W-Superalgebras

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W-algebras from affine Lie Superalgebras

- Coset
- BRST-cohomology
- Extended algebra

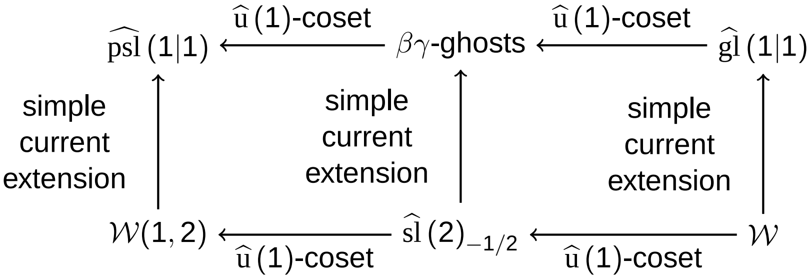
Today

- W-superalgebras extending $\widehat{\mathrm{gl}}(1|1)_k$ [Ridout, TC]
- Commutant realization of $W_n^{(2)}$ at critical level from $\widehat{\mathrm{psl}}(n|n)_0$ [Gao, Linshaw, TC]

The $\widehat{\mathrm{gl}}(1|1)$ WZW model

- [Rozansky, Saleur, Schomerus, Quella, Roenne, TC]
- Bulk correlation functions
- Boundary and bulk-boundary correlators
- Cardy boundary states
- Relation to symplectic fermions

The archetypical logarithmic CFTs



Lie superalgebras

$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with product $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and parity

$$|X| = \begin{cases} 0 & X \text{ in } \mathfrak{g}_0 \\ 1 & X \text{ in } \mathfrak{g}_1 \end{cases} .$$

$\mathfrak{g}$ is a **Lie superalgebra** if it satisfies antisupersymmetry and graded Jacobi identity:

$$0 = [X, Y] + (-1)^{|X||Y|}[Y, X] \qquad \text{and}$$

$$0 = (-1)^{|X||Z|}[X, [Y, Z]] + (-1)^{|Y||X|}[Y, [Z, X]] + (-1)^{|Z||Y|}[Z, [X, Y]]$$

for all X, Y and Z in $\mathfrak{g}$.

Example: gl(1|1)

$$[N, \psi^{\pm}] = \pm\psi^{\pm} \qquad \text{and} \qquad \{\psi^+, \psi^-\} = E\,.$$

Matrix representation

$$E = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \quad , \qquad N = \begin{pmatrix} n & 0 \\ 0 & n-1 \end{pmatrix} \,,$$
$$\psi^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \psi^- = \begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix} \quad .$$

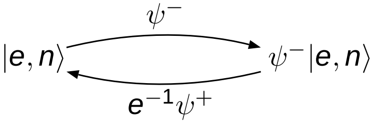
Invariant bilinear form is the supertrace $\mathrm{str}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a - d$.

Typical Representations

$$\begin{aligned} E|e,n\rangle &= e|e,n\rangle\ , \\ N|e,n\rangle &= n|e,n\rangle\ , \\ \psi^+|e,n\rangle &= 0 \end{aligned}$$

and ψ^- acts freely on this state, hence

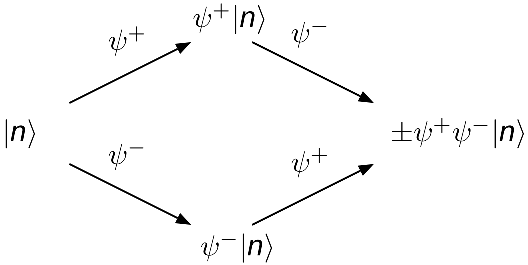
$$\psi^+\psi^-|e,n\rangle = e|e,n\rangle$$



Indecomposable but reducible representations

$$E|n\rangle = 0 \qquad , \qquad N|n\rangle = n|n\rangle$$

and ψ^+, ψ^- are acting freely on it.



