Edge modes, zero modes and conserved charges in parafermion chains

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Where you may have heard of parafermions:

- They provide an elegant description of integrable $\mathbb{Z}_n$ invariant integrable lattice models (e.g. the critical 3-state Potts model)

  Fradkin-Kadanoff; Fateev-Zamolodchikov

- The critical points are described by conformal field theories with important properties.

  non-abelian part of SU(2)-invariant models, W-algebras

  Zamolodchikov-Fateev; Fateev-Lukyanov

- Correlators in these CFTs are used to construct the Read-Rezayi wavefunctions for the fractional quantum Hall effect.

CFTs and the FQHE both play a central role in the search for systems with topological order.

Systems with topological order in 2+1 dimensions typically have anyonic/fractionalized/spin-charge separated excitations.

These quasiparticles can even have non-abelian braiding. The braiding/fusing rules of the anyons are those of a 2d RCFT.

A familiar example is Chern-Simons theory.

Lattice models are fundamental to both condensed matter physics and to integrable systems.

Maybe it would be a good idea to go back and see if the original lattice parafermions of Fradkin and Kadanoff have something to do with topological order…

I'll describe interacting lattice models with edge modes that are not perturbations of free fermions. This leads to a $\mathbb{Z}_N$-invariant interacting generalization of the Kitaev honeycomb model.

Outline

- Edge/zero modes in the Majorana chain

- Edge/zero modes in the 3-state (chiral) Potts chain using parafermions

  an unusual form of integrability

- Coupling chains to make 2d $\mathbb{Z}_n$ gauge theories

  generalizing the Kitaev honeycomb model

How to fermionize the quantum Ising chain

$$H = \sum_j \left[f\sigma_j^x \;\; + J\sigma_j^z \sigma_{j+1}^z \right]$$

flip term

interaction

Critical point is when $J = f$, ordered phase is $J > f$.

$\mathbb{Z}_2$ symmetry operator is flipping all spins:

$$\prod_j \sigma_j^x$$

Jordan-Wigner transformation in terms of Majorana fermions

$$\psi_j = \sigma_j^z \prod_{i<j} \sigma_i^x , \qquad \chi_j = \sigma_j^y \prod_{i<j} \sigma_i^x$$

string

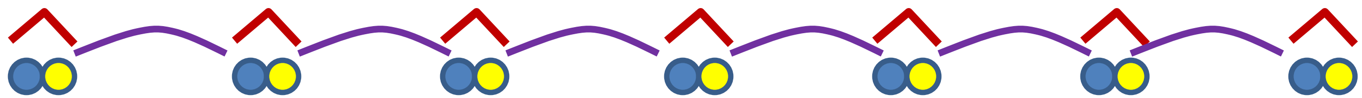
$$\{\psi_i , \psi_j\} = \{\chi_i , \chi_j\} = 2\delta_{ij}, \qquad \{\psi_i, \chi_j\} = 0$$

$\mathbb{Z}_2$ symmetry measures even or odd number of fermions:

$$(-1)^F = \prod_j \sigma_j^x = \prod_j (i\psi_j \chi_j)$$

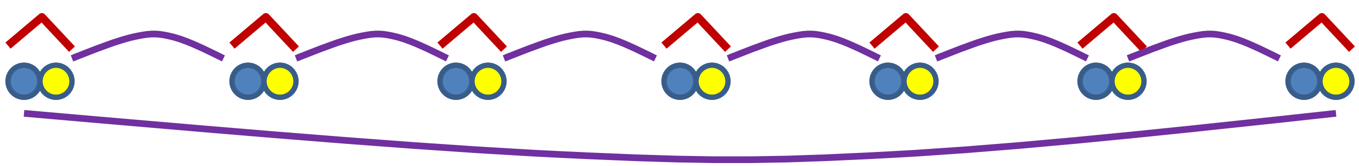
The Hamiltonian in terms of fermions

- with free boundary conditions:



$$H = if \sum_{j=1}^{N} \psi_j \chi_j + iJ \sum_{j=1}^{N-1} \chi_j \psi_{j+1}$$

- with periodic boundary conditions on the fermions:



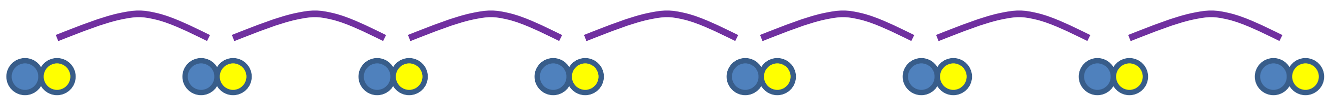
$$H = i \sum_{j=1}^{N} \left[f \psi_j \chi_j + J \chi_j \psi_{j+1} \right]$$

A catch: when written in terms of spins, this is twisted by $-(-1)^F$

Extreme limits:

- J=0 (disordered in spin language):

- f=0 (ordered in spin language):



The fermions on the edges, ψ_1 and χ_N, do not appear in H when f=0. They commute with H!

Gapless edge modes are a sign of topological order.

- When f=0, the operators χ_N and ψ_1 map one ground state to the other – they form an exact zero mode.

- The topological order persists for all $f<J$, even though for finite N, the two states split in energy.

- Can identify topological order (or lack thereof) for Hamiltonians of arbitrary fermion bilinears.

  Kitaev

What about periodic boundary conditions?

Simple way for 1d: can show it depends on $(-1)^F$ of ground state.

Even fancier way: compute sign of Pfaffian.

Heuristic way: find evidence that that the fermionic zero mode is still present. Can we find raising or lowering operator Ψ so that $[H, \Psi] = (\Delta E)\,\Psi$?

$$\Psi = \sum_j [\alpha_j \psi_j + \beta_j \chi_j]$$

Fermionic "zero" mode:

For free fermions, raising/lowering operators exist at any momentum. At k=0 and k=π,

$$\left[H, \sum_j (\psi_j \pm i\chi_j)(\pm 1)^j\right] = (\Delta E)\sum_j (\psi_j \pm i\chi_j)(\pm 1)^j$$

where $\Delta E = \pm f \pm J$. This is obvious here, but will be highly non-obvious for parafermions!

The 3-state (chiral) Potts model

The quantum chain version of the 3-state Potts model:

$$H = -\sum_j \left[f(\underset{\text{flip term}}{\tau_j + \tau_j^\dagger}) + J(\sigma_j^\dagger \sigma_{j+1} + \underset{\text{potential}}{\text{h.c.}}) \right]$$

$$\tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{-2\pi i/3} \end{pmatrix}$$

$$\tau^3 = \sigma^3 = 1, \quad \tau^2 = \tau^\dagger, \quad \sigma^2 = \sigma^\dagger$$

$$\boxed{\tau\sigma = e^{2\pi i/3}\sigma\tau}$$

Define parafermions just like fermions:

In a 2d classical theory, they're the product of order and disorder operators. In the quantum chain,

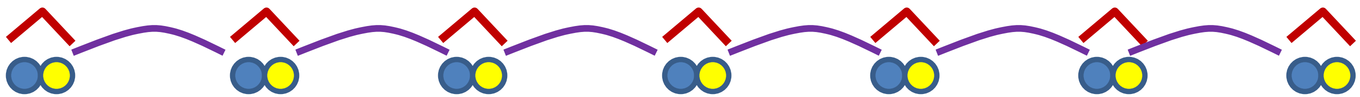
$$\psi_j = \sigma_j \prod_{i<j} \tau_i, \qquad \chi_j = \tau_j \sigma_j \prod_{i<j} \tau_i$$

$$\psi^3 = \chi^3 = 1, \quad \psi^2 = \psi^\dagger, \quad \chi^2 = \chi^\dagger$$

Instead of anticommutators, for $i < j$ and $\alpha = \chi$ or ψ,

$$\alpha_i \alpha_j = e^{2\pi i/3} \alpha_j \alpha_i$$

The Hamiltonian in terms of parafermions:

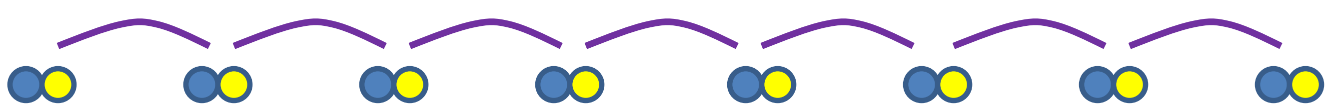


 $= f(\psi_j^\dagger \chi_j + \chi_j^\dagger \psi_j)$

 $= J(\psi_{j+1}^\dagger \chi_j + \chi_j^\dagger \psi_{j+1})$

Parafermions are not like free fermions – they cube to 1.
This isn't even integrable unless $J = f$.

