0, \mathbf{x}_j)$ and $(\mathbf{x}_i, \mathbf{x}_j)$. Then there exist a unique point x_{123} which simultaneously belongs to the three planes $(\mathbf{x}_1, \mathbf{x}_{12}, \mathbf{x}_{13})$, $(\mathbf{x}_2, \mathbf{x}_{12}, \mathbf{x}_{23})$ and $(\mathbf{x}_3, \mathbf{x}_{13}, \mathbf{x}_{23})$.



Geometry of quadrilateral lattices (where all faces are *planar* quadrilateral)

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Elementary geometry Theorem

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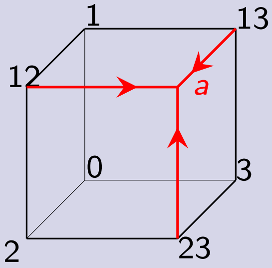
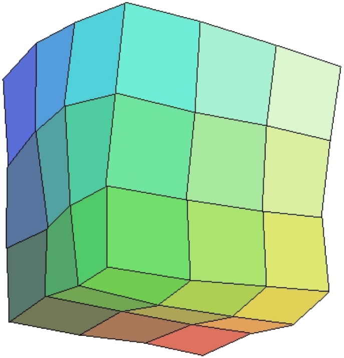


Evolution, time=0

“Ruler-and-compass” geometric evolution system (only the 2D-ruler is actually required)



Evolution, time=1

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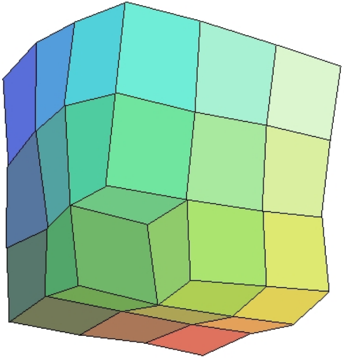
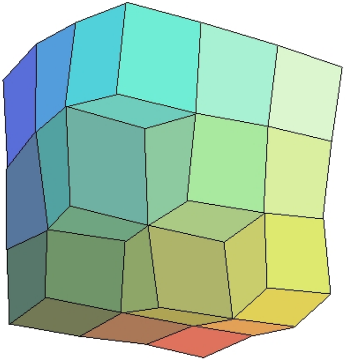


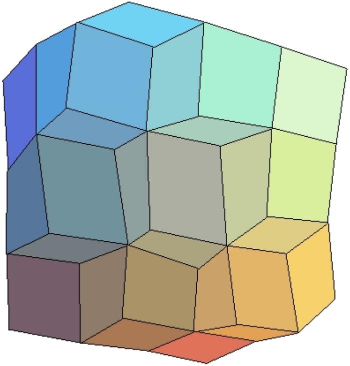
Evolution, time=2

“Ruler-and-compass” geometric evolution system (only the 2D-ruler is actually required)



Evolution, time=3

“Ruler-and-compass” geometric evolution system (only the 2D-ruler is actually required)



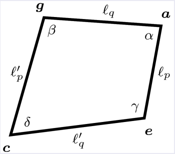
Suppose, $\mathbb{R}^M$ is Euclidean space: metric, lengths, angles...

- A quadrilateral is completely defined by five parameters: e.g. by three independent angles and by lengths of two sides:

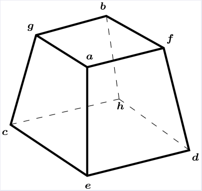
$$\alpha,\ \beta,\ \gamma,\ \delta\ :\quad \alpha+\beta+\gamma+\delta=2\pi$$

and

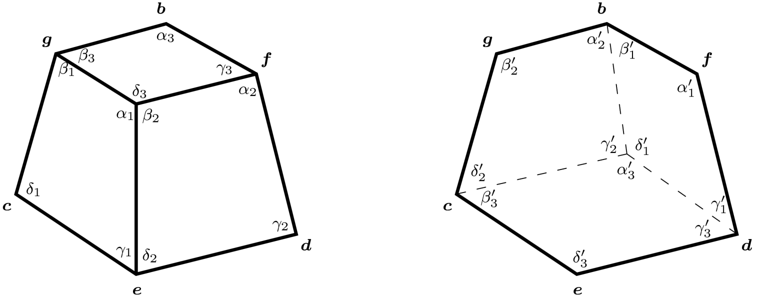
$$\ell_q = |ag|, \quad \ell_p = |ae|$$



- A hexahedron is completely defined by twelve parameters: e.g. by nine angles and by lengths of any three non-planar edges



The angles



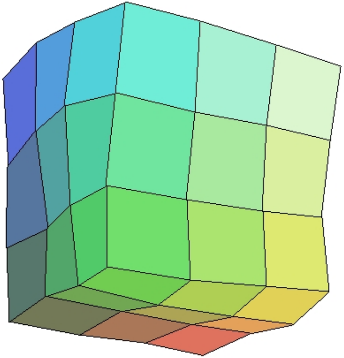
Given nine independent angles of the front faces, all the other angles of the cube may be calculated. Cosine theorem produces the map

$$\mathscr{R}_{123} \ : \ (\alpha_j, \beta_j, \gamma_j, \delta_j)_{j=1,2,3} \ \to \ (\alpha'_j, \beta'_j, \gamma'_j, \delta'_j)_{j=1,2,3}$$

$$\mathscr{R}_{123} \ \cdot \ F(\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3) \ = \ F(\mathscr{A}'_1, \mathscr{A}'_2, \mathscr{A}'_3) \ , \quad \mathscr{A}_j = (\alpha_j, \beta_j, \gamma_j, \delta_j)$$

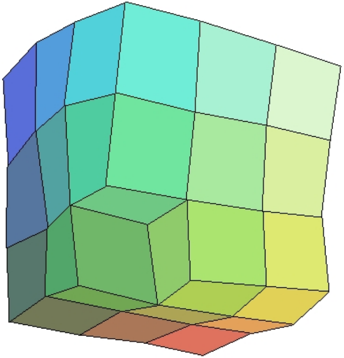
Evolution, time=0

"Ruler-and-compass" geometric evolution system (only the 2D-ruler is actually required)



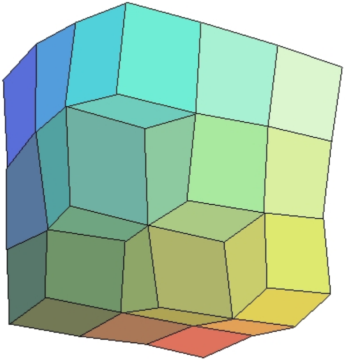
Evolution, time=1

“Ruler-and-compass” geometric evolution system (only the 2D-ruler is actually required)



Evolution, time=2

"Ruler-and-compass" geometric evolution system (only the 2D-ruler is actually required)



