Hall Viscosity

A Linear Response Approach

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Outline

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- Background - Non-dissipative viscosity and adiabatic transport
- Perturbed Hamiltonian at zero magnetic field - Strain operators
- The stress tensor
- Kubo formula for viscosity
- Example calculation - free electrons
- Zero Frequency viscosity of paired superfluids in 2 dimensions
- (Traceless) Strain operators at finite magnetic field
- (Traceless) Viscosity of the Landau level system
- Conclusion & Outlook

Background

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- Viscosity describes response of the stress tensor Π_{ij} to an applied strain $u_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$
  - $\langle \Pi_{ij} \rangle = \langle \Pi_{ij} \rangle_0 + \lambda_{ijk\ell} u_{k\ell} - \eta_{ijk\ell} \dot{u}_{k\ell}$
- Dissipation described by <u>symmetric part</u> $\eta^S_{ijk\ell} = \frac{1}{2}\left(\eta_{ijk\ell} + \eta_{k\ell ij}\right)$ of viscosity tensor
- Broken time reversal symmetry -> $\eta^A_{ijk\ell} = \frac{1}{2}\left(\eta_{ijk\ell} - \eta_{k\ell ij}\right) \neq 0$ in general
- How can we calculate $\eta^A_{ijk\ell}$?

Background

- Answer: for systems with a gap, can use adiabatic response (Avron, Seiler, Zograf, 1995, Levay)
- Idea – Time varying strain <-> Time varying metric
- Since $\Pi_{ij} = 2\frac{\delta H}{\delta g_{ij}}$, adiabatic theorem gives antisymmetric viscosity as the Berry Curvature associated with adiabatically deforming the metric
- In 2D: $\eta^A_{ijk\ell} = \eta^H \left(\delta_{jk}\epsilon_{i\ell} - \delta_{i\ell}\epsilon_{jk}\right)$
- Specific Systems:
  - IQH states: $\eta^H = \frac{\nu}{4}\bar{n}$
  - FQH states: $\eta^H = \frac{1}{2}\left(\frac{1}{2\nu} + h_\psi\right)\bar{n}$
  - ℓ-wave paired superfluids: $\eta^H = -\frac{1}{4}\ell\bar{n}$
  - General form (Read; Read & Rezayi): $\eta^H = \frac{1}{2}\bar{s}\bar{n}$
- Extensions to other systems (Hughes, Leigh, Fradkin ; Qi et. al.)

Motivation for this work

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- Viscosity is a linear transport coefficient
- Even for non-dissipative viscosity, should be able to make contact with traditional transport theory
- => There should be a Kubo formula for viscosity which can reproduce the results above
- We can't find it in the literature

Perturbed Hamiltonian for Viscosity

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- Goal – find a linear response formula for viscosity
- Starting Point: Time-varying metric perturbation at $B = 0$

$$H = g^{ij}(t) \sum_a \frac{p_i^a p_j^a}{2m} + \frac{1}{2} \sum_{a \neq b} V(\Lambda^T(t)(\mathbf{x}^a - \mathbf{x}^b))$$

$$g_{ij}(t) = \Lambda_{ik}(t)\Lambda_{jk}(t)$$

$$g^{ij}(t) = \Lambda^{-1}_{ki}(t)\Lambda^{-1}_{kj}(t)$$

$$\Lambda_{ij} = e^{\lambda_{ij}(t)}$$

- Look for a unitary transformation that diagonalizes the metric, i.e.

$$S^\dagger(t) H S(t) = \delta_{ij} \sum_a \frac{p_i^a p_j^a}{2m} + \frac{1}{2} \sum_{a \neq b} V(\mathbf{x}^a - \mathbf{x}^b) = H_0$$

