Minimal Model Holography

Matthias Gaberdiel
ETH Zürich

IHP Paris, 4 October 2011

based on work with Rajesh Gopakumar, arXiv:1011.2986
Tom Hartman, arXiv:1101.291
Rajesh Gopakumar, Tom Hartman &
Suvrat Raju, arXive:1106.1897

Motivation

Some years ago Klebanov & Polyakov proposed concrete version of AdS/CFT duality for a weakly coupled field theory

higher spin theory on AdS4 $\longleftrightarrow$ 3d O(N) vector model in large N limit

Recently: impressive checks of this duality. [Giombi & Yin] [Jevicki et al]

Motivation

Aim: try to find 3d/2d CFT version of such a duality.

This would be interesting since

- 2d CFTs well understood
- Higher spin theories simpler in 3d

  [Truncation to finitely many spins possible;
  separate scalar multiplets; ...]

The proposal

Our concrete proposal is as follows: [MRG,Gopakumar]

AdS3:
higher spin theory
with two complex
scalars of mass M

$\longleftrightarrow$

2d CFT:
$\mathcal{W}_N$ minimal models
in large N 't Hooft limit
with coupling λ

where $M^2 = -(1-\lambda^2)$

Comparison to KP

In contrast to Klebanov-Polykov:

1 parameter family of dual theories.

Special values:

- For $\lambda = 0$ the 2d CFT should be analogue of singlet sector of free fermion theory.
- For $\lambda = 1$ the resulting theory has linear $\mathcal{W}_\infty$ symmetry (free bosons).

Consistency checks

The proposal passes a number of non-trivial consistency checks:

- ✓ Symmetries agree

  [Henneaux & Rey]
  [Fredenhagen et al]
  [MRG,Gopakumar,Saha]
  [MRG & Hartman]

- ✓ Spectra agree

  [MRG,Gopakumar]
  [MRG, Gopakumar, Hartman, Raju]

- ✓ RG flows of 2d CFT reproduced from AdS

  [MRG,Gopakumar]

Overview

- The AdS description
- Matching with the CFT
- Checks of the proposal
- Conclusions

Higher spin theory on AdS3

Pure gravity in AdS3: Chern-Simons theory based on

$$sl(2,\mathbb{R})$$

[Achucarro & Townsend]
[Witten]

Higher spin description: replace

$$sl(2,\mathbb{R}) \to \mathrm{hs}[\mu]$$

[Vasiliev]

Higher spin algebra

Higher spin algebra can be defined as follows: consider associative algebra

$$B[\mu] = \frac{U(sl(2))}{\langle C_2 - \mu \mathbf{1} \rangle}$$

[Bordemann et.al.]
[Bergshoeff et.al.]
[Pope, Romans, Shen]

On this vector space then define Lie algebra with Lie brackets given by commutators; as vector space

$$B[\mu] = \mathrm{hs}[\mu] \oplus \mathbb{C} \ .$$

Higher spin algebra

Generators of $\mathrm{hs}[\mu]$:

$$V^s_n \ \text{ with } |n| < s \ , \ \ s = 2, 3, \ldots$$

`wedge algebra'

Commutation relations can be easily determined explicitly, e.g.

$$\begin{aligned} [V^3_2, V^3_1] &= 2\, V^4_3 \\ [V^3_2, V^3_{-1}] &= 6\, V^4_1 + \tfrac{1}{5}(3-4\mu)\, V^2_1 \end{aligned} \qquad \begin{aligned} [V^3_2, V^3_0] &= 4\, V^4_2 \\ [V^3_2, V^3_{-2}] &= 8\, V^4_0 + \tfrac{4}{5}(3-4\mu)\, V^2_0 \ . \end{aligned}$$

Truncation

Standard parametrisation of Casimir of sl(2) - write

$$\mu = h(h-1) = \tfrac{1}{4}(\lambda^2 - 1) \ .$$

Special cases:

- For $\mu = \frac{1}{4}(N^2-1), i.e.\ \lambda = N$: truncation to

$$\mathrm{sl}(N) = \mathrm{hs}[\tfrac{1}{4}(N^2-1)]/\chi_N$$

ideal of invariant form

Asymptotic symmetries

For these higher spin theories asymptotic symmetry group can be determined following Brown & Henneaux, leading to classical

$$\mathcal{W}_\infty[\mu] \ \text{algebra}$$

[Henneaux & Rey]
[Fredenhagen et al]
[MRG, Hartman]

Extends algebra `beyond the wedge':

$$\begin{array}{lccc} \text{pure gravity:} & \mathrm{sl}(2) & \longrightarrow & \text{Virasoro} \\ \text{higher spin:} & \mathrm{hs}[\mu] & \longrightarrow & \mathcal{W}_\infty[\mu]\ . \end{array}$$

[Figueroa-O'Farrill et.al.]

