Spin quantum Hall transition in the presence of multiple edge channels

Roberto Bondesan (IPhT, Saclay)

With: I. Gruzberg (Chicago), J. Jacobsen (Paris), H. Obuse (Karlsruhe), H. Saleur (Saclay)

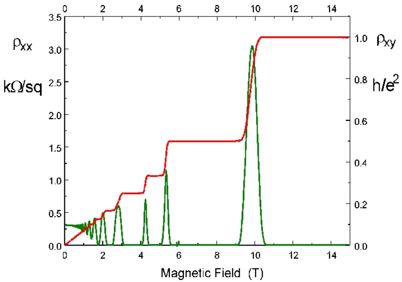
ACFTA - IHP 24/10/2011

Based on
BJS: arXiv:1101.4361, BGJOS: arXiv:1109.4866

Outline

Integer Quantum Hall Effect

- 2DEG at high B, low T
- Plateaus $\rho_{xy} = \dfrac{h}{e^2 \nu}$:



1. Disorder

2. Chiral edge states

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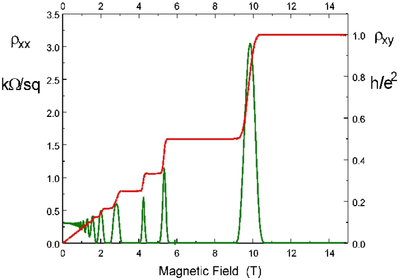

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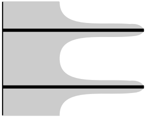
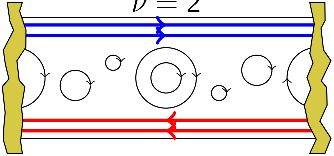
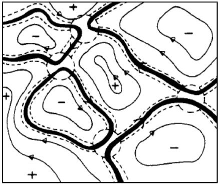
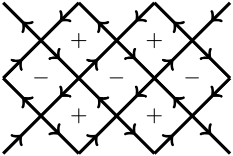


2. Chiral edge states



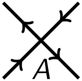
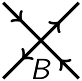
Network models: quantum percolation

-  CC (1988) → 

- Chalker-Coddington model:

    1. Quantum tunneling at nodes
    2. Random phases on links

CC model

- $U_e \in U(1)$
-  ,  : $S = \begin{pmatrix} \sqrt{1-t^2} & t \\ -t & \sqrt{1-t^2} \end{pmatrix}$

$\Rightarrow \mathcal{U}_{e,e'} = U_e^{1/2} S_{e,e'} U_{e'}^{1/2}$

-

$t = 1$: Insulator $\quad t_c = \frac{1}{\sqrt{2}}$ QCP $\quad t = 0$: QH state $\nu = 1$

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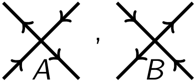
- 

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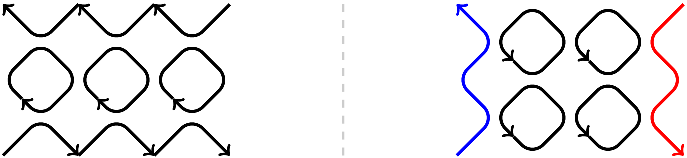
- 

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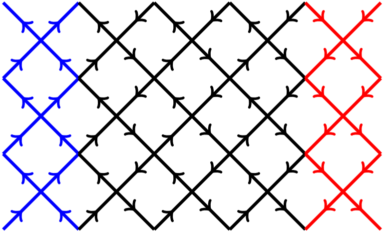
Edge states in CC model

- New network model:



  - Chiral extra edge channels
  - Higher plateaus: $\#(\text{edge states}) = \Delta(\text{top. numb.})$

Aim of the talk

Edge states for spin quantum Hall effect

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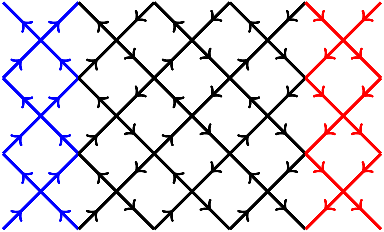

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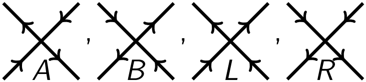
Edge states for spin quantum Hall effect

Spin quantum Hall effect (SQHE)

- Class C : $\not{T}$, C
- $d + id$ disordered superconductors
- Topological superconductor in 2D: $\sigma^{\text{spin}} \in 2\mathbb{Z}$
- Exactly solvable! (in some sense ...)

