CFT and Calogero-Sutherland model in the fractional quantum Hall effect

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work with
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Starting point

A fractional quantum Hall effect problem

- topological phases : gapped bulk, massless edge excitations $\Rightarrow$ **(chiral) conformal field theory**
- Bernevig, Haldane (2007) : many quantum Hall trial wavefunctions are Jack polynomials $\Rightarrow$ **Calogero-Sutherland Hamiltonian, integrable structure**

$\Rightarrow$ how to unify these two approaches ?

Motivations

Why do we want to know the CFT underlying Jack states ?

FQHE

- macroscopic (universal) properties (in particular the braiding)
- explicit quasi-holes wavefunctions
- unitarity issue

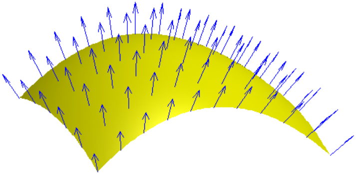
Mathematical physics

- emergence of the Calogero-Sutherland integrable structure in the CFT
- extra handle to manipulate the CFT underlying theses FQHE states

CFT and the Fractional quantum Hall effect

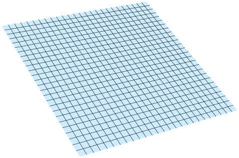
Setup of the quantum Hall effect

Particles (fermions or bosons) confined in 2D, in a strong perpendicular magnetic field, and low temperature



$$H = \frac{1}{2m}\left(\vec{p}_i - q\vec{A}\right)^2 + \sum_{i<j} V(\vec{x}_i - \vec{x}_j)$$

One body problem : Landau levels



Hamiltonian in the plane

$$H = \hbar\omega_c \left(a^\dagger a + \frac{1}{2} \right) \qquad \omega_c = \frac{|qB|}{m}$$

Discrete spectrum, huge degeneracy

- a and $a^\dagger$ change the Landau level :

$$a = \sqrt{2}\left(\bar{\partial} + \frac{z}{4}\right) \qquad a^\dagger = \sqrt{2}\left(-\partial + \frac{\bar{z}}{4}\right)$$

- Each Landau level is infinitely degenerate (translations)

The Lowest Landau level (LLL) is made of all holomorphic functions

$$\Psi_n(z,\bar{z}) = z^n e^{-z\bar{z}/4} \qquad n = 0, 1, \cdots \infty \qquad (\text{or } n \leq N_\Phi)$$

LLL projection (low energy physics) $\Rightarrow$ dimensional reduction !

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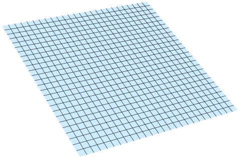

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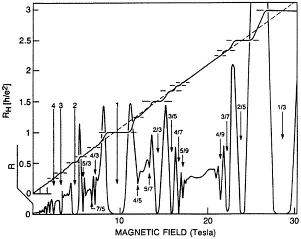
Fractional filling : N particles in N_Φ orbital/states

N body wavefunctions in the LLL : (anti)symmetric polynomials

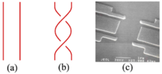
$$P(z_1, \ldots, z_N) e^{-\sum_i z_i \bar{z}_i/4} \qquad \text{filling fraction} \qquad \nu = N/N_\phi$$

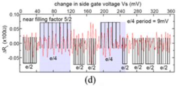


Strongly correlated systems, emergence of exotic phases : fractional charges, non-abelian braiding.

Electrons interactions

- Interactions can lift the degeneracy $\Rightarrow$ incompressible state
- we are interested in the low energy properties

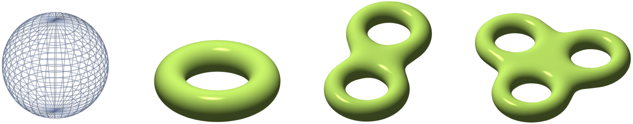




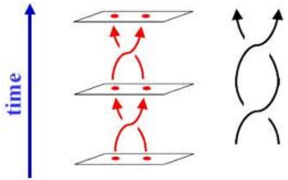
Topological phases and non-abelian states

A system is in a topological phase if, at low energy, all observables are invariant under smooth deformation of the underlying space-time manifold, i.e. when its low energy effective field theory is a TQFT (with a gap).