However, when f=0, there are edge modes!



Does the zero mode remain for $J > f > 0$?

Take periodic boundary conditions on parafermions.
Can we find a Ψ so that $[H, \Psi] = (\Delta E)\,\Psi$?

$$\Psi = \sum_j \left[\alpha_j \psi_j + \beta_j \chi_j\right]$$

The answer is yes only if the couplings obey an interesting constraint!

Generalize to the chiral Potts model:

$$= f(e^{i\phi}\psi_j^\dagger\chi_j + e^{-i\phi}\chi_j^\dagger\psi_j) \qquad\qquad = J(e^{i\theta}\psi_{j+1}^\dagger\chi_j + e^{-i\theta}\chi_j^\dagger\psi_{j+1})$$

Then there is an exact "zero" mode Y if the couplings obey:

$$f\cos(3\phi) = J\cos(3\theta)$$

This is strong evidence that non-abelian topological order exists for all $f < J$ in this interacting system.

$$f\cos(3\phi) = J\cos(3\theta)$$

This calculation is the world's easiest way of finding the couplings of the integrable chiral Potts model.

Howes, Kadanoff and den Nijs; von Gehlen and Rittenberg; Albertini, McCoy, Perk and Tang; Baxter; Bazhanov and Stroganov

The integrable chiral Potts model is quite peculiar. The Boltzmann weights of the 2d classical analog are parametrized by higher genus Riemann surfaces instead of theta functions. They satisfy a generalized Yang-Baxter equation with no difference property.

The “superintegrable” line $\theta=\phi=\pi/6$ is very special. It is halfway between ferro and antiferromagnet, and so the spectrum is invariant under $H \rightarrow -H$.

Here the “zero” mode occurs for any value of f and J.

Along the superintegrable line the model a direct way of finding the infinite number of conserved charges is to use the Onsager algebra.

Dolan and Grady; von Gehlen and Rittenberg; Davies

This algebra can be rewritten in a more intuitive fashion.

A key observation is that the "zero" mode Ψ has definite U(1) "charge" (and dual "charge") as well.

For Ising, the transverse magnetization and its dual are

$$M = \sum_i \sigma_i^x \ , \qquad \widehat{M} = \sum_i \sigma_i^z \sigma_{i+1}^z$$

Acting with a sum of raising operators such as Ψ increases *M* by one.

The Hamiltonian is just the sum of these two charges:

$$H = fM + J\widehat{M}$$

These U(1) charges do not commute, so do not commute with *H*, and so are only conserved mod 2.

This however does suggest analyzing their algebra. This is what Onsager did!

What Onsager (effectively) did:

Decompose $\widehat{M}$ into sectors with distinct U(1) charges, i.e. those that conserve the transverse magnetization, those that violate it by +2, and those that violate it by -2:

$$\widehat{M} = B_1^0 + B_1^+ + B_1^-$$

This implies

$$[M, \widehat{M}] = B_1^+ - B_1^-$$

Now keep going.

Let

$$M = B_0^0, \quad B_0^+ = B_0^- = 0$$

and define a sequence of operators via

$$[B_1^-,\ B_n^+] = 2\left(B_{n+1}^0 - B_{n-1}^0\right) \ ,$$

$$\pm[B_1^0,\ B_n^\pm] = B_{n+1}^\pm - B_{n-1}^\pm$$

Now we commute these new operators with each other. The miracle is that we get nothing new!

The resulting Onsager algebra is remarkably beautiful:

$$[B^a_m, B^a_n] = 0; \qquad a = 0, \pm,$$

$$[B^+_m,\ B^-_n] = 2\left(B^0_{n+m} - B^0_{n-m}\right)$$

$$\pm[B^\pm_m,\ B^0_n] = B^\pm_{n+m} - B^\pm_{n-m}$$

Using this makes it easy to find the infinite number of conserved charges commuting with the Hamiltonian

$$H = \alpha B^0_0 + \beta B^0_1 + \gamma(B^+_1 + B^-_1)$$

There's an easy explanation of the miracle.

Or better said, another miracle explains this one:

The Onsager algebra is simply that of (zero-momentum) fermion bilinears!