W-algebra

Resulting algebra generated by V_n^s , but now there is no restriction on n any longer.

In the generic case, the resulting W-algebra is non-linear, except for

[MRG,Hartman]

$$\mu = 0 \leftrightarrow \lambda = 1 \ .$$

Then we have

$$\mathcal{W}_\infty[0] = \mathcal{W}_\infty$$

linear W-algebra of Pope, Romans & Shen.

Special cases

Henneaux-Rey: analysis for

$$\mu = -\frac{3}{16} \ , \ \text{i.e.} \ \lambda = \frac{1}{2} \ .$$

For this value

$$\text{hs}[-\tfrac{3}{16}] = \text{hs}(1,1) \ .$$

(original higher spin theory of Vasiliev)

Fredenhagen et.al.: analysis for sl(N) and formal large N limit.

Overview

- The AdS description
- Matching the CFT
- Checks of the proposal
- Conclusions

Dual CFT

Dual CFT should therefore have

$$\mathcal{W}_\infty[\lambda] \equiv \mathcal{W}_\infty[\mu = \tfrac{1}{4}(\lambda^2 - 1)] \text{ symmetry.}$$

Basic idea:

$$\mathcal{W}_\infty[\lambda] \; = \; \lim_{N\to\infty} \mathcal{W}_N \qquad \text{with} \qquad \lambda = \frac{N}{N+k} \; .$$

't Hooft limit of 2d CFT!

The 2d CFTs

$\mathcal{W}_N$ minimal models are cosets

$$\frac{\mathfrak{su}(N)_k \oplus \mathfrak{su}(N)_1}{\mathfrak{su}(N)_{k+1}}$$

[Bais et al]

with central charge

$$c_N(k) = (N-1)\left[1 - \frac{N(N+1)}{(N+k)(N+k+1)}\right] .$$

Symmetry algebra generated by fields of spin=2,3,...,N.

[For N=2: Virasoro minimal models (c<1).]

Level rank duality

The relation between 't Hooft limit and $\mathcal{W}_\infty[\lambda]$ is some sort of level rank duality

[Kuniba, Nakanishi, Suzuki]
[Bauer, Altschuler, Saleur]

$$\frac{\mathfrak{su}(N)_k \oplus \mathfrak{su}(N)_1}{\mathfrak{su}(N)_{k+1}} \cong \frac{\mathfrak{su}(M)_l \oplus \mathfrak{su}(M)_1}{\mathfrak{su}(M)_{l+1}}$$

where

$$\mathrm{hs}[\lambda] \equiv \mathrm{sl}(\lambda)$$

$$M = \frac{N}{N+k} \equiv \lambda \,, \qquad N = \frac{M}{M+l}$$

$$\mathcal{W}_\infty[\lambda]$$

Level rank duality

We have tested this relation extensively, and it seems to be true. In particular, we have compared

- Eigenvalues of generators
- Characters

and they match perfectly.

[MRG,Hartman]
[MRG, Gopakumar, Hartman, Raju]

Symmetries

So the symmetries suggest that we should have

HS on AdS3

=CS with

$\mathrm{hs}[\lambda]$

$\longleftrightarrow$

2d CFT with

$\mathcal{W}_\infty[\lambda]$

symmetry

= 't Hooft limit of minimal models

Spectrum

However, higher spin fields themselves correspond only to the vacuum representation of the W-algebra!

Indeed, the 1-loop determinant of the spin s field on thermal AdS3 equals

[MRG, Gopakumar, Saha]

$$Z^{(s)} = \prod_{n=s}^{\infty} \frac{1}{|1-q^n|^2} \ .$$

[Generalisation of Giombi, Maloney & Yin calculation to higher spin, using techniques developed in David, MRG, Gopakumar.]

1-loop partition function

The complete higher spin theory therefore contributes

$$Z_{\mathrm{hs}} = \prod_{s=2}^{\infty} \prod_{n=s}^{\infty} \frac{1}{|1-q^n|^2} \ .$$

This agrees precisely with vacuum character of generic vacuum representation!

--- not consistent by itself.....

Representations

Indeed, the full CFT also has the representations labelled by (from coset description)

$$(\rho, \mu; \nu)$$

rep of $\mathfrak{su}(N)_k$ $\mathfrak{su}(N)_1$ $\mathfrak{su}(N)_{k+1}$

Compatibility constraint: $\rho + \mu - \nu \in \Lambda_R(\mathfrak{su}(N))$

fixes μ uniquely.