- Ground state degeneracy depends on the genus



- Excitations ("quasi-holes") with fractional charges, possibly non-abelian anyons (non trivial action of the braid group)



Link between $2+1$ TQFT and $1+1$ CFT

Quasi-hole wavefunctions are conformal blocks.

- degeneracy = number of conformal blocks
- braiding = monodromies

(Chiral) CFT and trial wavefunctions

Ground-state wavefunction [Moore and Read (1992)]

$$P(z_1, \ldots, z_N) = \langle V_e(z_1) \ldots V_e(z_N) \rangle$$

- $U(1)$ <u>**Laughlin state**</u> $V_e =: e^{i\sqrt{r}\varphi}:$

$$P = \prod_{i<j} (z_i - z_j)^r$$

- $\mathrm{SU}(2)_2$ <u>**Moore-Read state**</u> $V_e = \Psi : e^{i\varphi} :$

$$P = \mathrm{Pf}\frac{1}{z_i - z_j} \prod_{i<j} (z_i - z_j)$$

- $\mathrm{SU}(2)_k$ <u>**Read-Rezayi state**</u> $V_e = \Psi_1 : e^{i\sqrt{2/k}\varphi} :$

Quasi-hole wavefunction as a conformal block

$$P(z_1,\ldots,z_N|w_1,\ldots,w_M) = \langle V_e(z_1)\ldots V_e(z_N)V_{qh}(w_1)\ldots V_{qh}(w_M)\rangle$$

- $U(1)$ **Laughlin state** $V_{qh} =: e^{i1/\sqrt{r}\varphi}:$

$$P(z|w) = \prod_{i<j}(z_i - z_j)^r \prod_{i,l}(z_i - w_l) \prod_{l<m}(w_l - w_m)^{1/r}$$

- $\mathrm{SU}(2)_k$ **Read-Rezayi state** $V_e = \sigma : e^{i\sqrt{1/2k}\varphi}:$

$$\langle\Psi(z_1)\ldots\Psi(z_N)\sigma(w_1)\cdots\sigma(w_M)\rangle \prod_{i<j}(z_i - z_j)^{2/k} \prod_{i,l}(z_i - w_l)^{1/k}$$

Several conformal blocks $\Rightarrow$

- **degeneracy of quasi-hole states, and non-abelian braiding (!)**

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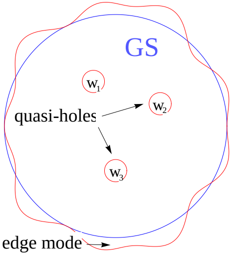
Bulk-edge correspondence

Operator Product Expansion (OPE) of electrons fields

$$V_e(z_1)\cdots V_e(z_N)|0\rangle \;=\; \sum_\lambda P_\lambda(z_1,\cdots,z_N)|\lambda\rangle$$

This is a mapping between :

- **edge states :** (descendant) states $|\lambda\rangle$ in the CFT Hilbert space
- **zero-energy bulk modes :** $P_\lambda = \langle\lambda|V_e(z_1)\cdots V_e(z_N)|0\rangle$



Quasi-hole wavefunctions describe the same space of zero-energy modes P_λ :

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What are the corresponding interactions between electrons ?

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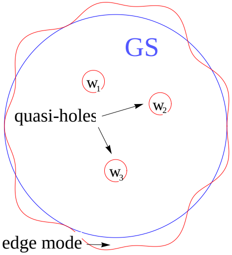

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Model interactions and clustering (for bosons)

Laughlin state $\nu = 1/2$

The ground state wavefunction

$$P = \prod_{i<j} (z_i - z_j)^2$$

is the densest zero energy mode of a repulsive δ interaction.