- This is a <u>strain transformation</u>

Strain Generators

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- We can write $S = \exp\left(-i\lambda_{ij}(t) J_{ij}\right)$
- The condition $S^\dagger(t) H S(t) = H_0$ gives the following commutation relations:

$$i\left[J_{ij}, p_k^a\right] = \delta_{ki} p_j^a$$

$$i\left[J_{ij}, x_k^a\right] = -\delta_{kj} x_i^a$$

- These imply that

$$\boxed{J_{ij} = -\frac{1}{2}\sum_a \left\{x_i^a, p_j^a\right\}}$$

- Note that $i\left[J_{ij}, J_{k\ell}\right] = \delta_{i\ell} J_{kj} - \delta_{jk} J_{i\ell}$
  - These are the commutation relations of $\mathfrak{sl}(d, \mathbb{R})$
  - The J_{ij} operators generate deformations of shape - Strain generators
  - To avoid boundary issues, we will work in the infinite plane

The Transformed System

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- Since $\lambda_{ij}(t)$ is time dependent, we must transform the time-dependent Schrödinger Equation $i\frac{\partial}{\partial t}|\psi\rangle = H(t)|\psi\rangle$
- $|\psi\rangle = S|\psi'\rangle$ implies the transformed Schrödinger equation is

$$i\frac{\partial|\psi'\rangle}{\partial t} = H_0|\psi'\rangle - iS^{\dagger}\frac{\partial S}{\partial t}|\psi'\rangle$$

- Linear response -> keep only first order terms -> the Hamiltonian is

$$H = H_0 + H_1$$

$$H_1 = -\frac{\partial \lambda_{ij}}{\partial t}J_{ij}$$

- We have transformed the metric perturbation into a "potential" perturbation
  - Analogous to the case of an electric field, where a gauge transformation takes us from $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$ to $\mathbf{E} = -\nabla\phi$

- Can gain some insight by looking at the continuity equation for momentum density $\mathbf{g}(\mathbf{r}) = \frac{1}{2}\sum_a \{\mathbf{p^a}, \delta(\mathbf{r}-\mathbf{x^a})\}$

$$\frac{\partial g_j(\mathbf{r})}{\partial t} + \partial_i \Pi_{ij}(\mathbf{r}) = f_j^{ext}(\mathbf{r}) = 0$$

  - $\mathbf{f}^{ext}(\mathbf{r})$ is the external force density, which is zero for the Hamiltonian H_0

- Fourier transforming, and expanding to leading order in $\mathbf{q}$,

$$\frac{\partial}{\partial t}\sum_a \left(p_j^a - \frac{iq_i}{2}\{p_j^a, x_i^a\}\right) + iq_i \Pi_{ij}(q=0) = 0$$

- We notice that

$$\boxed{\begin{aligned} J_{ij} &= -\int d^d\mathbf{r}\, r_i g_j(\mathbf{r}) \\ \Pi_{ij} \equiv \Pi_{ij}(q=0) &= -\frac{\partial J_{ij}}{\partial t} \end{aligned}}$$

- Observation - the perturbation $H_1 = -\frac{\partial \lambda_{ij}}{\partial t} J_{ij}$ is a position-dependent velocity field - c.f. classical viscosity

Linear Response Theory

- Can now use standard linear response theory to find response of the stress Π_{ij} to the strain perturbation $H_1 = -\frac{\partial \lambda_{ij}}{\partial t} J_{ij}$
- Recall the Kubo Formula:
  - Response of an operator A to a perturbation $U(t) = f(t)B$ can be written

$$\langle A \rangle = \langle A \rangle_0 - V \int_{-\infty}^{\infty} dt' \, \chi(t-t') f(t')$$

  - where the response function $\chi(t-t')$ is

$$\chi(t-t') = -\frac{i}{V} \lim_{\epsilon \to 0^+} \Theta(t-t') \, \langle [A(t), B(t')] \rangle_0 \, e^{-\epsilon(t-t')}$$