Representations

Indeed, the full CFT also has the representations labelled by (from coset description)

$$(\rho;\nu)$$

Conformal dimension:

$$h(\rho;\nu)=\frac{C_N(\rho)}{N+k}+\frac{C_N(\mu)}{N+1}-\frac{C_N(\nu)}{N+k+1}+n\ .$$

\`height'

C_N : quadratic Casimir

Simple representations

For example, the simplest representations are (in ‘t Hooft limit)

$$h(0;\mathrm{f}) = \frac{1}{2}(1-\lambda) = h(0;\bar{\mathrm{f}})$$

fundamental anti-fundamental

$$h(\mathrm{f};0) = \frac{1}{2}(1+\lambda) = h(\bar{\mathrm{f}};0)$$

These four representations generate all W-algebra representations upon taking fusion products.

Full theory

The full theory should therefore have partition function

$$Z_{\rm CFT} = \sum_{(\rho;\nu)} \left| \chi_{(\rho;\nu)} \right|^2 \ .$$

sum over all reps

CFT-character
of corresponding
representation

whereas

$$Z_{\rm hs} = \prod_{s=2}^{\infty} \prod_{n=s}^{\infty} \frac{1}{|1-q^n|^2} = \left| \chi_{(0;0)} \right|^2 \ .$$

Proposal

[MRG,Gopakumar]

Contribution from all remaining characters is accounted for by adding to the hs theory two complex scalar fields (i.e. 2 scalar matter multiplets) of the same mass

$$-1 \leq M^2 \leq 0 \quad \text{with} \quad M^2 = -(1-\lambda^2) \ .$$

Compatible with hs symmetry since theory based on $\mathrm{hs}[\mu]$ has massive scalar multiplet with mass

$$M^2 = 4\mu = 4h(h-1) = (\lambda^2 - 1) \ .$$

[Vasiliev]

Two quantisations

In terms of dual CFT mass is related to conformal dim.

$$M^2 = \Delta(\Delta - 2)\ .$$

For masses in above window, there are two quantisations

[Klebanov & Witten]

$$\Delta_\pm = 1 \pm \lambda\ .$$

One multiplet will be quantised in (+) quantisation,
the other in (-) quantisation.

Multiplets

Multiplet with (+) quantisation has

$$h = \bar{h} = \tfrac{1}{2}(1+\lambda) \ .$$

This agrees with the conformal dimension of the representations

$$(\mathrm{f};0) \quad \text{and} \quad (\bar{\mathrm{f}};0)$$

in ‘t Hooft limit.

Multiplets

Similarly, multiplet with (-) quantisation has

$$h = \bar{h} = \tfrac{1}{2}(1-\lambda) \ .$$

This agrees with the conformal dimension of the representations

$$(0;\mathrm{f}) \quad \text{and} \quad (0;\bar{\mathrm{f}})$$

in ‘t Hooft limit.

Overview

- The AdS description
- Matching the CFT
- Checks of the proposal
- Conclusions

1-loop computation

Main evidence from 1-loop calculation:

Contribution of single scalar to 1-loop determinant is

[Giombi, Maloney & Yin]

$$Z^{(1)}_{\rm scalar} = \prod_{l=0,l'=0}^{\infty} \frac{1}{(1 - q^{h+l}\bar{q}^{h+l'})} ,$$

where, depending on the quantisation,

$$h = h_{\pm} = \frac{1}{2}\Delta_{\pm} = \frac{1}{2}(1 \pm \lambda) .$$

Total 1-loop partition function

The total 1-loop partition function of our AdS theory is then

$$Z^{(1)}_{\rm total} = \prod_{s=2}^{\infty}\prod_{n=s}^{\infty}\frac{1}{|1-q^n|^2} \times \prod_{l_1=0,l_1'=0}^{\infty}\frac{1}{(1-q^{h_-+l_1}\bar{q}^{h_-+l_1'})^2} \times \prod_{l_2=0,l_2'=0}^{\infty}\frac{1}{(1-q^{h_++l_2}\bar{q}^{h_++l_2'})^2}\ .$$

Claim: this agrees exactly with CFT partition function in ‘t Hooft limit!

[MRG,Gopakumar]
[MRG,Gopakumar, Hartman, Raju]

Total 1-loop partition function

Basic idea:

$$Z^{(1)}_{\text{total}} = \prod_{s=2}^{\infty}\prod_{n=s}^{\infty}\frac{1}{|1-q^n|^2} \times \prod_{l_1=0,l_1'=0}^{\infty}\frac{1}{(1-q^{h_-+l_1}\bar{q}^{h_-+l_1'})^2} \times \prod_{l_2=0,l_2'=0}^{\infty}\frac{1}{(1-q^{h_++l_2}\bar{q}^{h_++l_2'})^2} \ .$$

$\mathcal{W}_N$ modes

$$(0;\mathrm{f})^{\otimes s_1} \otimes (0;\bar{\mathrm{f}})^{\otimes s_2} \qquad (\mathrm{f};0)^{\otimes r_1} \otimes (\bar{\mathrm{f}};0)^{\otimes r_2}$$

every rep is generated by these!