Evolution, time=3

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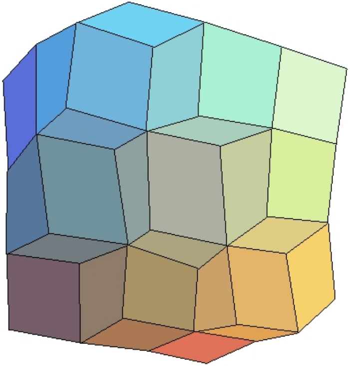
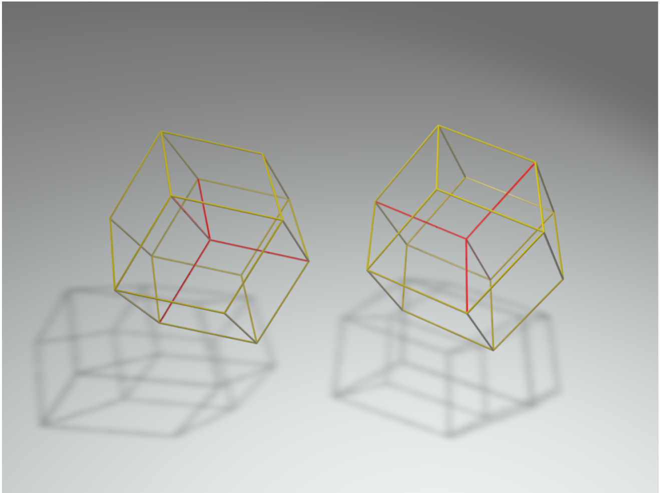
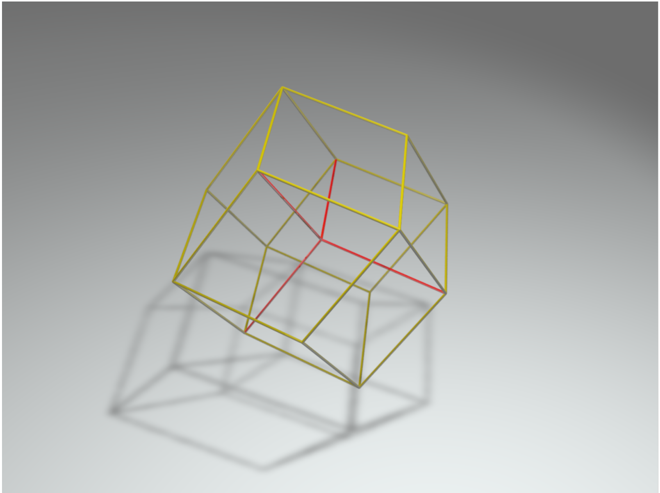




Rhombic dodecahedron can be dissected into four hexhahedra in two non-equivalent ways

(Proof follows from the mere existence of 4D cube)

Zamolodchikov tetrahedron eq. $\mathscr{R}_{123} \cdot \mathscr{R}_{145} \cdot \mathscr{R}_{246} \cdot \mathscr{R}_{356} = \mathscr{R}_{356} \cdot \mathscr{R}_{246} \cdot \mathscr{R}_{145} \cdot \mathscr{R}_{123}$



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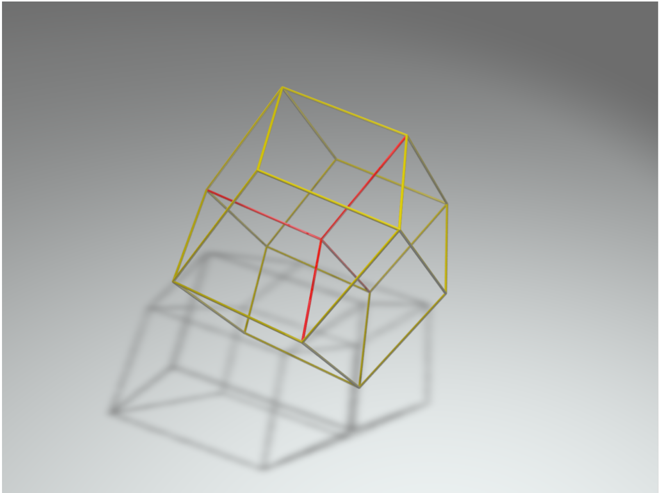

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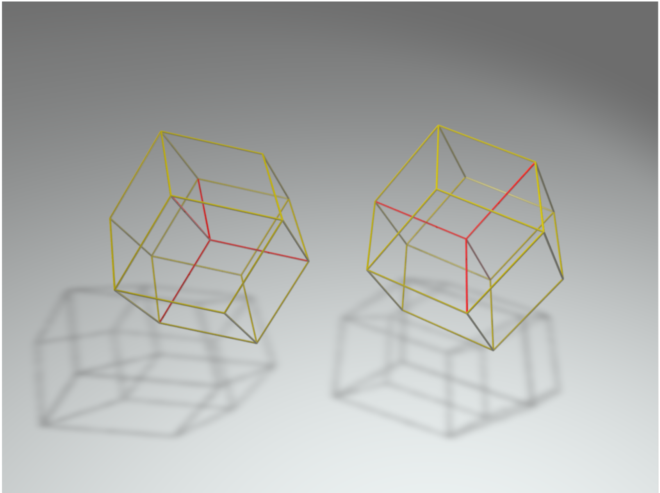