Define the usual complex fermions $c_j = \psi_j + i\chi_j$ so

$$\{c_i^\dagger, c_j\} = 2\delta_{ij}; \qquad \{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0$$

Then the zero-momentum fermion bilinears

$$B_m^+ = \sum_j c_j^\dagger c_{j+m}^\dagger; \qquad B_m^- = (B_m^+)^\dagger$$

$$B_m^0 = \sum_j \left(c_j^\dagger c_{j+m} - c_j c_{j+m}^\dagger \right)$$

satisfy the Onsager algebra!

The miracle of the superintegrable chiral Potts model is that despite its not being a free-fermion theory, the algebra generated in the same way is the same Onsager algebra!

Certain combinations of parafermions behave identically to fermion bilinears.

Maybe there is a Pfaffian-ish formula to detect topological order here ?!?

To see this, work in a basis where τ is diagonal and σ is not. Then notice that

$$e^{-i\pi/6}\begin{pmatrix}1 & 0 & 0\\ 0 & \omega & 0\\ 0 & 0 & \omega^{-1}\end{pmatrix}+e^{i\pi/6}\begin{pmatrix}1 & 0 & 0\\ 0 & \omega & 0\\ 0 & 0 & \omega^{-1}\end{pmatrix}=2\cos(\pi/6)\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$

The superintegrable chiral Potts is thus of the form

$$H = fM + J\widehat{M}$$

where M is now the magnetization of a spin-1 particle!

The interaction term can also be rewritten in terms of the usual spin-1 matrices, e.g.

$$\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = S^+ + (S^-)^2$$

Then split it into terms that conserve the U(1) symmetry generated by *M* and those violating it by +3 or -3:

$$\sum_j (\sigma_j^\dagger \sigma_{j+1} e^{i\pi/6} + h.c.) \equiv B_1^0 + B_1^+ + B_1^-$$

terms such as $S_i^+ (S^+)^2_{i+1} e^{i\pi/6}$

The resulting generators then satisfy the same Onsager algebra!

More interesting stuff happens. The U(1)-invariant $\gamma = 0$ case can be solved via the standard Bethe ansatz, with the Bethe equations related to those of the XXZ chain at $J_z/J_x = 1/2$, where the ground state has remarkable combinatorial properties studied by many Parisiens.

This XXZ chain has a hidden supersymmetry, where the supersymmetry generator obeys $Q^2 = 0$.

Fendley, Nienhuis and Schoutens; Hagendorf and Fendley

Here, we have $\Psi^3 = 0$!

Topological order in 2d

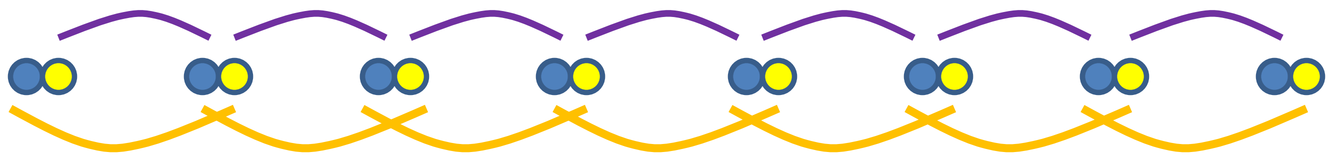
The Ising/Majorana chain has an elegant generalization to 2d via the Kitaev honeycomb model.

This is a spin model that can be mapped to free fermions coupled to a background gauge field.

I'll describe the analog for parafermions.