Network model SQHE with edge channels

- $U_e \in SU(2)$
- 

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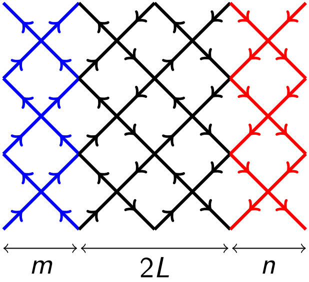

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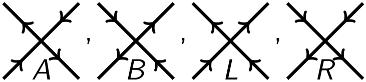
- 

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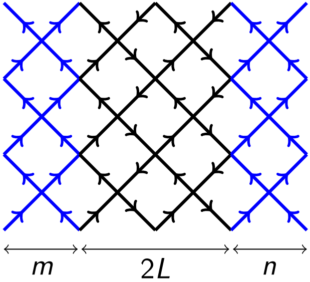

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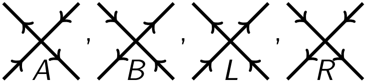
-  : $S_S = \mathbf{1} \otimes \begin{pmatrix} \sqrt{1-t_S^2} & t_S \\ -t_S & \sqrt{1-t_S^2} \end{pmatrix}$
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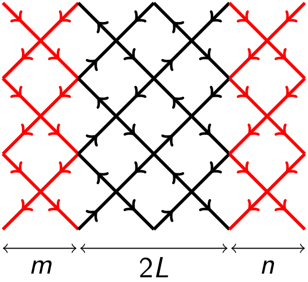

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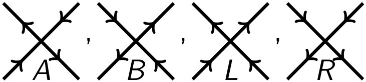
- 

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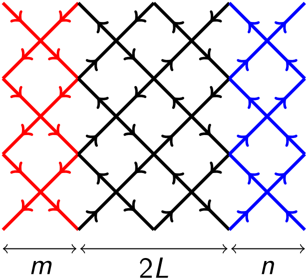
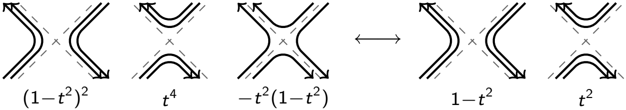


Disorder average

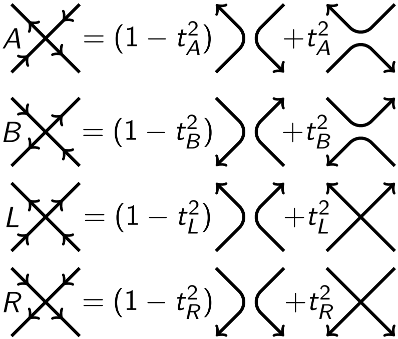
- $G(e,e',z) = \langle e| (1 - z\mathcal{U})^{-1} |e'\rangle = \sum_{\gamma(e,e')} \cdots z U_{e_j} s_j \cdots$
- $\overline{G(e,e',z)}$: [Beamond,Cardy,Chalker '02, Gruzberg,Read,Ludwig '99, Cardy '04]

1. Link: 0, 2 times: $\overline{U^q} = c_q \mathbf{1}\,,\quad c_q = \begin{cases} 1 & \text{if } q = 0 \\ -\frac{1}{2} & \text{if } q = \pm 2 \\ 0 & \text{otherwise.} \end{cases}$
2. Node: 0, 2, 4 times



New loop model

- Decomposition:



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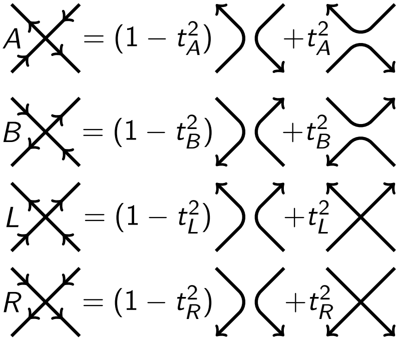