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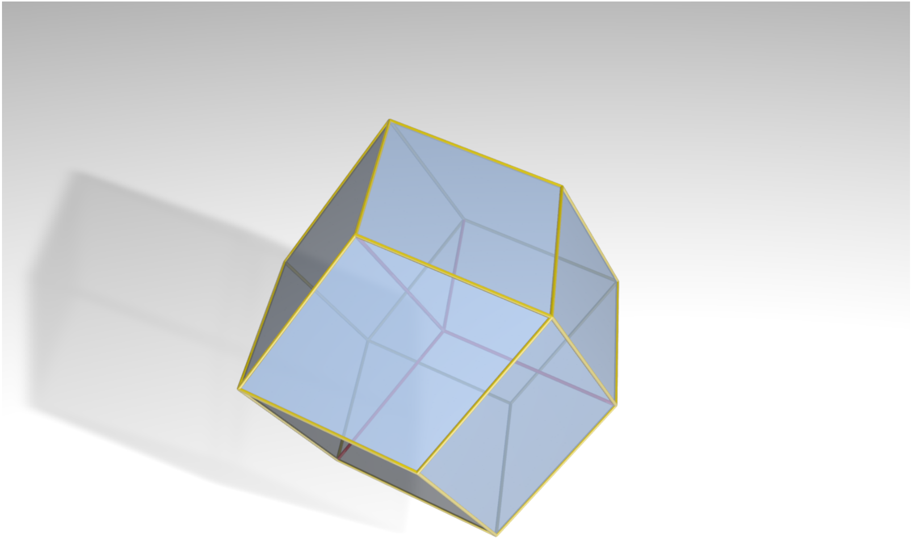
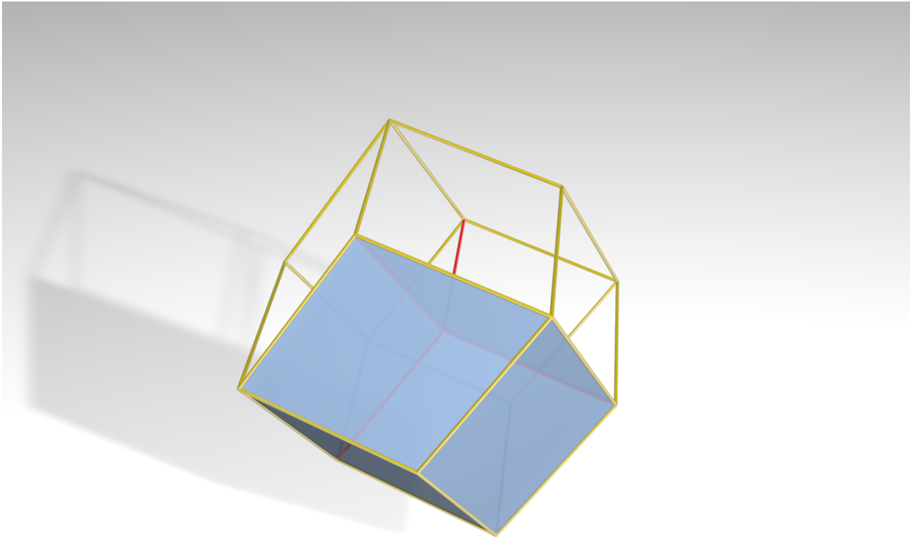
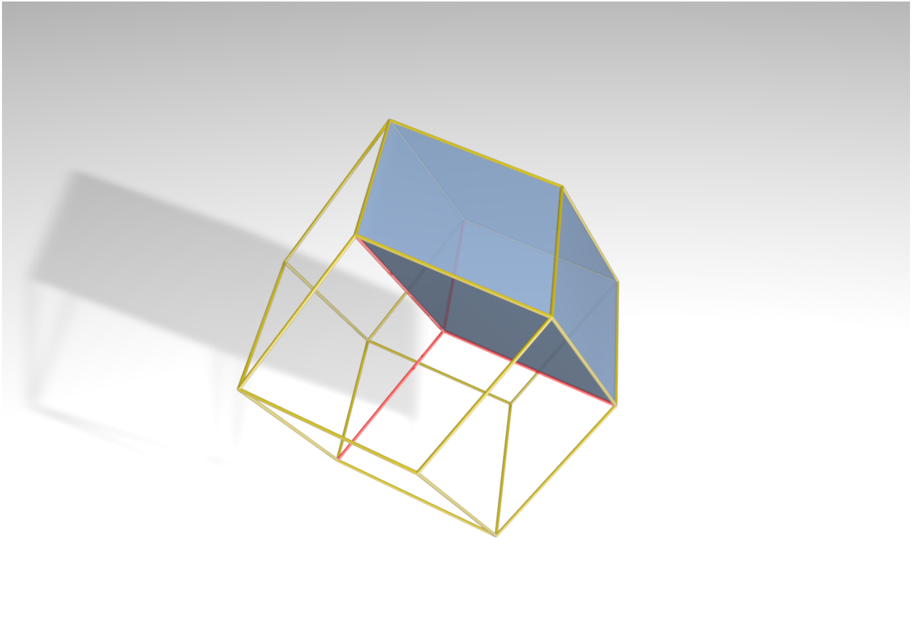
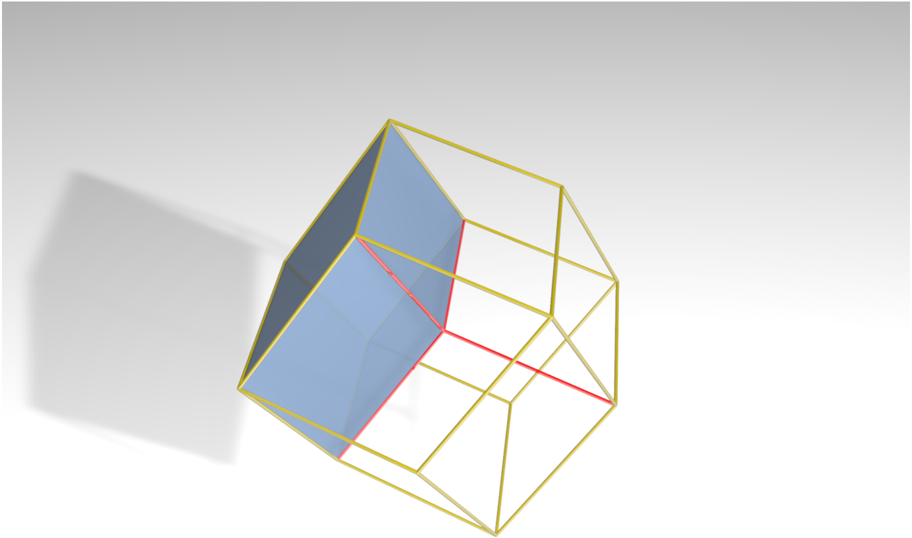
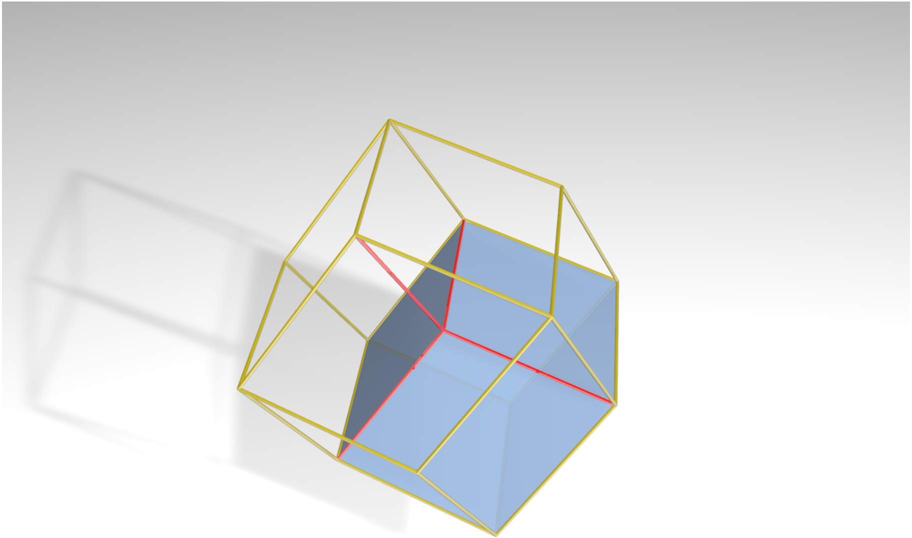
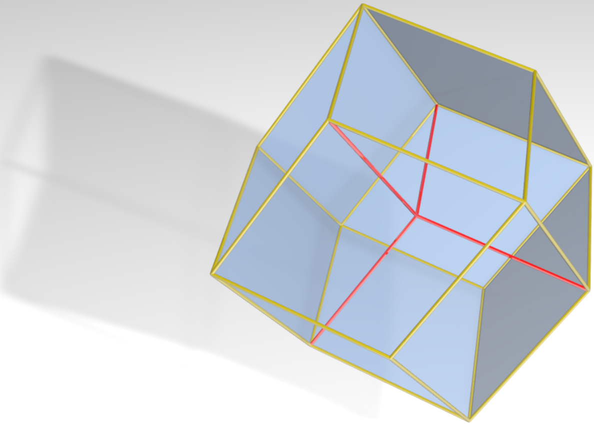