Moore-Read state $\nu = 1$

$$P = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)$$

3- body δ interaction.

Read-Rezayi state : $k+1$-body δ interaction

The zero energy modes P_λ are all polynomials that vanish when $k+1$ particles come to the same point.

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Jack states

Jack (symmetric) polynomials $J_\lambda^\alpha(z_1, \cdots, z_N)$

Indexed by a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$, and a real parameter α :

$$J_\lambda^\alpha = m_\lambda + \sum_{\mu<\lambda} u_{\lambda\mu}(\alpha) m_\mu .$$

Jacks are independent polynomials (at fixed α) !

Eigenvectors of the Calogero-Sutherland Hamiltonian

$$\mathcal{H}_{\mathrm{CS}}^{(\alpha)} = \sum_{i=1}^{N} \left(z_i \frac{\partial}{\partial z_i} \right)^2 + \frac{1}{\alpha} \sum_{i<j} \frac{z_i + z_j}{z_i - z_j} \left(z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j} \right)$$

- the CS model is integrable $\Rightarrow$ hierarchy of commuting Hamiltonians
- the CS Hamiltonian $\mathcal{H}_{\mathrm{CS}}$ has nothing to do with the physical FQHE Hamiltonian, it is merely a tool to characterize the Jacks.

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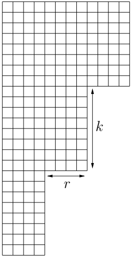
Generalized Pauli principle and clustering property

Clustering of Jacks at $\alpha = -\frac{(k+1)}{(r-1)}$ [Feigin et al (2001)]

For λ (k, r)-admissible, the Jack $J_\lambda^{(\alpha)}$ enjoys (k, r) **clustering :**
it vanishes with power r when $k+1$ particles come together !

$$(k, r)\text{-admissible partitions : } \lambda_i - \lambda_{i+k} \geq r$$



Generalized Pauli principle

At most k particles in r consecutive orbitals.

- Densest such partition is $[k \underbrace{0 \cdots 0}_{r-1} k \underbrace{0 \cdots 0}_{r-1} k \cdots]$.
- To each admissible partition λ corresponds a "zero-energy" mode $J_\lambda^{(\alpha)}$
- Describes a bosonic FQHE state at filling $\nu = k/r$.

Jack states and underlying CFT

Jack polynomials with (k, r)-admissible partitions provide trial wavefunctions for the FQHE.

Special cases include

- $k = 1$, r even : Laughlin
- $k = 2$, $r = 2$: Moore-Read
- $k > 2$, $r = 2$: Read-Rezayi

$\Rightarrow$ **Jacks give a unified description, and provide a natural extension to $k \neq 1$ and $r > 2$**].

Underlying CFT for the (k, r)-Jack state [Bernevig,Gurarie,Simon '09]

The underlying CFT is a $\mathcal{W}$-algebra based on $\mathrm{SU}(k)$, namely

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WA_{k-1} conformal field theory, basic facts

(2D) CFT : field theory invariant under $z \to f(z)$

The stress-energy tensor $T(z)$ encodes the reaction to metric deformations.

WA_{k-1} : extended conformal symmetry

$k-1$ chiral fields $\{T(z) = W^{(2)}(z), W^{(3)}(z), \cdots, W^{(k)}(z)\}$

- $\underline{k = 2 :}$ only $T(z)$, pure Virasoro algebra
- $\underline{k > 2 :}$ $\mathcal{W}_k$ algebra

Minimal models $\mathrm{WA}_{k-1}(p, p')$

Finite number of primary fields, unitary for $p' = p \pm 1$.

$$\mathrm{WA}_{k-1}(k+1, k+r) \qquad \text{non-unitary for } r > 2$$

$k = 2$ case (pure Virasoro), minimal model $M(3, 2 + r)$

Standard CFT result

Degeneracy (null-vector)
$\Rightarrow$ **differential equation for conformal blocks**