  - with time-evolution and averages taken w.r.t. the unperturbed hamiltonian
  - Can write this in frequency space as

$$\boxed{\chi(\omega) = -\frac{i}{V} \lim_{\epsilon \to 0+} \int_0^{\infty} dt \, e^{i\omega^+ t} \, \langle [A(t), B(0)] \rangle_0}$$

$$\omega^+ = \omega + i\epsilon$$

Kubo Formula for Viscosity

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- Adapting this to the stress-strain response, we find that at $B = 0$

$$\langle \Pi_{ij} \rangle (t) - \langle \Pi_{ij} \rangle_0 = -V \int_{-\infty}^{\infty} dt' \eta_{ijkl}(t - t') \frac{\partial \lambda_{kl}}{\partial t}$$

$$\boxed{\eta_{ijk\ell}(\omega) = -\frac{1}{V} \lim_{\epsilon \to 0^+} i \int_0^{\infty} dt e^{i\omega^+ t} \langle [\Pi_{ij}(t), J_{k\ell}(0)] \rangle_0}$$

- Using time-translation invariance and $\Pi_{ij} = -\frac{\partial J_{ij}}{\partial t}$, can write

$$\eta_{ijk\ell}(\omega) = \lim_{\epsilon \to 0^+} \frac{1}{\omega^+ V} \left(\langle [\Pi_{ij}(0), J_{k\ell}(0)] \rangle_0 + \int_0^{\infty} dt e^{i\omega^+ t} \langle [\Pi_{ij}(t), \Pi_{k\ell}(0)] \rangle_0 \right)$$

$$\eta_{ijk\ell}(\omega) = \frac{1}{V} \lim_{\epsilon \to 0} \left(-i \langle [J_{ij}(0), J_{k\ell}(0)] \rangle_0 + \omega^+ \int_0^{\infty} e^{i\omega^+ t} \langle [J_{ij}(t), J_{k\ell}(0)] \rangle_0 \, dt \right)$$

- Three equivalent expressions for the viscosity

Simple Example: Free Electron Gas

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- $H_0 = \sum_a \frac{p_i^a p_i^a}{2m}$, so the stress tensor is $\Pi_{ij} = \sum_a \frac{p_i^a p_j^a}{m}$ and it is constant in time
- State of interest - N electrons in a large box
- Can use stress-stress form of viscosity, and only the contact term contributes. Thus,

$$\eta_{ijk\ell}(\omega) = \lim_{\epsilon \to 0^+} \frac{1}{V\omega^+} \sum_{ab} \frac{-1}{2m} \left\langle \left[p_i^a p_j^a, \left\{ x_k^b, p_\ell^b \right\} \right] \right\rangle$$
$$= \lim_{\epsilon \to 0^+} \frac{2i}{dV\omega^+} \langle E \rangle \left(\delta_{jk}\delta_{i\ell} + \delta_{j\ell}\delta_{ik} \right)$$

- Real part $\Re\left[\eta_{ijk\ell}(\omega)\right] \propto \delta(\omega)(\delta_{jk}\delta_{i\ell} + \delta_{j\ell}\delta_{ik})$
  - Interpretation: infinite response of free electrons to static perturbation because of acceleration. C.f. diamagnetic conductivity
  - For non-trivial systems, we will see that these divergences cancel with terms in the correlators

More Interesting Example: Paired States in 2D

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- $H_0 = \int d^2r\, \psi^\dagger(r)\left(-\frac{1}{2m}\nabla^2 - \mu\right)\psi(r) + \frac{1}{2}\int\int d^2r d^2r' \left(\Delta(r-r')\psi^\dagger(r)\psi^\dagger(r') + h.c.\right)$
  - pair-potential $\Delta(r-r')$ transforms as an ℓ-pole under rotations
  - Energies $E_k = \sqrt{\left(\frac{k^2}{2m} - \mu\right)^2 + |\Delta_k|^2}$
- Second-quantized strain operator

$$J_{ij} = -\int d^2r\, r_i g_j(\mathbf{r})$$
$$= \frac{i}{2}\int d^2r\, r_i \left(\psi^\dagger(r)\frac{\partial\psi(r)}{\partial r_j} - \frac{\partial\psi^\dagger(r)}{\partial r_j}\psi(r)\right)$$