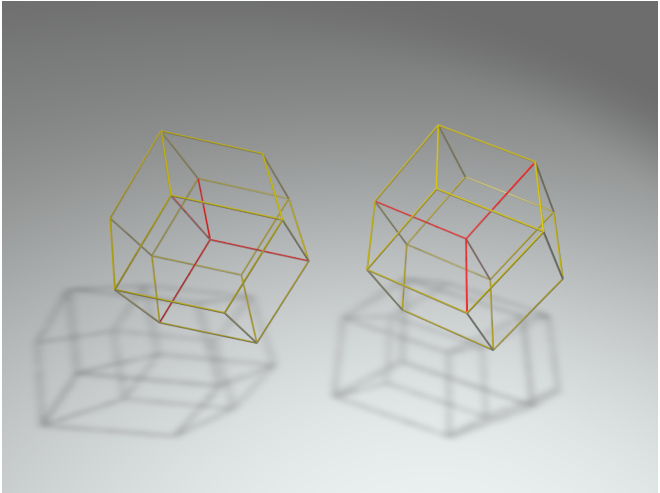










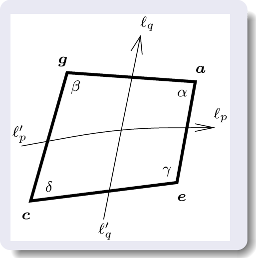


Rhombic dodecahedron can be dissected into four hexhahedra in two non-equivalent ways. 3D analog of the Yang-Baxter equation (3D tilings, space filling polyhedra, zonotopes)

(Proof follows from the mere existence of 4D cube)

Elementary relation for a quadrilateral:

$$\begin{pmatrix} \ell_p' \\ \ell_q' \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\sin\gamma}{\sin\delta} & \frac{\sin(\delta+\beta)}{\sin\delta} \\ \frac{\sin(\delta+\gamma)}{\sin\delta} & \frac{\sin\beta}{\sin\delta} \end{pmatrix}}_{X_{pq}} \begin{pmatrix} \ell_p \\ \ell_q \end{pmatrix}$$



Here

$$X_{pq} = X_{pq}[\mathscr{A}]\ , \quad \mathscr{A} = (\alpha, \beta, \gamma, \delta)$$

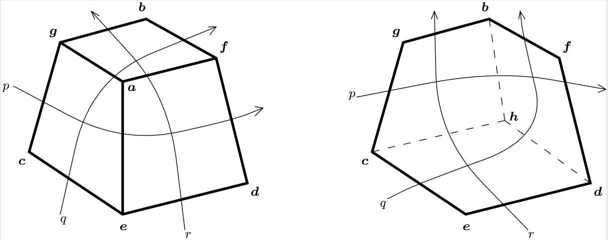
Main algebraic property of 3-parameters matrix X:

$$X_{pq} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \ : \quad AD - BC = \frac{AB - CD}{DB - AC}$$

Note: $(\vec{n}_p, \vec{n}_q) = (\vec{n}_p', \vec{n}_q') \cdot X_{pq}$, where $\vec{n}_{p,q}^{\#}$ are normal vectors for the sides.

Zero curvature equation

I. Korepanov, 1992



$$\begin{bmatrix} \ell_{g,c} \\ \ell_{c,e} \end{bmatrix} = X_{pq} \begin{bmatrix} \ell_{a,e} \\ \ell_{g,a} \end{bmatrix}, \quad \begin{bmatrix} \ell_{a,e} \\ \ell_{e,d} \end{bmatrix} = X_{pr} \begin{bmatrix} \ell_{f,d} \\ \ell_{a,f} \end{bmatrix}, \quad \begin{bmatrix} \ell_{g,a} \\ \ell_{a,f} \end{bmatrix} = X_{qr} \begin{bmatrix} \ell_{b,f} \\ \ell_{g,b} \end{bmatrix}$$

$$\Rightarrow$$

$$\begin{bmatrix} \ell_{c,e} \\ \ell_{e,d} \end{bmatrix} = X'_{qr} \begin{bmatrix} \ell_{h,d} \\ \ell_{c,h} \end{bmatrix}, \quad \begin{bmatrix} \ell_{g,c} \\ \ell_{c,h} \end{bmatrix} = X'_{pr} \begin{bmatrix} \ell_{b,h} \\ \ell_{g,b} \end{bmatrix}, \quad \begin{bmatrix} \ell_{b,h} \\ \ell_{h,d} \end{bmatrix} = X'_{pq} \begin{bmatrix} \ell_{f,d} \\ \ell_{b,f} \end{bmatrix}$$

$$X_{pq}[\mathscr{A}_1]X_{pr}[\mathscr{A}_2]X_{qr}[\mathscr{A}_3] = X_{qr}[\mathscr{A}'_3]X_{pr}[\mathscr{A}'_2]X_{pq}[\mathscr{A}'_1] \quad \leftarrow \quad \text{all cosine theorems together}$$

Functional Tetrahedron equation

R. M. Kashaev, I. G. Korepanov and S. M. Sergeev, Theoretical and Mathematical Physics 117 (1998) 370-384

Using the map defined above for $\mathscr{A}_j = (\alpha_j, \beta_j, \gamma_j, \delta_j)$,

$$\mathscr{R}_{123} \cdot F(\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3) = F(\mathscr{A}_1', \mathscr{A}_2', \mathscr{A}_3') ,$$

one may rewrite the zero curvature relation in the Tetrahedral Zamolodchikov algebra form

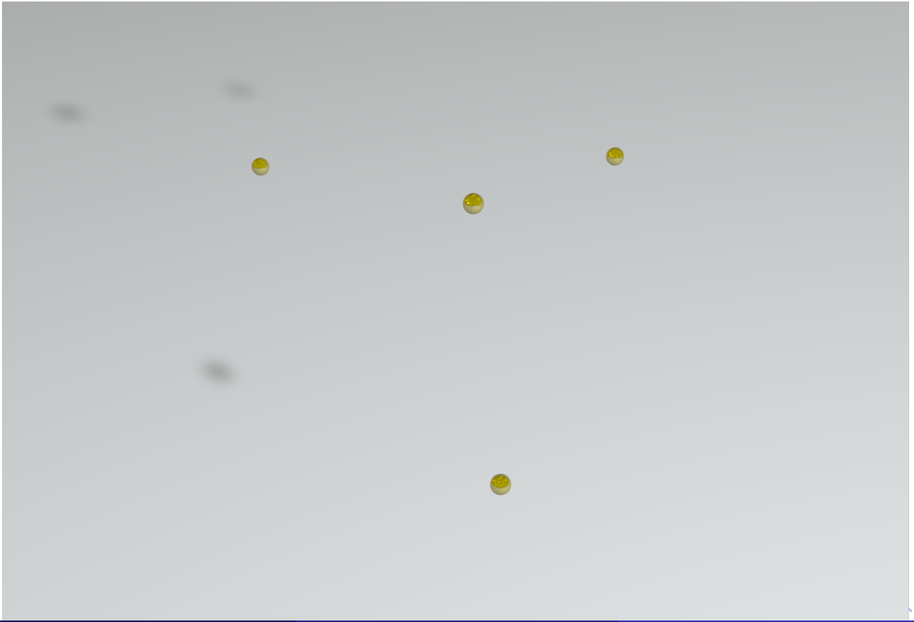
$$X_{pq}[\mathscr{A}_1] X_{pr}[\mathscr{A}_2] X_{qr}[\mathscr{A}_3] = \mathscr{R}_{123} \cdot X_{qr}[\mathscr{A}_3] X_{pr}[\mathscr{A}_2] X_{pq}[\mathscr{A}_1]$$

Therefore, the map $\mathscr{R}_{123}$ satisfies the Functional Tetrahedron equation

$$\mathscr{R}_{123} \cdot \mathscr{R}_{145} \cdot \mathscr{R}_{246} \cdot \mathscr{R}_{356} \cdot F = \mathscr{R}_{356} \cdot \mathscr{R}_{246} \cdot \mathscr{R}_{145} \cdot \mathscr{R}_{123} \cdot F$$

Here the dot sign stands for the superposition of maps.