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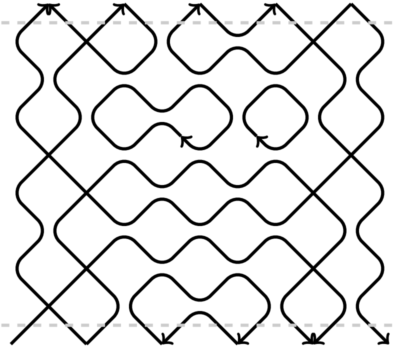

Conductance

- Landauer: $g = \mathrm{Tr}\, \mathbf{t}\mathbf{t}^\dagger$, $\mathbf{t}_{ij} = \langle e_i^{\text{out}} | \left(1 - \mathcal{U}\right)^{-1} | e_j^{\text{in}} \rangle$.

$\Rightarrow$ $\overline{g} = 2 \sum_{e \in C^{\text{in}}} \sum_{e' \in C^{\text{out}}} P(e', e)$

$\Rightarrow$ Loops $\equiv$ transport

What next:

Solve loop model, then go back to SQH.

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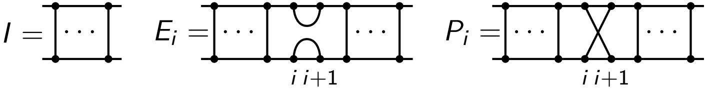
Algebraic remarks

- $\mathrm{ES}_{2L,m,n}(1)$. Generators:



$\Rightarrow$ Anisotropic limit, criticality:

$$H = -u\sum_{i=0}^{m-1} P_i - \sum_{i=m}^{2L+m-2} E_i - v\sum_{i=2L+m-1}^{2L+m+n-2} P_i$$

bulk

boundary

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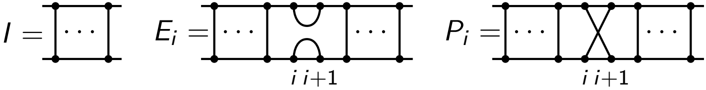

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Superspin chains

- SUSY rep. of $\mathrm{ES}_{2L,m,n}(1)$
- $\uparrow \equiv V, \downarrow \equiv V^\star$ reps. of $\mathfrak{sl}(2|1)$

$$\Rightarrow \mathcal{H}^{\mathcal{L},\mathcal{R}} = \begin{cases} V^{\otimes m} \otimes (V \otimes V^\star)^{\otimes L} \otimes (V^\star)^{\otimes n} & (m;n) \\ V^{\otimes m} \otimes (V \otimes V^\star)^{\otimes L} \otimes V^{\otimes n} & (m;-n) \\ (V^\star)^{\otimes m} \otimes (V \otimes V^\star)^{\otimes L} \otimes (V^\star)^{\otimes n} & (-m;n) \\ (V^\star)^{\otimes m} \otimes (V \otimes V^\star)^{\otimes L} \otimes V^{\otimes n} & (-m;-n). \end{cases}$$

Topological super σ-models

- Continuum limit periodic chain [Read,Saleur '01]:
  - Target: $\mathbb{CP}^{1|1} = \frac{\mathrm{U}(2|1)}{\mathrm{U}(1)\times\mathrm{U}(1|1)}$
  - $S \;=\; \frac{1}{2g_\sigma^2}\int \mathrm{d}^2 z\, D_\mu^\dagger Z_\alpha^\dagger D_\mu Z_\alpha - \frac{i\theta}{2\pi}\int \mathrm{d}^2 z\, \epsilon^{\mu\nu}\partial_\mu a_\nu$

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  - At $\theta = \pi(\mathrm{mod}\ 2\pi)$, $g_\sigma = O(1)$, LogCFT $c = 0$:

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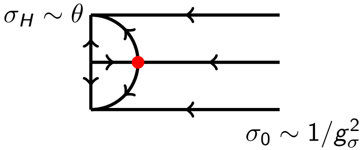