- Calculation is easiest using strain-strain form of viscosity
  - Work in the ground state at zero temperature
  - focus only on the $\omega \to 0$ limit
  - Need to evaluate $\eta_{ijk\ell}(\omega \to 0) = \frac{1}{V}\lim_{\omega\to 0}\left(-i\left\langle[J_{ij}(0), J_{k\ell}(0)]\right\rangle_0 + \omega^+\int_0^\infty e^{i\omega^+ t}\left\langle[J_{ij}(t), J_{k\ell}(0)]\right\rangle_0 dt\right)$
- Strategy - Use Bogoliubov transformation to evaluate time dependence of J_{ij} , carefully take the limit

Paired State Viscosity II

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- Fourier modes of $\langle [J_{ij}(t), J_{k\ell}(0)] \rangle_0 \propto f(\mathbf{k}) e^{2iE_k t} + g(\mathbf{k}) e^{-2iE_k t}$
- So modes of $\omega^+ \int_0^\infty \langle [J_{ij}(t), J_{k\ell}(0)] \rangle_0 e^{i\omega^+ t} dt \propto \frac{\omega^+ F(\mathbf{k},\omega^+)}{(\omega^+)^2 - 4E_k^2}$ with F linear in ω^+
- => This term vanishes in the static limit, leaving

$$\eta_{ijk\ell}(\omega \to 0) = -\frac{i}{V} \langle [J_{ij}(0), J_{k\ell}(0)] \rangle_0$$

$$= \frac{1}{V} \left(\delta_{kj} \langle J_{i\ell} \rangle_0 - \delta_{i\ell} \langle J_{kj} \rangle_0 \right)$$

- Rotational covariance => $\langle J_{xx} \rangle_0 = \langle J_{yy} \rangle_0$ and $\langle J_{xy} \rangle_0 = -\langle J_{yx} \rangle_0$
- Thus,

$$\eta_{ijk\ell}(0) = \frac{1}{2V} \langle J_{xy} - J_{yx} \rangle_0 \left(\delta_{kj}\epsilon_{i\ell} - \delta_{i\ell}\epsilon_{kj} \right)$$

$$\eta^H = \frac{1}{2V} \langle J_{xy} - J_{yx} \rangle_0 = -\frac{\langle L_z \rangle_0}{2V}$$

$$= \frac{1}{2}\bar{s}\bar{n}$$

as expected!

Strain in a Nonzero Magnetic Field

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- Switching on a magnetic field, we have for a general 2D system

$$H_0 = \sum_a \frac{\pi_i^a \pi_i^a}{2m} + \frac{1}{2}\sum_{a \neq b} V(\mathbf{x}^a - \mathbf{x}^b)$$

$$\mathbf{g}(\mathbf{r}) = \frac{1}{2}\sum_a \{\boldsymbol{\pi}^a, \delta(\mathbf{r} - \mathbf{x}^a)\}$$

$$\boldsymbol{\pi}^a = \mathbf{p}^a - \mathbf{A}$$

- We may think that, just as before, the strain could be defined as $\tilde{J}_{ij} = -\int d^d r \, r_j g_j(\mathbf{r}) = -\frac{1}{2}\sum_a \{x_i^a, \pi_j^a\}$
- Unfortunately, this breaks all the nice commutation relations derived earlier: $i[\tilde{J}_{ij}, \pi_k^a] = \delta_{ik}\pi_j^a + B\epsilon_{jk}x_i^a$, $\tilde{J}_{ij} \notin \mathfrak{sl}(2, \mathbb{R})$
  - Why? - Lorentz Force
  - Changes the continuity equation: $\frac{\partial g_j(\mathbf{r})}{\partial t} + \partial_i \Pi_{ij}(\mathbf{r}) = \frac{B}{m}\epsilon_{jk} g_k(\mathbf{r})$
  - Equivalently, $[\pi_i^a, \pi_j^b] = iB\delta_{ab}\epsilon_{ij} \neq 0$