Miquel theorem (1838)



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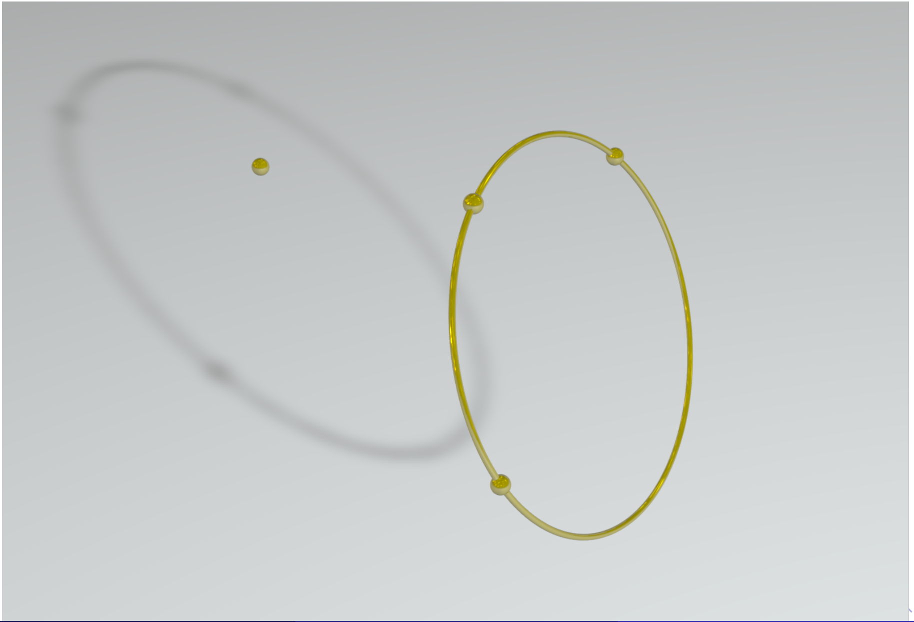

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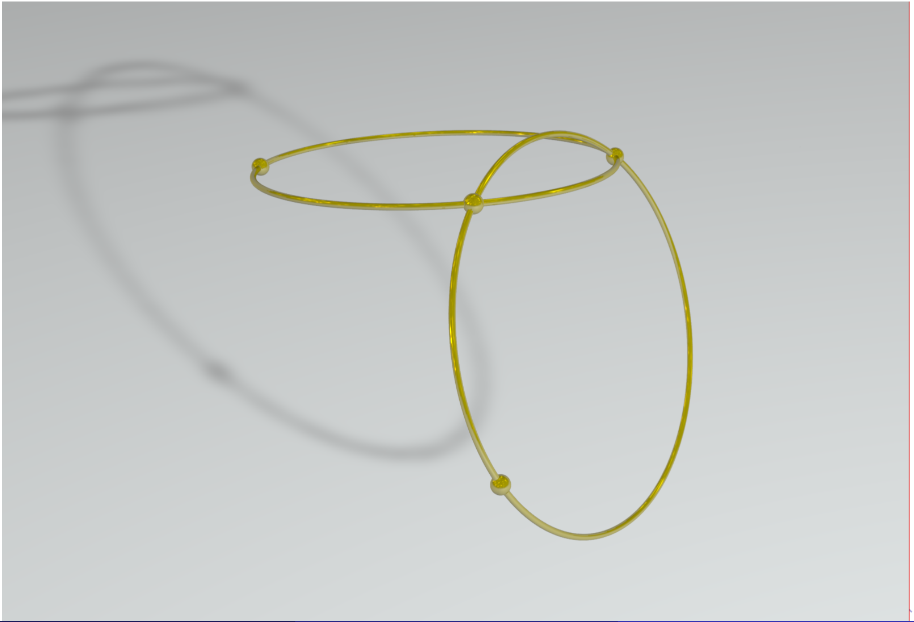

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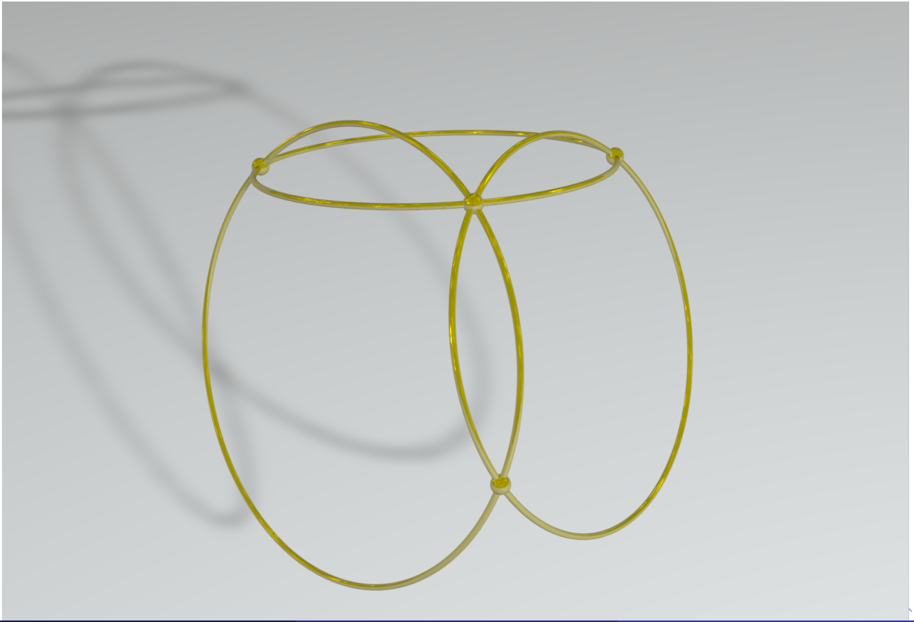