Link with Calogero-Sutherland [Cardy (2004)]

Using the level 2 degeneracy of $\Phi_{(1|2)}$

$$3L_{-1}^2\Phi_{(1|2)} = (r+2)L_{-2}\Phi_{(1|2)}$$

one recovers the Calogero-Sutherland Hamiltonian at coupling $\alpha = -\frac{2+1}{r-1}$.

$$\langle \Phi_{(1|2)}(z_1) \dots \Phi_{(1|2)}(z_N) \rangle \prod_{i<j} (z_i - z_j)^{r/2} = J^{-3/(r-1)}_{[20^{r-1}20^{r-1}\dots 2]}$$

A similar analysis holds for $\mathcal{W}_k$ theories [B.E. et R.Santachiara (2009)].

What is really going on in the CFT

emergence of the Calogero-Sutherland hierarchy

Including the $U(1)$ part

Enhanced algebra $Vir(g) \otimes U(1)$

It turns out to be essential to include the $U(1)$ part.

$$[a_n, a_m] = n\delta_{n+m,0}$$
$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}$$
$$[L_n, a_m] = 0$$

where g is a convenient parametrization of the central charge

$$c(g) = 1 - 6(g-1)^2/g = c(1/g)$$

The electron operator is

$$V_e(z) = \Phi_{(1|2)}(z) : e^{i\sqrt{g/2}\varphi(z)} : \qquad \in \mathrm{Vir}(g) \otimes \mathrm{U}(1)$$

Main result : a very nice basis $|\lambda\rangle$

The null-vector $\left(L_{-2} - gL_{-1}^2\right) V_e = 0$ induces an action of the Calogero-Sutherland Hamiltonian in the Hilbert space of the CFT.

A Calogero-Sutherland compatible basis of descendents

There exists a basis $|\lambda\rangle$ such that **all** functions P_λ in the OPE

$$V_e(z_1)\cdots V_e(z_N)|0\rangle \;=\; \sum_\lambda P_\lambda(z_1,\cdots,z_N)|\lambda\rangle$$

are eigenfunctions of the CS Hamiltonian with coupling g

$$H_2^{(g)} = \sum_i (z_i\partial_i)^2 + g(1-g)\sum_{i\neq j}\frac{z_i z_j}{z_{ij}^2}$$

(This is the same Hamiltonian as before, up to conjugation with a Jastrow factor)

CS action in the Hilbert space of $Vir(g) \otimes U(1)$

This basis $|\lambda\rangle = |\lambda\rangle_{g,\pm}$ is defined as the eigenbasis of

$$I_3^{(\pm)}(g) = 2(1-g)\sum_{m\geq 1} m a_{-m} a_m \pm \sqrt{2g}\sum_{m\geq 1}(a_{-m}L_m + L_{-m}a_m)$$
$$\pm\sqrt{\frac{g}{2}}\left(\sum_{m,k\geq 1} a_{-m-k}a_m a_k + a_{-m}a_{-k}a_{m+k}\right)$$

This is the CS Hamiltonian acting directly in the CFT Hilbert-space.

Some remarks

- I_3 is hermitian (while CS might not be)
- it clearly mixes $Vir(g)$ and $U(1)$
- the label $\pm$ refers to $\mathrm{U}(1)$ charge conjugation
- the basis is independant of the number N of vertex operators V_e

Some easy properties of I_3

The operator I_3 commutes with

$$I_1 = a_0 \qquad\qquad I_2 = \sum_{m>0} a_{-m} a_m + L_0$$

i.e. it preserves $U(1)$ charge and conformal dimension. $\Rightarrow$ I_3 **is block diagonal at every level of descendants.**

For instance at level 2 there are 3 descendants : $a_{-1}^2|0\rangle, a_{-2}|0\rangle$ and $L_{-2}|0\rangle$

$$I_3^{\pm}(g) = \begin{pmatrix} 4(1-g) & \pm\sqrt{2g} & 0 \\ \pm\sqrt{2g} & 8(1-g) & \pm\sqrt{2gc} \\ 0 & \pm\sqrt{2gc} & 0 \end{pmatrix}$$

$I_1, I_2, I_3 \cdots$ Are there any higher order integrals of motion ?