The affine Lie superalgebra $\widehat{\mathfrak{gl}}(1|1)$

Generators $E_n, N_n, \psi_n^{\pm}$ (n in $\mathbb{Z}$) and K, d, where K is central and d is a derivation

$$[d, X_n] = nX_n \qquad \text{for } X \in \{E, N, \psi^{\pm}\}.$$

The non-vanishing relations of the remaining generators are

$$\begin{aligned}
[E_n, N_m] &= Kn\delta_{n+m,0}\,, \\
[N_n, \psi_m^{\pm}] &= \psi_{n+m}^{\pm} \text{ and} \\
\{\psi_n^{-}, \psi_m^{+}\} &= E_{n+m} + Kn\delta_{n+m,0}\,.
\end{aligned}$$

Representations of $\widehat{\mathrm{gl}}(1|1)$ at level k

- typical irreducible highest-weight representations $T_{e,n}$ for $e \neq mk, m \in \mathbb{Z}$
- indecomposable but reducible representations $P_{m,n}$ for $e = mk, m \in \mathbb{Z}$
  Contain irreducible atypical submodules $A_{m,n}$
- Characters of these representations form a representation of SL(2,$\mathbb{Z}$) $\Leftrightarrow$ $(e, n) \in \mathbb{R}^2$

Question: Is it possible to have a smaller sectrum?
Answer: Yes, by extending $\widehat{\mathrm{gl}}(1|1)$.

Fusion

- $T_{e,n}$ and $P_{m,n}$ close under fusion
  OPE contains logarithmic singularities

- Atypical irreducibles close under fusion
  OPE doesnot contain logarithmic singularities

Idea: Extend $\widehat{\mathrm{gl}}(1|1)$ by some atypical modules

Question: What is the extended algebra?

The $\widehat{\mathrm{gl}}(1|1)$ WZW model

2 ways to treat the $\widehat{\mathrm{gl}}(1|1)$ WZW model:

- Wakimoto-type free field realization [Saleur, Schomerus]
- with symplectic fermions [Roenne, TC]

$\widehat{\mathrm{gl}}(1|1)$ and symplectic fermions

Symplectic fermions $\chi^{\pm}$ and bosons Y, Z

$$\chi^{+}(z)\chi^{-}(w) = \frac{1}{(z-w)^2} \quad , \quad \partial Y(z)\partial Z(w) = \frac{1}{(z-w)^2}$$

The $\widehat{\mathrm{gl}}(1|1)$ currents are

$$E(z) = k\partial Y(z), \qquad N(z) = \partial Z(z), \qquad \psi^{\pm}(z) = \sqrt{k}e^{\pm Y(z)}\chi^{\pm}(z),$$

$\widehat{\mathrm{gl}}(1|1)$ and symplectic fermions

Operator products

$$N(z)E(w) = \frac{k}{(z-w)^2}$$

$$\psi^+(z)\psi^-(w) = \frac{k}{(z-w)^2} + \frac{E(w)}{(z-w)}$$

$$N(z)\psi^\pm(w) = \pm\frac{\psi^\pm(w)}{(z-w)}$$

$\widehat{\mathrm{gl}}(1|1)$ and symplectic fermions

- Twisted modules of symplectic fermions are used to get typical $\widehat{\mathrm{gl}}(1|1)$ modules.
- Untwisted modules give indecomposable but reducible ones.
- Operator Algebra can be computed.

Some extended algebras

- dimension $1/2$: $\beta\gamma$-ghosts plus a pair of free fermions.
- dimension 1: $\widehat{\mathrm{sl}}(2|1)_1$ and $\widehat{\mathrm{sl}}(2|1)_{-1/2}$.
- dimension $3/2$: Three different W-algebras containing $N = 2$ superconformal algebra at $c = \pm 1$ and $W_3^{(2)}$ algebra at level $k = 0, -5/3$.
- These cannot be obtained by DS-reduction from affine Lie superalgebras.