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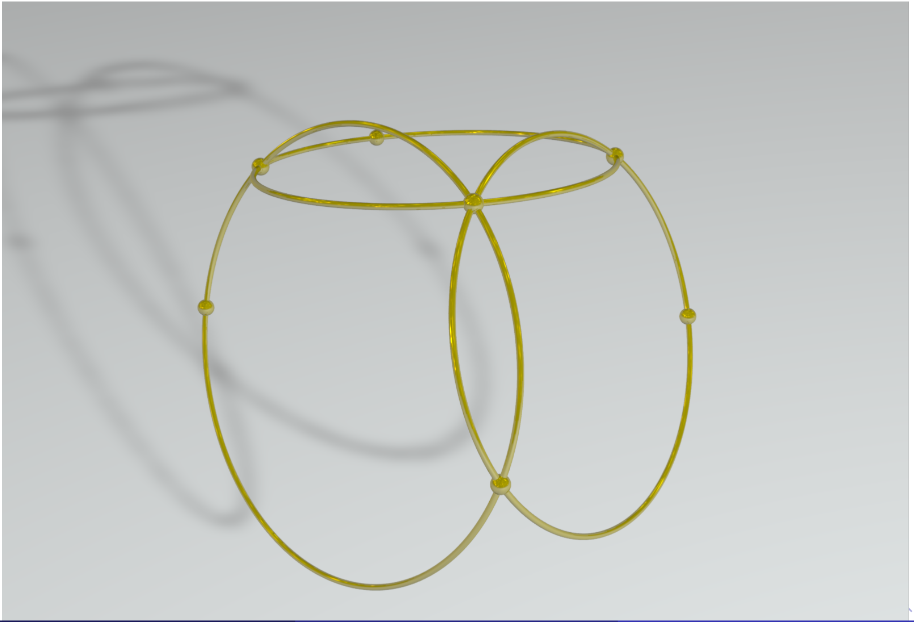

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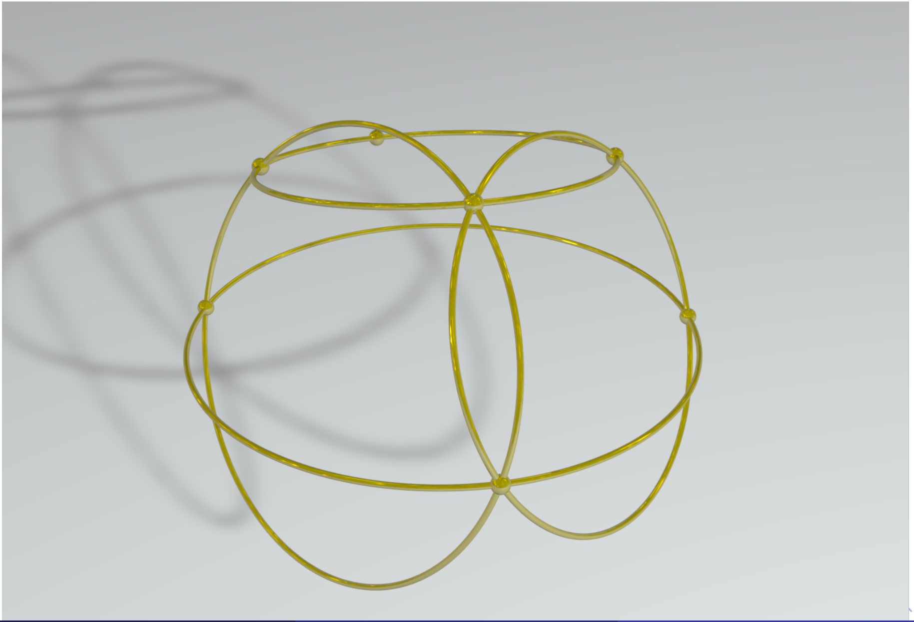

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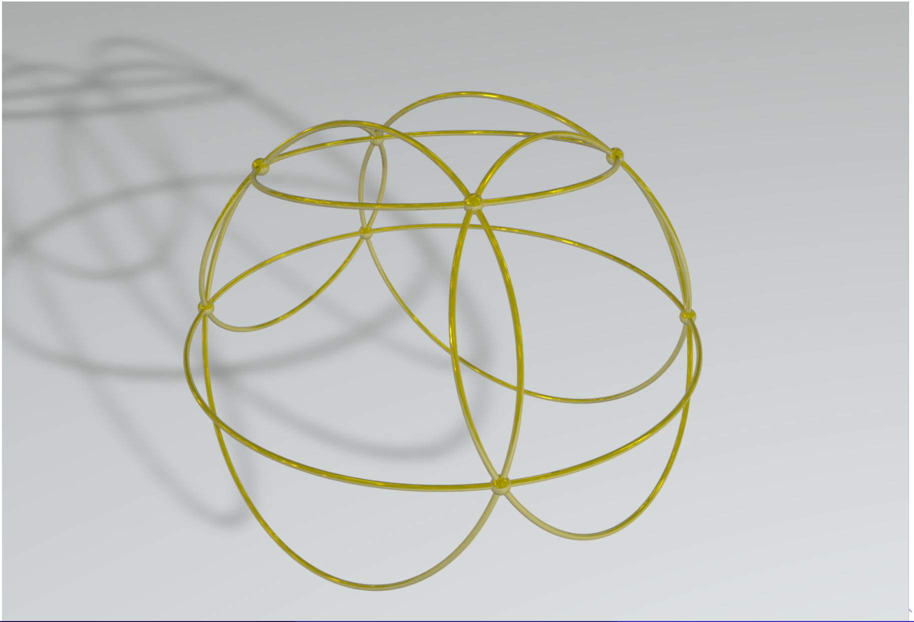
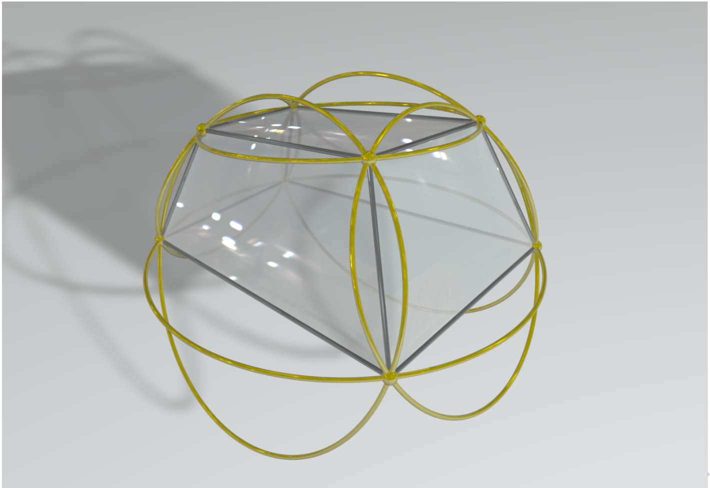


Miquel theorem defines circular lattices.

Miquel theorem

If three adjacent quadrilaterals faces of a hexahedron can be inscribed into circles, then the whole hexahedron can be inscribed into a sphere.