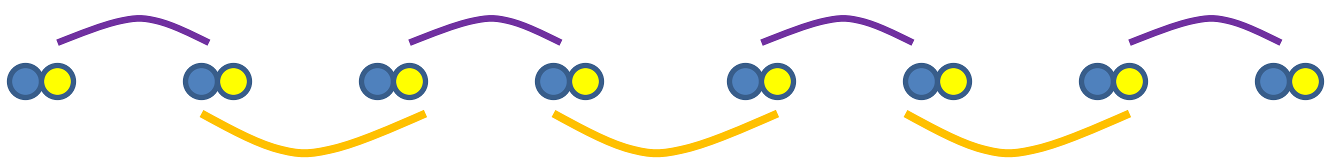
View the 2d model as coupled 1d chains

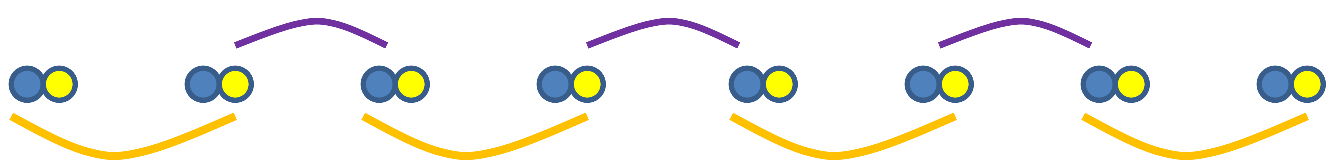
The quantum YZ chain



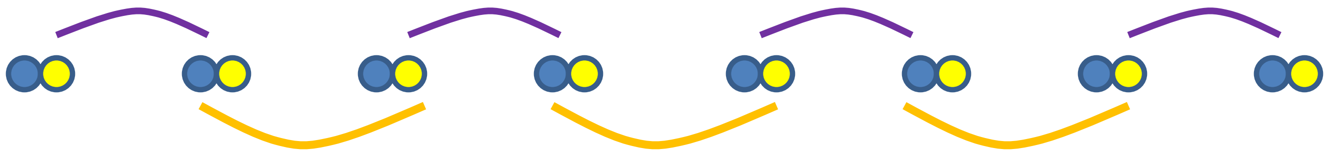
 $= iJ_z \sigma_i^z \sigma_{i+1}^z$  $\equiv iJ_y \sigma_i^y \sigma_{i+1}^y$

is comprised of two commuting Hamiltonians:



\+ 

Consider one of these Hamiltonians:

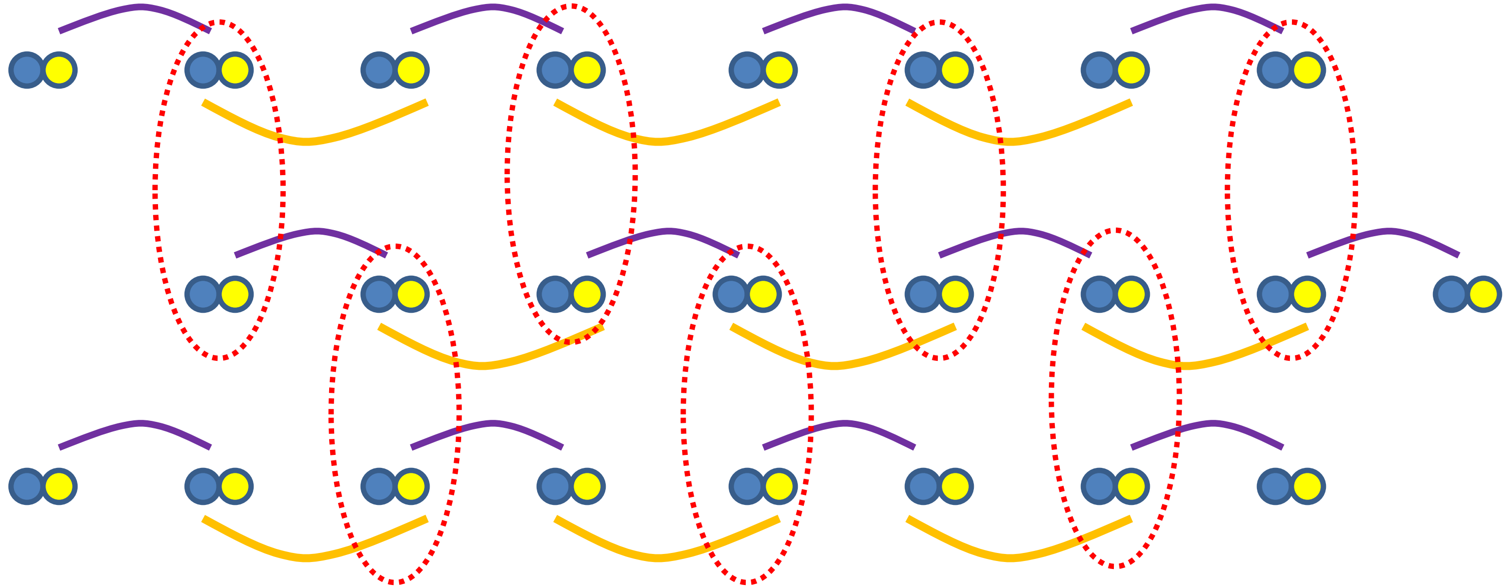


$$H^{(1)} = \sum_j \left[\sigma^z_{2j-1}\sigma^z_{2j} + \sigma^y_{2j}\sigma^y_{2j+1}\right] = i\sum_j \left[\chi_{2j-1}\psi_{2j} + \psi_{2j}\chi_{2j+1}\right]$$

Just like the edge modes, the fermions ψ_{2j-1} and χ_{2j} do not appear!