Strain in a Nonzero Magnetic Field II

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- Additional difficulty - dilations/compressions (trace part of strain) change the total flux through the system
- Work around - consider only traceless deformations

$$J_{ij} = \tilde{J}_{ij} - \frac{1}{2}\mathrm{tr}(\tilde{J})\delta_{ij} + \frac{B}{2}\epsilon_{j\ell}\sum_a x_i^a x_\ell^a$$
$$= \sum_a -\frac{1}{2}\left\{x_i^a, \pi_j^a\right\} + \frac{1}{4}\delta_{ij}\left\{x_k^a, \pi_k^a\right\} + \frac{B}{2}\epsilon_{j\ell}x_i^a x_\ell^a$$

- Gives the traceless part of the strain transformations, restores $\mathfrak{sl}(d,\mathbb{R})$ commutation relations
  - Interpretation: For $\lambda_{ij}(t)$ traceless, $\tilde{S}(t) = \exp(-i\lambda_{ij}(t)\tilde{J}_{ij})$ implements a strain transformation combined with a gauge transformation. Extra term undoes this gauge transformation
  - Subtracts out Lorentz force contribution to the traceless part of the first moment of $\frac{\partial \mathbf{g}}{\partial t}$

(Traceless) Viscosity of the Landau Level System

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- Now we specialize to $H_0 = \sum_a \frac{\pi_i^a \pi_i^a}{2m}$
- Traceless part of stress tensor given by

$$\begin{aligned} \Pi_{ij} &= -i\left[H_0, J_{ij}\right] \\ &= \frac{1}{2m}\sum_a \left\{\pi_i^a, \pi_j^a\right\} - \delta_{ij} H_0 \end{aligned}$$

- Can calculate the (traceless) viscosity from any of the 3 forms of the Kubo formula presented earlier
  - Particularly instructive in stress-stress and strain-strain forms.
  - Will work at zero temperature
  - State of interest: very large system with ν filled LLs

Reminder: Some Useful Operators

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- Recall that we can diagonalize the LL Hamiltonian by introducing raising and lowering operators:

$$b_q = \frac{1}{\sqrt{2B}}\left(\pi_x^q + i\pi_y^q\right)$$

$$a_q = \sqrt{\frac{B}{2}}\left(x^q - iy^q\right) + ib_q^\dagger$$

- In these terms, $H_0 = \sum_q \omega_c \left(b_q^\dagger b_q + \frac{1}{2}\right)$

- Expressing the stress and strain in terms of raising and lowering operators, we find

$$\Pi_{ij} = \sum_q \frac{\omega_c}{2}\left(\left({b_q^\dagger}^2 + b_q^2\right)\sigma_{ij}^z + i\left({b_q^\dagger}^2 - b_q^2\right)\sigma_{ij}^x\right)$$

$$J_{ij} = \sum_q \frac{i}{4}\left({b_q^\dagger}^2 - b_q^2 + a_q^2 - {a_q^\dagger}^2\right)\sigma_{ij}^z - \frac{1}{4}\left(b_q^2 + {b_q^\dagger}^2 + a_q^2 + {a_q^\dagger}^2\right)\sigma_{ij}^x + \frac{1}{2}\left(b_q^\dagger b_q - a_q^\dagger a_q\right)\epsilon_{ij}$$

- Time dependence is now trivial, Kubo formulas easy to evaluate

(Traceless) Viscosity from Stress-Stress Formula

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$$\eta_{ijk\ell}(\omega) = \lim_{\epsilon\to 0^+} \frac{1}{\omega^+ V}\left(\langle [\Pi_{ij}(0), J_{k\ell}(0)]\rangle_0 + \int_0^\infty dt e^{i\omega^+ t}\langle [\Pi_{ij}(t), \Pi_{k\ell}(0)]\rangle_0 \right)$$