Simplification of $\mathscr{A}$ for all quadrilaterals:

$$\alpha + \delta = \pi\,, \quad \beta + \gamma = \pi \quad : \quad \mathscr{A} = (\alpha, \beta)$$

$$k = \frac{\sin\beta}{\sin\alpha}, \quad a = \frac{\sin(\alpha+\beta)}{\sin\alpha}, \quad a^* = \frac{\sin(\alpha-\beta)}{\sin\alpha}, \qquad k^2 = 1 - aa^*$$

$$\mathscr{R}_{123} \quad : \quad \begin{cases} (k_2 a_1^*)' = \; k_3 a_1^* - k_1 a_2^* a_3, & (k_2 a_1)' = k_3 a_1 - k_1 a_2 a_3^*, \\ (a_2^*)' = a_1^* a_3^* + k_1 k_3 a_2^*, & (a_2)' = a_1 a_3 + k_1 k_3 a_2, \\ (k_2 a_3^*)' = k_1 a_3^* - k_3 a_1 a_2^*, & (k_2 a_3)' = k_1 a_3 - k_3 a_1^* a_2, \end{cases}$$

Theorem (VB & Sergeev 2006)

The map $\mathscr{R}_{123}$ is a canonical transformation preserving the (ultra-local) Poisson algebra

$$\{\alpha_i, \beta_j\} = \delta_{ij}\,, \quad \{\alpha_i, \alpha_j\} = \{\beta_i, \beta_j\} = 0\,.$$

Local Poisson algebra for k, a, a^* is given by

$$\{a, a^*\} = 2k^2, \quad \{k, a\} = -ka, \quad \{k, a^*\} = ka^*$$

Quantization: the automorphism of tensor cube of quantum q-oscillator algebra

$$\mathcal{H}_q : \qquad q^{-1}aa^* - qa^*a = 2, \qquad aa^* = 1 - qk^2, \quad a^*a = 1 - q^{-1}k^2$$

is given exactly by the same map.

3D quantum R-matrix (solution to tetrahedron equation)

Fock representation

$$|n>=(a^*)^n\,|0>, \qquad a|0>=0,$$

$$X'=\mathcal{R}_{123}\circ X, \qquad X'=R_{123}\,X\,R_{123}^{-1}, \qquad X\in\mathcal{H}_q^{\otimes 3}$$

$$\langle n_1,n_2,n_3|R\,|n_1',n_2',n_3\rangle=\delta_{n_1+n_2,n_1'+n_2'}\delta_{n_2+n_3,n_2'+n_3'}\sqrt{\frac{(q^2;q^2)_{n_1'}\,(q^2;q^2)_{n_2'}\,(q^2;q^2)_{n_3'}}{(q^2;q^2)_{n_1}\,(q^2;q^2)_{n_2}\,(q^2;q^2)_{n_3}}}$$

$$\times\,\frac{(-1)^{n_2}\,q^{(n_1'-n_2)(n_3'-n_2)}}{(q^2;q^2)_{n_2'}}\;\frac{(q^{2(1-n_2'+n_3)};q^2)_\infty}{(q^{2(1+n_3)};q^2)_\infty}\;{}_2\phi_1(q^{-2n_2'},q^{2(1+n_3')},q^{2(1-n_2'+n_3)};q^2,q^{2(1+n_1)})\;, \qquad (1)$$

where

$$(x;q^2)_n=(1-x)(1-q^2x)\cdots(1-q^{2(n-1)}x)\;, \qquad (2)$$

and

$${}_2\phi_1(a,b,c;q^2,z)\;=\;\sum_{n=0}^{\infty}\frac{(a;q^2)_n(b;q^2)_n}{(q^2;q^2)_n(c;q^2)_n}z^n \qquad (3)$$

is the q-deformed Gauss hypergeometric series. In the quasi-classical limit $q=e^{\hbar}$, $\hbar\to 0$

$$\langle n_1,n_2,n_3|R\,|n_1',n_2',n_3\rangle=e^{-\mathcal{L}(k_1',k_2',k_3'|k_1,k_2,k_3)/\hbar}, \qquad k_j=q^{n_j}$$

gives Lagrangian density and variational principle for circular lattices.

The classical action

The action is a sum of Lagrangian density over all hexahedrons of the lattice,

$$\mathcal{S} \;=\; \sum_{\text{lattice}} \mathscr{L}(k,k')$$

Another choice of variables corresponds to the Legendre transform of the Lagrangian,

$$\mathscr{L}(k,k') \equiv \sum_j \log k_j \log v_j + \mathcal{L}(v,v') - \sum_j \log k'_j \log v'_j, \qquad v_j = \frac{a_j^*}{a_j}$$

The answer:

$$\mathscr{L}(v,v') = \sum_{i=0}^{3} \Lambda_h(\Omega_i) + \Lambda_h(\Omega'_i),$$

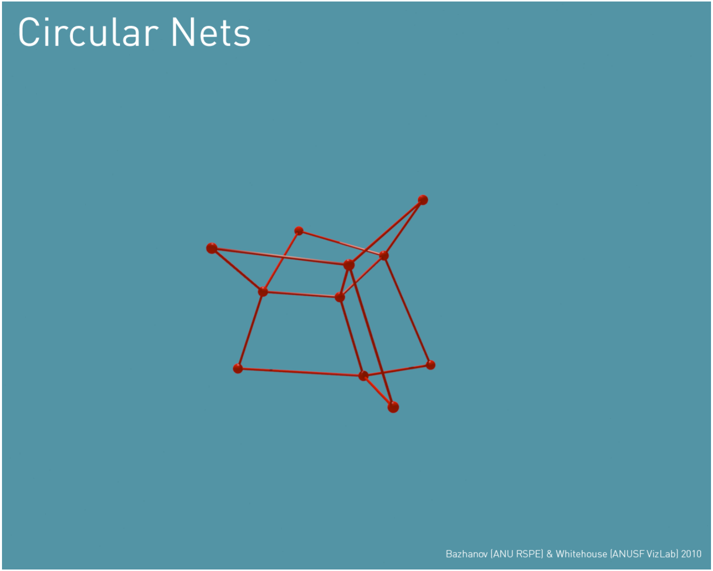
where

$$\Lambda_h(\Omega) \;=\; -\int_0^{\Omega} \log|2\sinh x|\, dx$$

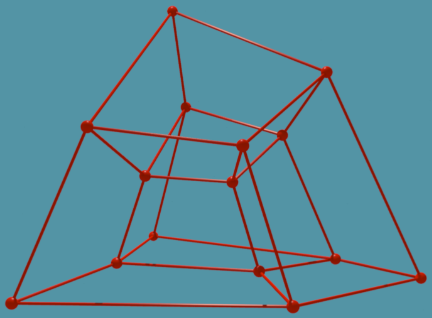
is the hyperbolic Lobachevski function and

$$\frac{v_2}{v_1 v_3} = e^{-2\Omega'_2}, \quad \frac{v'_2}{v'_1 v_3} = e^{-2\Omega'_1}, \quad \frac{v_2}{v'_1 v'_3} = e^{-2\Omega'_0}, \quad \frac{v'_2}{v_1 v'_3} = e^{-2\Omega'_3},$$

$$\frac{v'_2}{v'_1 v'_3} = e^{2\Omega_2}, \quad \frac{v_2}{v_1 v'_3} = e^{2\Omega_1}, \quad \frac{v'_2}{v_1 v_3} = e^{2\Omega_0}, \quad \frac{v_2}{v'_1 v_3} = e^{2\Omega_3}.$$


Circular Nets
Bazhanov (ANU RSPE) & Whitehouse (ANUSF VizLab) 2010

Circular Nets



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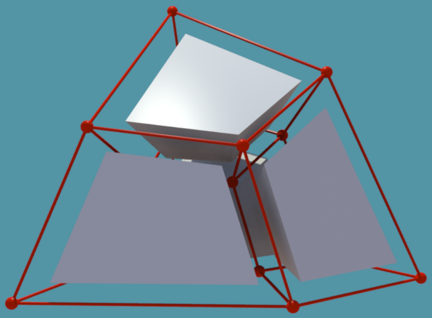