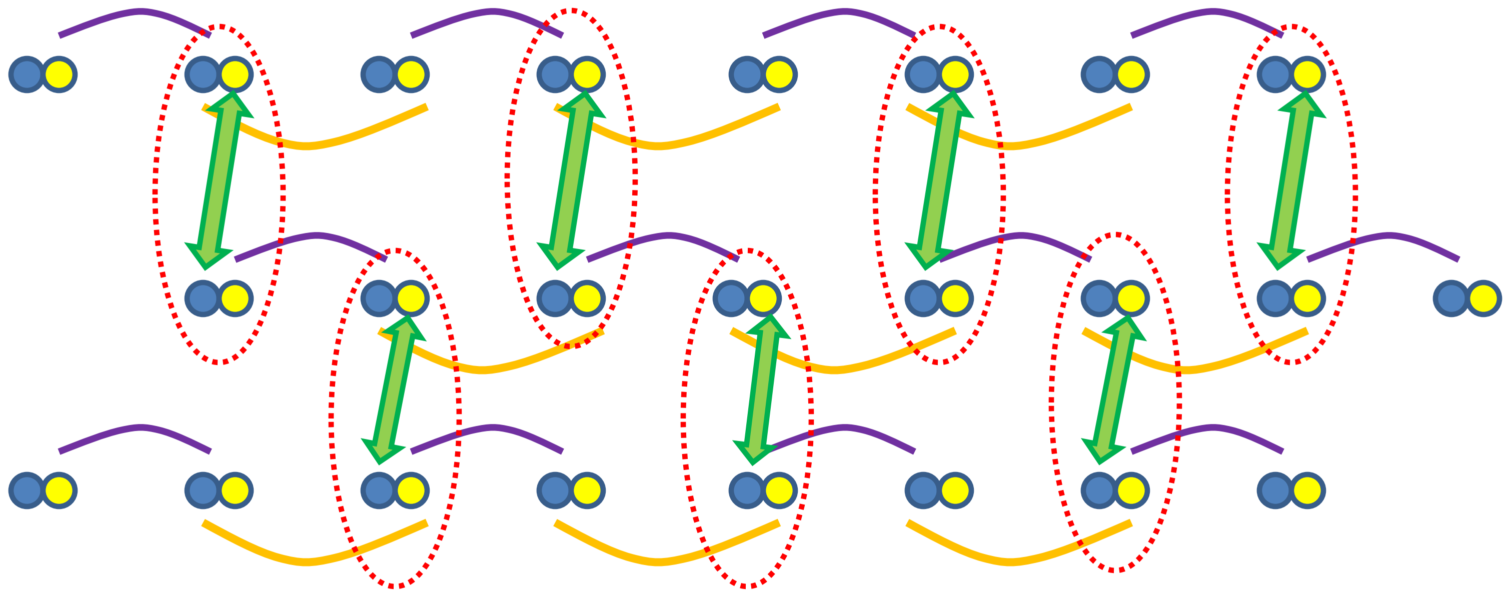
They commute with each individual term in $H^{(1)}$.

Now couple chains together into a honeycomb lattice:



$$H = \sum \psi\psi\chi\chi \; + \; \sum \psi\chi \; + \; \sum \chi\psi$$

$$H = \sum \sigma^x\sigma^x \; + \; \sum \sigma^y\sigma^y \; + \; \sum \sigma^z\sigma^z$$



Each fermion bilinear commutes with each term in H.

The $\mathbb{Z}_2$ gauge flux is the product around a hexagon. $= \sigma^z \sigma^x \sigma^y \sigma^z \sigma^y \sigma^x$

The flux through each plaquette can be chosen individually, and is not dynamical.

Thus the Kitaev honeycomb model is simply free fermions coupled to a background $\mathbb{Z}_2$ gauge field.