- Contact term gives $\frac{1}{V\omega^+}\langle [\Pi_{ij}(0), J_{k\ell}(0)]\rangle = \frac{i}{V\omega^+}\langle H_0\rangle \left(\delta_{i\ell}\delta_{jk} - \epsilon_{i\ell}\epsilon_{jk}\right)$
  - divergent as $\omega \to 0$
- Time integral gives

$$\frac{1}{V\omega^+}\int_0^\infty e^{i\omega^+ t}\langle [\Pi_{ij}(t), \Pi_{k\ell}(0)]\rangle dt = \frac{\langle H_0\rangle \omega_c}{V\left({\omega^+}^2 - 4\omega_c^2\right)}\left(\frac{4i\omega_c}{\omega^+}\left(\delta_{i\ell}\delta_{jk} - \epsilon_{i\ell}\epsilon_{jk}\right) - 2\left(\delta_{jk}\epsilon_{i\ell} - \delta_{i\ell}\epsilon_{kj}\right)\right)$$

- Combining terms, we get for the traceless viscosity

$$\eta_{ijk\ell}(\omega) = \frac{\langle H_0\rangle}{V\left({\omega^+}^2 - 4\omega_c^2\right)}\left[i\omega^+\left(\delta_{i\ell}\delta_{jk} - \epsilon_{i\ell}\epsilon_{jk}\right) - 2\omega_c\left(\delta_{jk}\epsilon_{i\ell} - \delta_{i\ell}\epsilon_{kj}\right)\right]$$

  - no divergence as $\omega \to 0$
- In the static limit, we recover the Hall viscosity

$$\eta_{ijk\ell}(\omega \to 0) = \frac{\langle H_0\rangle}{2V\omega_c}\left(\delta_{jk}\epsilon_{i\ell} - \delta_{i\ell}\epsilon_{kj}\right) \qquad \eta^H = \frac{\langle H_0\rangle}{2V\omega_c} = \frac{\nu}{4}\bar{n}$$

(Traceless) Viscosity from Strain-Strain Formula

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- Strain-Strain Kubo formula must give the same result, but it's interesting to see how
- Contact term gives $-\frac{i}{V}\langle[J_{ij}(0), J_{k\ell}(0)]\rangle = \frac{1}{V}\left(\frac{\langle H_0\rangle}{2\omega_c} - \frac{\langle a^\dagger a + \frac{1}{2}\rangle}{2}\right)(\delta_{jk}\epsilon_{i\ell} - \delta_{i\ell}\epsilon_{kj})$
  - Ill-defined due to infinite LL degeneracy
- Time integral gives $\frac{\omega^+}{V}\int_0^\infty e^{i\omega^+ t}\langle[J_{ij}(t), J_{k\ell}(0)]\rangle\, dt = \frac{1}{V}\left(\frac{\langle a^\dagger a + \frac{1}{2}\rangle}{2} - \frac{\langle H_0\rangle\,\omega^{+2}}{2\omega_c\left(\omega^{+2} - 4\omega_c^2\right)}\right)(\delta_{jk}\epsilon_{i\ell} - \delta_{i\ell}\epsilon_{kj})$
  $+\frac{i\omega^+\langle H_0\rangle}{V(\omega^{+2} - 4\omega_c^2)}(\delta_{i\ell}\delta_{jk} - \epsilon_{i\ell}\epsilon_{jk})$

- As required, after combining terms we recover the result from the previous slide, and in particular $\eta^H = \frac{\nu}{4}\bar{n}$
- Intra-LL terms cancel exactly!
  - In the static limit, corresponds to adiabatic transport of *degenerate subspaces* (Read & Rezayi)
  - Ensures independence of final result from boundary conditions

Conclusions & Outlook

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- At zero magnetic field, we have derived a general Kubo formula for the viscosity
- At finite magnetic field, we have a Kubo formula for the traceless part of the viscosity tensor
- For IQH systems and 2D paired superfluids, we have derived the Hall-viscosity from elementary linear response theory
- What remains to be done:
  - Bulk viscosity of systems in finite magnetic fields
  - Frequency-dependent viscosity of paired states
  - Relationship between (Hall) viscosity and conductivity (c.f. Hoyos & Son)