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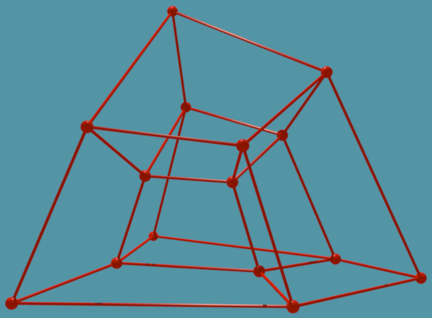

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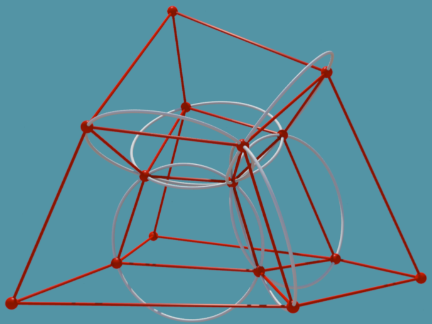

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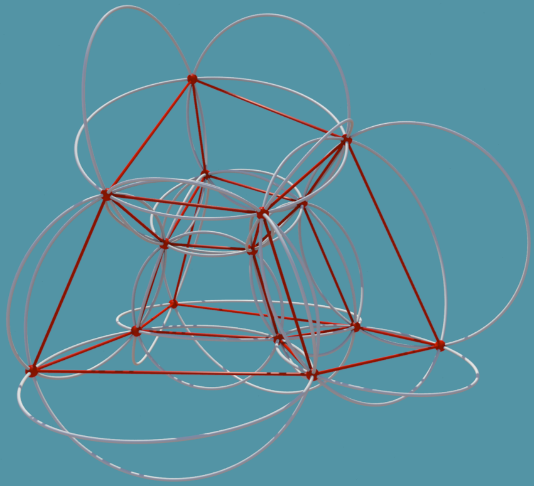

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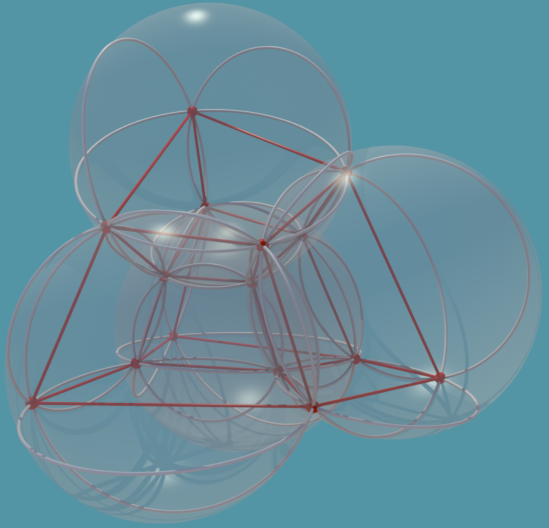

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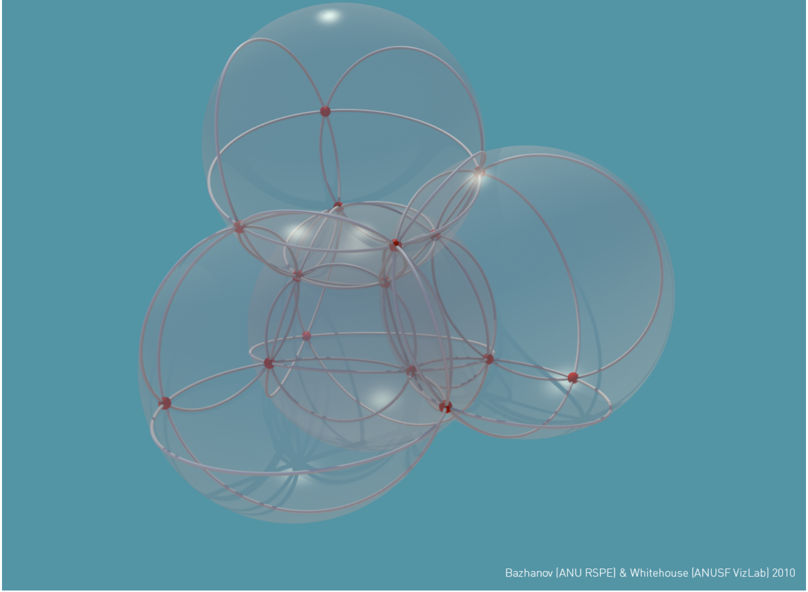
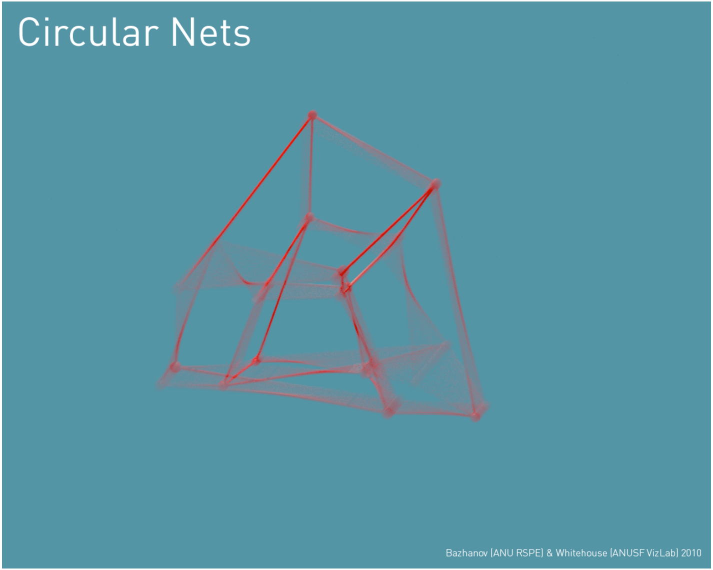


Circular nets are exactly solvable both in classical and quantum theory



Conclusion and outlook

- 3D circular lattices describe discrete analogs of orthogonal coordinate systems (Bobenko, Doliwa-Santini, Konopelchenko-Schief). Continuous case (Lamé, Egorov, Darboux, Dubrovin, Kaup, Zaknarov-Manakov, Krichiver, Novikov, ...)
- Quantization leads to 3D integrable models (VB, Mangazeev & Sergeev (2008)):
  - compact case: generates all solutions of the Yang-Baxter equations associated with $U_q(\widehat{sl}(n))$, n — number of layers of the 3D lattice (VB & Sergeev (2006)), and for $U_q(\widehat{sl}(m|n))$ (Sergeev 2009)
  - non-compact analog of the N-state generalized Zamolodchikov model (Zamolodchikov (1979), VB & Baxter (1992))
  - 3D integrable models with POSITIVE Boltzmann weights
- Question: Are there non-trivial solutions of 4-simplex equation? (e.g., related to $\mathcal{N} = 4$ Yang-Mills and AdS/CFT)
- Is molecular geometry integrable? (see the last slide)

- The algebraic structures of the theory of integrable quantum system in statistical mechanics and quantum field theory, such as the Yang-Baxter equations and quantum groups naturally arise from quantization of the simplest models of the discrete geometry.
- Angle of circular quadrilaterals are canonically conjugated variables!

C_{60} fullerene molecule as a circular lattice