The Calogero-Sutherland model is integrable !

Higher order CS Hamiltonians ($\partial_i = \partial/\partial z_i$)

$$H_1 = \sum_i z_i \partial_i \qquad\qquad I_2$$

$$H_2^{(g)} = \sum_i (z_i\partial_i)^2 + g(1-g)\sum_{i\neq j}\frac{z_i z_j}{z_{ij}^2} \qquad\qquad I_3^{(g)}$$

$$H_3^{(g)} = \sum_i (z_i\partial_i)^3 + \frac{3}{2}g(1-g)\sum_{i\neq j}\frac{z_i z_j}{z_{ij}^2}(z_i\partial_i - z_j\partial_j) \qquad\qquad I_4^{(g)}$$

$$\dots \qquad\qquad \dots$$

(Conjecture) infinite hierarchy $I_n^{(g)}$, second quantized CS.

The same integrals of motion appear in the AGT conjecture [Alba, Fateev, Litvinov, Tarnopolsky (2011)].

What does it mean for FQHE states ?

Back to FQHE : consequences for Jack states

Upon specializing to the minimal model $M(3, r+2)$, the field $\Phi_{(1|2)}$ becomes single channeled :

$$\Phi_{(1|2)} \times \Phi_{(n|m)} = \Phi_{(n|3-m)} \qquad\qquad m = 1, 2$$

$\Rightarrow$ the functions $\langle\lambda|V_e(z_1)\cdots V_e(z_N)|0\rangle$ are simply polynomials.

Bulk-edge correspondence for Jack states

$$V_e(z_1)\cdots V_e(z_N)|0\rangle = \sum_\lambda J_\lambda(z_1, \cdots, z_N)|\lambda\rangle$$

where J_λ are Jacks, i.e. the zero-energy modes of the FQHE Jack state.

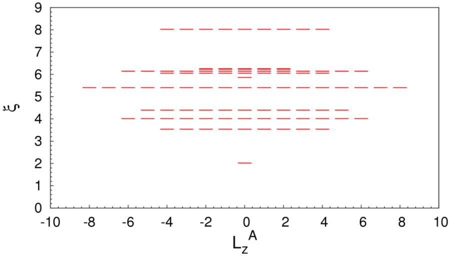
In the thermodynamic limit ($N \to \infty$) this is an explicit 1-to-1 mapping

- between bulk zero energy modes J_λ
- and states in the edge CFT $|\lambda\rangle$

Entanglement spectrum and Li-Haldane conjecture

Entanglement spectrum : spectrum of the reduced dentisy matrix after "cutting" the system in half.

$\Rightarrow$ counting of the edge modes (Li-Haldane conjecture).



Edge modes counting :

1	1	3	$\cdots$
$\|0\rangle$	$a_{-1}\|0\rangle$	$a_{-1}^2\|0\rangle$	
		$a_{-2}\|0\rangle$	
		$L_{-2}\|0\rangle$	

FIGURE: PES for the Moore-Read state, $N_A = N_B = 4$ particles (N.Regnault).

Particle entanglement spectrum

Cut the system into $N+M$ particles.

Expanding in the $|\lambda\rangle$ basis

$$V_e(w_1)\cdots V_e(w_M)|0\rangle = \sum_\lambda J_\lambda(w_1,\cdots,w_M)\,|\lambda\rangle_+$$

$$\langle 0|V_e(z_1)\cdots V_e(z_N) = \sum_\lambda J_{\lambda^c}(z_1,\cdots,z_N)\,{}_-\langle\lambda|$$

$$\langle V_e(z_1)\cdots V_e(z_N)V_e(w_1)\cdots V_e(w_M)\rangle = \sum_{\lambda,\mu} J_{\lambda^c}(z)A_{\lambda,\mu}J_\mu(w)$$

where $A_{\lambda,\mu} = {}_+\langle\lambda|\mu\rangle_-$ is a change of CFT bases $\Rightarrow$ rank saturated!