Edge states as conformal boundaries

- Symmetric CBC labelled by edge states [Candu *et al.* '10]

$$(\partial_y + ia_y)Z_\alpha = \Theta_1 g_\sigma^2(\partial_x + ia_x)Z_\alpha \ ,$$
$$(\partial_y - ia_y)Z_\alpha^\dagger = -\Theta_1 g_\sigma^2(\partial_x - ia_x)Z_\alpha^\dagger$$

$\Theta_1 = (2\mathcal{L} + \theta/\pi)$, $\Theta_2 = (2\mathcal{R} + \theta/\pi)$

$\Rightarrow$ Dep. on exact value of θ(= Hall conductance):

$$\theta \to \theta + 2\pi p \Longleftrightarrow (V \otimes V^\star)^{\otimes L} \to V^{\otimes p} \otimes (V \otimes V^\star)^{\otimes L} \otimes (V^\star)^{\otimes p}$$

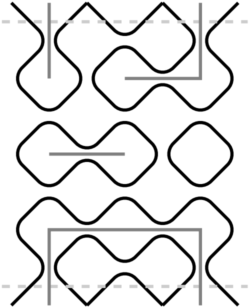
$\mathcal{L} = \mathcal{R} = 0$: Percolation

- $\mathrm{ES}_{2L,0,0}(1) = \mathrm{TL}_{2L}(1)$ planar
- loops = percolation hulls
- Sectors:
  $2j = \#$ (through lines = legs)

$$V_{2j} \underset{\times}{\rightleftarrows} V_{2(j-1)}$$

$\Rightarrow$ $2j$-leg exp. $h_{1,1+2j}$

$$h_{r,s} = \frac{\left((3r-2s)^2-1\right)}{24}$$



Boundary loop models

- Blob algebra
  [Martin,Saleur '94]
  - $\bigcirc = n$
  - $H = -\lambda b - \sum E_i$
  - $h_{r(n),r(n)+2j}$
    [Jacobsen,Saleur '06]
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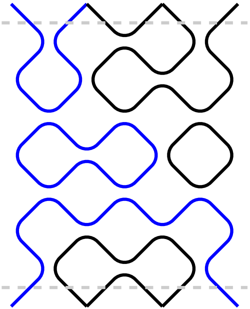

- Two-boundary case
  - Exponents [Dubail,Jacobsen,Saleur '09]
  - Rich boundary critical phenomena

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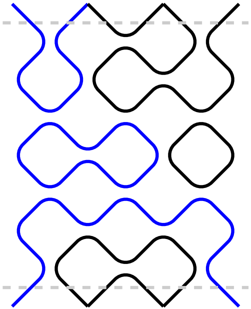

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Relation blob-edge states

- $$H = -u \sum_{i=0}^{m-1} P_i - \sum_{i=m}^{2L+m-2} E_i - v \sum_{i=2L+m-1}^{2L+m+n-2} P_i$$

$$\cong$$

$$\tilde{H} = -u \frac{1}{(m+1)!} \sum_{\sigma \in S_{\text{left}}} \sigma - \sum_{i=m}^{2L+m-2} E_i - v \frac{1}{(n+1)!} \sum_{\sigma \in S_{\text{right}}} \sigma$$

- $\tilde{H}$ rep. of blob algebras, weights dep. on m, n

What we do

Compute leading exponents $h^{m,n}(k)$ in sector k

- Example:

$m = 2, n = 2, 2L = 4$
$k = 2, \#(\text{legs}) = 4$

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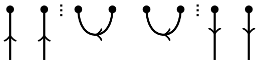
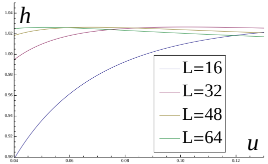


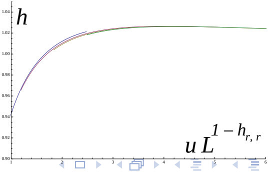
Critical exponents

k	$\#(\text{legs})$	$h^{m,n}(k)$
0	$m-n$	$h_{r_0,r_0}=0$
1	$m-n+2$	h_{r_1,r_1}
$\vdots$	$\vdots$	$\vdots$
n	$n+m$	h_{r_n,r_n}
$n+1$	$n+m+2$	$h_{1,3}$
$\vdots$	$\vdots$	$\vdots$
$n+j$	$n+m+2j$	$h_{1,1+2j}$
$\vdots$	$\vdots$	$\vdots$

$$r_k=\frac{6}{\pi}\arccos\left(\frac{\sqrt{3}}{2}\sqrt{\frac{(n+1-k)(m+1+k)}{(m+1)(n+1)}}\right)$$