Summary and Outlook

- Atypical moduels of $\widehat{\mathrm{gl}}(1|1)$ can be used to extend $\widehat{\mathrm{gl}}(1|1)$
- One finds interesting algebras
- What are the modular properties of the modules of the extended algebras?
  Leads to Appell Lerch sums

$W_n^{(2)}$ at critical level

From a supercoset to invariant theory

[Peng Gao, Andrew Linshaw, TC]

Berkovits Fermionic Coset

Berkovits non-linear sigma model is a pure spinor version of the fermionic coset

$$\frac{PSU(2,2|4)}{SU(2,2) \times SU(4)}$$

It has a common limit with the $AdS_5 \times S^5$ sigma model

Action

$\mathrm{psl}\,(4|4) = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$

$\chi_\pm, \beta_\pm \in \mathfrak{g}_\pm$

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_{J\bar{J}} \\
\mathcal{L}_0 &= \mathrm{str}(\partial\chi_+, \bar{\partial}\chi_-) + \mathrm{str}(\partial\chi_-, \bar{\partial}\chi_+) + \mathrm{str}(\beta_+\bar{\partial}\beta_-) + \mathrm{str}(\bar{\beta}_+\partial\bar{\beta}_-) \\
\mathcal{L}_{J\bar{J}} &= \frac{1}{2}\mathrm{str}(\partial\chi_+, [[\bar{\partial}\chi_+, \chi_-], \chi_-]) + \frac{1}{2}\mathrm{str}(\bar{\partial}\chi_-, [[\partial\chi_+, \chi_-], \chi_-]) + \\
&\quad \mathrm{str}(\{\beta_+, \beta_-\}, \{\bar{\beta}_+, \bar{\beta}_-\}) + \\
&\quad \mathrm{str}(\{\beta_+, \beta_-\}, [\bar{\partial}\chi_+, \chi_-]) + \mathrm{str}(\{\bar{\beta}_+, \bar{\beta}_-\}, [\partial\chi_+, \chi_-])
\end{aligned}$$

Perturbation of 16 pairs of symplectic fermions and $\beta\gamma$ systems

A free field realization of $\widehat{\mathrm{pgl}}(4|4)_0$

Symplectic fermion CFT contains *bc*-ghosts as subalgebra

$$
\begin{aligned}
J_B^{E_+^{\alpha\beta}} &= \beta^+_{\beta\gamma}\beta^-_{\gamma\alpha} \\
J_B^{E_-^{\alpha\beta}} &= -\beta^-_{\beta\gamma}\beta^+_{\gamma\alpha}
\end{aligned}
$$

These realize critical level $\widehat{\mathrm{sl}}(4)_{-4} \oplus \widehat{\mathrm{sl}}(4)_{-4} \oplus \widehat{\mathrm{gl}}(1)$

$$
J_B^{E_+^{\alpha\alpha}} + J_B^{E_-^{\alpha\alpha}} = 0\,.
$$

$$
\begin{aligned}
J^{E_\epsilon^{\alpha\beta}} &= J_B^{E_\epsilon^{\alpha\beta}} - \delta_{\epsilon,+} b_{\beta\gamma}c_{\gamma\alpha} + \delta_{\epsilon,-} b_{\gamma\alpha}c_{\beta\gamma}\,, \\
J^{F_-^{\alpha\beta}} &= -b_{\beta\alpha}\,, \\
J^{F_+^{\alpha\beta}} &= -c_{\beta\gamma}J_B^{E_+^{\alpha\gamma}} - c_{\gamma\alpha}J_B^{E_-^{\gamma\beta}} - b_{\gamma\delta}c_{\beta\gamma}c_{\delta\alpha}\,.
\end{aligned}
$$

A current-current perturbation

- $J \in \mathrm{psl}(4|4)$ with components $J^{t_a} = \mathrm{str}(t_a J)$

$$\mathcal{L}_{J\bar{J}} = \langle J, \bar{J} \rangle$$

- Perturbation preserves diagonal $\mathrm{psl}(4|4)$ symmetry
- What more symmetry does the perturbation preserve?
- $Com(\widehat{\mathrm{psl}}(4|4)_0, V_{bc}^{\otimes 16} \otimes V_{\beta\gamma}^{\otimes 16})$
- Expect a lot, as the center is allready large