In the thermodynamic limit $(N,M\to\infty)$:

entanglement spectrum = edge (CFT) counting

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Jack quasi-hole wavefunctions

Quasi-hole operator

The FQHE wavefunction for the $(2, r)$ Jack state involving N electrons and M quasi-holes is the following conformal block

$$\langle\sigma(\omega_1)\cdots\sigma(\omega_M)\Psi(z_1)\cdots\Psi(z_N)\rangle\prod_{i<j}(z_i-z_j)^{\frac{r}{2}}\prod_{i,l}(z_i-\omega_l)^{\frac{1}{2}}$$

where $\sigma = \Phi_{(2|1)}$ is the quasi-hole field and $\Psi = \Phi_{(1|2)}$ the electron field.

Quasi-hole operator $\Phi_{(2|1)}$

σ has the same degeneracy as Ψ

$\Rightarrow$ Calogero-Sutherland acting on $\langle\sigma(w_1)\cdots\sigma(w_M)\rangle\prod_{i<j}(w_i-w_j)^{\frac{\tilde{r}}{k}}$.

Calogero-Sutherland at dual coupling $\tilde{\alpha} = 1-\alpha$.

Quasi-hole operator

The FQHE wavefunction for the $(2, r)$ Jack state involving N electrons and M quasi-holes is the following conformal block

$$\langle \sigma(\omega_1) \cdots \sigma(\omega_M) \Psi(z_1) \cdots \Psi(z_N) \rangle \prod_{i<j} (z_i - z_j)^{\frac{r}{2}} \prod_{i,l} (z_i - \omega_l)^{\frac{1}{2}}$$

where $\sigma = \Phi_{(2|1)}$ is the quasi-hole field and $\Psi = \Phi_{(1|2)}$ the electron field.

Quasi-hole operator $\Phi_{(2|1)}$

σ has the same degeneracy as Ψ
$\Rightarrow$ Calogero-Sutherland acting on $\langle \sigma(w_1) \cdots \sigma(w_M) \rangle \prod_{i<j} (w_i - w_j)^{\frac{\tilde{r}}{k}}$.

Calogero-Sutherland at dual coupling $\tilde{\alpha} = 1 - \alpha$.

Double Calogero-Sutherland structure

B.E., B.A.Bernevig et R.Santachiara, Phys. Rev. B82 (2010)

Jack wavefunction with both electrons and quasi-holes

$$F^{(a)}(w,z) \equiv \langle \sigma(w_1)\cdots\sigma(w_M)\Psi(z_1)\cdots\Psi(z_N)\rangle^{(a)} \times \mathrm{U}(1)$$

Very simple differential equation : double CS structure

$$\mathcal{H}_{\mathrm{CS}}^{(\alpha)}(z)F^{(a)}(w,z) = \mathcal{H}_{\mathrm{CS}}^{(\tilde{\alpha})}(\omega)F^{(a)}(w,z)$$

avec $\alpha = 1-\tilde{\alpha} = -\frac{k+1}{r-1}$.

The most generic quasi-hole wavefunction can be decomposed as

$$F^{(a)}(\omega,z) = \sum_\lambda f_\lambda^{(a)}(\omega)J_\lambda(z) \quad \text{where the } J_\lambda \text{ are } (k,r)\text{-admissible Jacks}$$

and the $f^{(a)}(w)$ are non-Abelian solutions of Calogero-Sutherland with dual coupling $\tilde{\alpha} = 1-\alpha$.

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Transformation $g \to 1/g$

Let us go back to $Vir(g) \times U(1)$:

$$V_e(z) = \Phi_{(1|2)}(z) : e^{i\sqrt{g/2}\varphi(z)} :$$
$$V_{qh}(w) = \Phi_{(2|1)}(w) : e^{i1/\sqrt{2g}\varphi(w)} :$$

Is there an extra (dual) hierarchy $I_n(1/g)$?