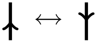
- Indep. of m, n for $\#(\text{legs}) > n+m$
- Irrational
- Indep. of couplings: boundary RG flow

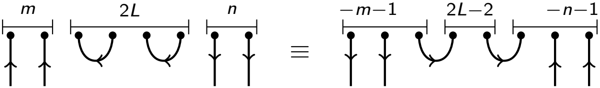




Symmetries and other cases

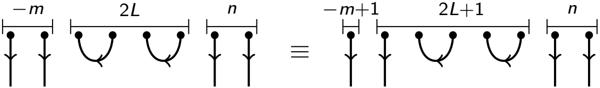
1. Left $\leftrightarrow$ Right: $h^{\mathcal{L},\mathcal{R}}(k) = h^{\mathcal{R},\mathcal{L}}(k)$

2. 

- 

$\Rightarrow\ h^{-m,-n}(k) = h^{m-1,n-1}(k)$

3. $\mathcal{L}\cdot\mathcal{R} < 0,\ \#(\text{legs}) \geq |\mathcal{L}| + |\mathcal{R}|$

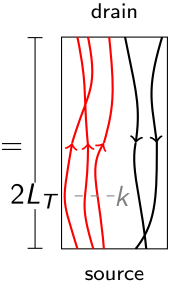
- 

$\Rightarrow\ h^{-m,n}(k) = h^{m,-n}(k) = h_{1,2+2k} = \frac{k(2k+1)}{3}$

Critical conductance in a strip

- Bottom-top $\bar{g}^{\mathcal{L},\mathcal{R}} = 2\max(0, \mathcal{L}-\mathcal{R}) + 2\sum_{k=1}^{\infty} kP(k, L_T/L)$

$P(k, L_T/L) =$



$$\approx e^{-\pi h^{m,n}(k)\frac{L_T}{L}}, \quad \frac{L_T}{L} \to \infty$$

- In quasi 1D geometry:

$$\bar{g}^{\mathcal{L},\mathcal{R}} \sim 2\max(0, \mathcal{L}-\mathcal{R}) + Ce^{-\pi h^{\mathcal{L},\mathcal{R}}(1)\frac{L_T}{L}}$$

Numerics for network model

- $g^{\mathcal{L},\mathcal{R}}$ from transfer matrices
  - Fit $\bar{g}^{\mathcal{L},\mathcal{R}} \sim g_\infty + Ce^{-\lambda \frac{L_T}{L}}$
  - Typically $L_T/L \in [2, 40]$, disorder $\mathcal{O}(10^5) \sim \mathcal{O}(10^6)$

$\Rightarrow$ Confirmed $h^{\mathcal{L},\mathcal{R}}(1)$

$\Rightarrow$ Verified indep. on bdry couplings (even random)

$\mathcal{L},\mathcal{R}$	numerics $h^{\mathcal{L},\mathcal{R}}(1)$	analytical $h^{\mathcal{L},\mathcal{R}}(1)$
$0, 0$	0.3333(12)	1/3
$0, 1$	0.3330(7)	1/3
$0, 10$	0.3325(24)	
$1, 1$	0.03775(25)	0.037720
$2, 2$	0.01600(2)	0.015906
$1, 2$	0.0520(25)	0.052083
$2, 4$	0.02954(7)	0.029589
$-2, -2$	0.0377(4)	0.037720
$-3, -2$	0.0522(2)	0.052083
$-1, 0$	0.999(9)	1
$-2, 0$	0.999(3)	
$-2, 1$	0.998(3)	

Conclusions and Outlooks

- Mapping SQH extra edge channels to classical loop model
- Exact critical exponents of boundary CFT
- Verified predictions of decay conductance
- Outlooks
  - Geometrical description of edge states in other systems
  - Exact conductance