The Commutant is bosonic

$$J^{F_-^{\alpha\beta}} = -b_{\beta\alpha}\,,$$
$$J^{F_+^{\alpha\beta}} = -c_{\beta\gamma}J_B^{E_+^{\alpha\gamma}} - c_{\gamma\alpha}J_B^{E_-^{\gamma\beta}} - b_{\gamma\delta}c_{\beta\gamma}c_{\delta\alpha}.$$

- Invariance under $J^{F_-^{\alpha\beta}}$ implies that commutant is independent of $c_{\alpha\beta}$
- Passing to a graded structure one can show that invariance under $J^{F_+^{\alpha\beta}}$ implies that commutant is independent of $b_{\alpha\beta}$
- It follows

$$Com(\widehat{\mathrm{psl}}(4|4)_0\,, V_{bc}^{\otimes 16}\otimes V_{\beta\gamma}^{\otimes 16}) = Com(\widehat{\mathrm{sl}}(4)_{-4}\oplus\widehat{\mathrm{sl}}(4)_{-4}\,, V_{\beta\gamma}^{\otimes 16})$$

The generators

- $\widehat{\mathrm{u}}(1)$-current $C_1 = J_B^{E_+^{\alpha\alpha}}$
- The Casimirs C_2, C_3, C_4
- The determinants $D =$det $\beta^+, D' =$det β^-

Relation (Quantum analogue of Newton-Girard)

$$
\begin{aligned}
: DD' :=& \frac{1}{256}C_4 + \frac{1}{32} : C_1C_3 : + \frac{1}{32} : C_2C_1C_1 : + \\
& - \frac{1}{256} : C_1C_1C_1C_1 : + \frac{1}{8} : \partial C_1C_2 : - \frac{3}{32} : \partial C_1C_1C_1 : + \\
& - \frac{1}{4} : \partial^2 C_1C_1 : - \frac{3}{16} : \partial C_1 \partial C_1 : - \frac{1}{4}\partial^3 C_1
\end{aligned}
$$

Operator product algebra

$$
\begin{aligned}
&C_1(z)C_1(w) \sim -16(z-w)^{-1}\,,\\
&C_1(z)D(w) \sim -4D(w)(z-w)^{-1}\,,\\
&C_1(z)D'(w) \sim 4D'(w)(z-w)^{-1}\,,\\
&D(z)D'(w) \sim 24(z-w)^{-4} + 6C_1(w)(z-w)^{-3} +\\
&\quad + \Big(- C_2(w) + \frac{3}{4} :C_1(w)C_1(w): + 3\partial C_1(w)\Big)(z-w)^{-2} +\\
&\quad + \Big(-\frac{1}{8}C_3(w) - \frac{1}{4} :C_2(w)C_1(w): + \frac{1}{16} :C_1(w)C_1(w)C_1(w): +\\
&\quad + \frac{3}{4} :C_1(w)\partial C_1(w): + \partial^2 C_1(w)\Big)(z-w)^{-1}\,.
\end{aligned}
$$

Operator product algebra of $W_4^{(2)}$ at critical level
[Feigin,Semikhatov]

Remarks

- The Commutant is strongly generated by D, D', C_1, C_2, C_3
- The Commutant is a quotient of $W_4^{(2)}$ at critical level by a one-dimensional ideal
- The Zhu functor relates the (quasi) CFT to some associative algebra, the representation theory of the Zhu algebra of our commutant is similar to that of $\mathrm{sl}(2)$ and we can find all irreducible finite-dimensional representations
- These give all irreducible admissible representations of our quotient of $W_4^{(2)}$ at critical level
- Proofs use to pass to a commutative graded algebra and some classical problem and a reconstruction theorem [Linshaw, Schwarz, Song]

Generalizations

- Free field realization of $\widehat{\mathrm{psl}}(n|n)_0$ inside n^2 bc and $\beta\gamma$ systems
- The commutant is a quotient by a one-dimensional ideal of $W_n^{(2)}$ at critical level
- Irreducible representations are similar to $\mathrm{sl}(2)$ and can be classified
- In fact $\widehat{\mathrm{sl}}(2)_k$ is $W_2^{(2)}$ at level k