No, there is not !

$$I_3^{(+)}(g) = -gI_3^{(-)}(1/g)$$

Basically electrons and quasi-holes share the same CFT basis.

The bases for g and $1/g$ are $U(1)$ charge conjugate from one another

$$|\lambda\rangle_{g,+} = |\lambda\rangle_{1/g,-}$$

Quasi-hole wave-function

Inserting a complete basis between quasi-holes and electrons operators

$$\langle V_{qh}(w_1)\cdots V_{qh}(w_M)V_e(z_1)\cdots V_e(z_N)\rangle = \sum_\lambda f_\lambda^{(1/g)}(w)J_\lambda^{(g)}(z)$$

For Laughlin's wavefunction this is Gaudin's formula :

$$\prod_{i,j}(w_i - z_j) = \sum_\lambda J_{\lambda^t}^{(1/g)}(-1/w)J_\lambda^{(g)}(z)$$

A remark

$$\langle V_e(w_1)\cdots V_e(w_M)V_e(z_1)\cdots V_e(z_N)\rangle = \sum_{\mu,\lambda} J_{\mu^c}^{(g)}(w)A_{\mu,\lambda}J_\lambda^{(g)}(z)$$

Inserting qh or e$^-$ (PES) span the same space of zero-energy modes $J_\lambda(z)$.

An explicit calculation : one extra flux quantum

k quasi-holes wavefunction for the (k, r) Jack state

$$F(w_1, \ldots, w_k | z_1, \ldots, z_N) \propto \langle \sigma(w_1) \cdots \sigma(w_k) \Psi(z_1) \cdots \Psi(z_N) \rangle \times \mathrm{U}(1)$$

Using the double CS structure we know

$$F(w|z) = \sum_{\mu} a_\mu J_\mu^{\tilde{\alpha}}(w) J_{\lambda(\mu)}^{\alpha}(z)$$

with some mapping $\mu \to \lambda(\mu)$. Moreover translation invariance gives

$$a_\mu = \prod_{i=1}^{k} \left(- \frac{k+1-i}{i} \right)^{\mu_i}$$

which turns out to be quite simple (does not depend on N or even r).

Conclusion and perspectives

Conclusion

- Degeneracy of Ψ and $\sigma \Rightarrow$ Calogero-Sutherland Hamiltonian(s)
- Two CS hierarchies in $\mathcal{W}_k(g) \times U(1)$, related through $g \to 1/g$.
- same integrable structure as in AGT

Applications to FQHE Jack states (including Laughlin, MR and RR)

- we can build up the bulk-edge correspondence, with an integrable handle

$$V_e(w_1) \cdots V_e(w_M)|0\rangle = \sum_\lambda J_\lambda(w_1, \cdots, w_M)\,|\lambda\rangle$$

- proof that $\mathcal{W}_k(g) \times U(1)$ is the CFT underlying the Jack states
- formal duality between electrons and quasi-holes $\Rightarrow$ wavefunctions
- Particle entanglement spectrum $=$ edge state counting (bulk-edge correspondence)

Perspectives

- **FQHE :**
  - Entanglement spectrum and FQHE correlation functions
  - FQHE with internal degrees of freedom (spin,valley, etc) : Halperin and Nass states, partially symmetric Jack polynomial, link with $\mathrm{SU}(n)_k$ [B.E. and B.A.Bernevig arXiv :1107.2534]

Beyond Jacks

FQHE trial wave-functions coming from unitary parafermionic CFTs [B.E., N.Regnault et R.Santachiara, Nucl. Phys. B824 : 539-562(2010)]

  - Hamiltonians, quasi-hole braiding, edge states counting . . .unknown

- **Mathematical physics :**
  - better understanding of the higher integrals of motion I_n
  - going supersymmetric, q deformation . . .

Thank you for your attention !

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