A magnetic field destroys the solvability, but causes non-abelian topological order.

On the Fisher lattice, non-abelian topological order occurs without the magnetic field.

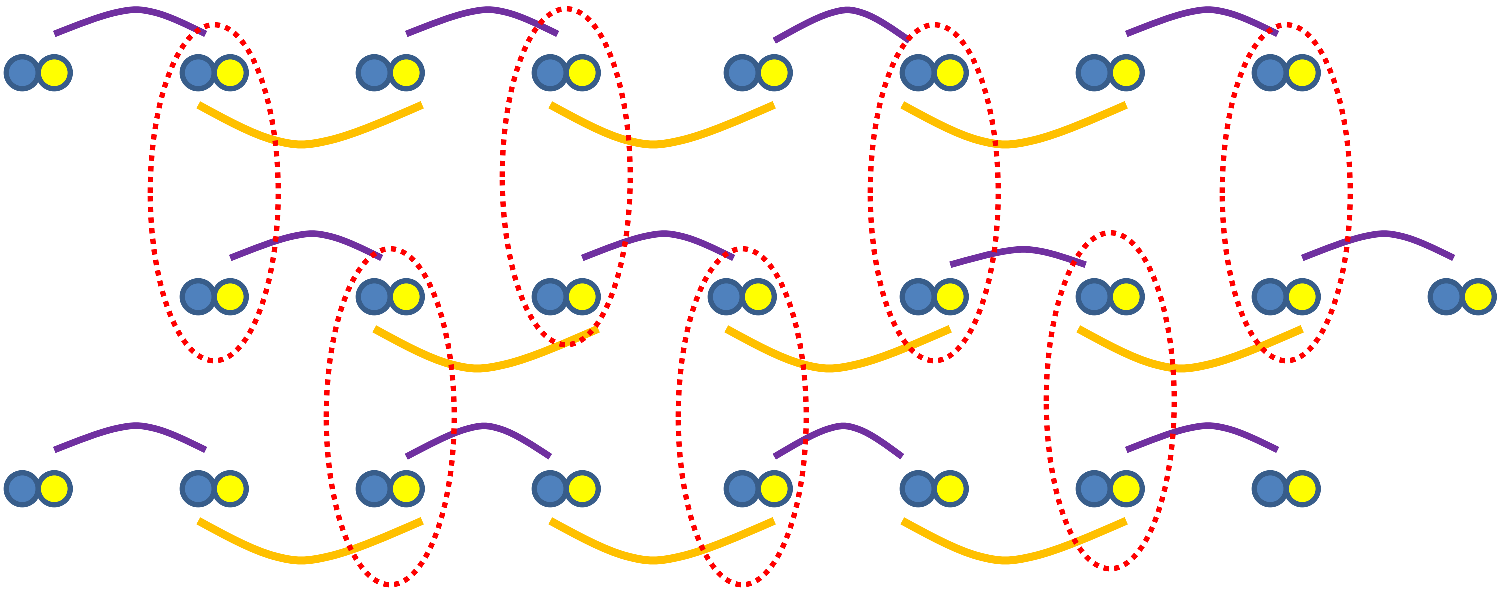
Yao and Kivelson

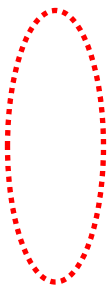
So what about parafermions?

The same trick yields a "YZ" Hamiltonian that doesn't involve half the parafermions:

 $= (\tau_i)^\dagger \tau_{i+1} + \text{h.c.}$

 $= (\tau_i \sigma_i)^\dagger \tau_{i+1} \sigma_{i+1} + \text{h.c.}$



 $= \sigma_k^\dagger \sigma_{k+1} + \text{h.c.}$

$$\tau^\dagger \quad \sigma \quad (\tau\sigma)$$

The $\mathbb{Z}_3$ gauge flux is = around a hexagon.

$$(\tau^\dagger \sigma^\dagger) \quad \sigma^\dagger \quad \tau$$

Questions

- The handwaving arguments for topological order work for parafermions. Presumably non-abelian?
- Is there a formula for the parafermions generalizing the Pfaffian/Chern number for fermions?
- Is there a connection to 2+1d integrable models?
- Should work for all $\mathbb{Z}_N$, what about $U(1)$?
- What's with the Onsager algebra?