Lowest orders

For single scalar first non-trivial terms (including higher spin mode contributions) are

$$
\begin{aligned}
Z^{(1)} &= q^h \bar{q}^h \left(1 + q + 2q^2 + 4q^3 + \cdots\right) \left(1 + \bar{q} + 2\bar{q}^2 + 4\bar{q}^3 + \cdots\right) \\
&\quad + q^{2h} \bar{q}^{2h} \left(1 + q + 3q^2 + \cdots\right) \left(1 + \bar{q} + 3\bar{q}^2 + \cdots\right) \\
&\quad + q^{2h+1} \bar{q}^{2h+1} \left(1 + q + \cdots\right) \left(1 + \bar{q} + \cdots\right) + \cdots \ .
\end{aligned}
$$

This should be of the form

$$
Z^{(1)} = \chi_{h_1}(q)\chi_{h_1}(\bar{q}) + \chi_{h_2}(q)\chi_{h_2}(\bar{q}) + \chi_{h_3}(q)\chi_{h_3}(\bar{q}) + \cdots
$$

characters of $\mathcal{W}_N$ reps

Lowest orders

For $h = h_- = \frac{1}{2}(1-\lambda)$ we get

[MRG,Gopakumar]

$$
\begin{aligned}
\chi_{h_1}(q) &= q^{h_-}\Big(1 + q + 2q^2 + 4q^3 + \cdots\Big) && = \chi(0;\mathrm{f}) \\
\chi_{h_2}(q) &= q^{2h_-}\Big(1 + q + 3q^2 + \cdots\Big) && = \chi(0;[0,1,0^{N-3}]) \\
\chi_{h_3}(q) &= q^{2h_-+1}\Big(1 + q + \cdots\Big) && = \chi(0;[2,0^{N-2}]) \ ,
\end{aligned}
$$

calculated from first principles in CFT!

\`single' particle: (0;f)

\`two particle' states:

$$(0;\mathrm{f}) \otimes (0;\mathrm{f}) = (0;[0,1,0^{N-3}]) \oplus (0;[2,0^{N-2}])$$

Higher terms

Subsequently we have found an analytic proof that this identity holds to arbitrary order. More specifically, by rewriting both sides in terms of U(N) characters, we have shown that

$$Z_{\rm hs} \cdot \prod_{l,l'=0}^{\infty} \frac{1}{(1-q^{h_-+l}\bar{q}^{h_-+l'})} = \sum_{\nu} |\chi_{(0;\nu)}|^2 \ .$$

contribution from one scalar

all reps with finitely many boxes in Young tableaux

[MRG, Gopakumar, Hartman, Raju]

Partition function

The analysis works similarly for the other three scalars, e.g.

$$Z_{\mathrm{hs}} \cdot \prod_{l,l'=0}^{\infty} \frac{1}{(1 - q^{h_+ + l} \bar{q}^{h_+ + l'})} = \sum_{\nu} |\chi_{(\nu;0)}|^2 \ .$$

Young tableaux with finitely many boxes/anti-boxes

Partition function

and thus one would expect that we can put everything together as

$$Z_{\rm hs} \cdot \prod_{l,l'=0}^{\infty} \frac{1}{(1-q^{h_+ + l}\bar{q}^{h_+ + l'})^2 (1-q^{h_- + l}\bar{q}^{h_- + l'})^2}$$

$$= \sum_{(R_1;R_2)} \left|\chi_{(R_1;R_2)}\right|^2 .$$

finitely many boxes and anti-boxes

However, this is not quite true....

Subtlety

A problem arises for example for the representation (f;f) for which conformal dimension of primary equals

$$h(\mathrm{f};\mathrm{f}) \quad = \quad \frac{N^2-1}{2N(k+N)(k+N+1)} = \mathcal{O}(\tfrac{1}{N})$$

$$\neq \quad h(\mathrm{f};0) + h(0;\mathrm{f}) = h_+ + h_- = 1 \ .$$

this is what gravity calculation would suggest
[Conformal dimension of multiparticle states add.]

Subtlety

More specifically, the ‘t Hooft limit of the CFT character equals

$$\chi_{(\mathrm{f};\mathrm{f})} = \left(1 + \frac{q}{(1-q)^2}\right) \prod_{s=2}^{\infty} \prod_{n=s}^{\infty} \frac{1}{(1-q^n)}$$

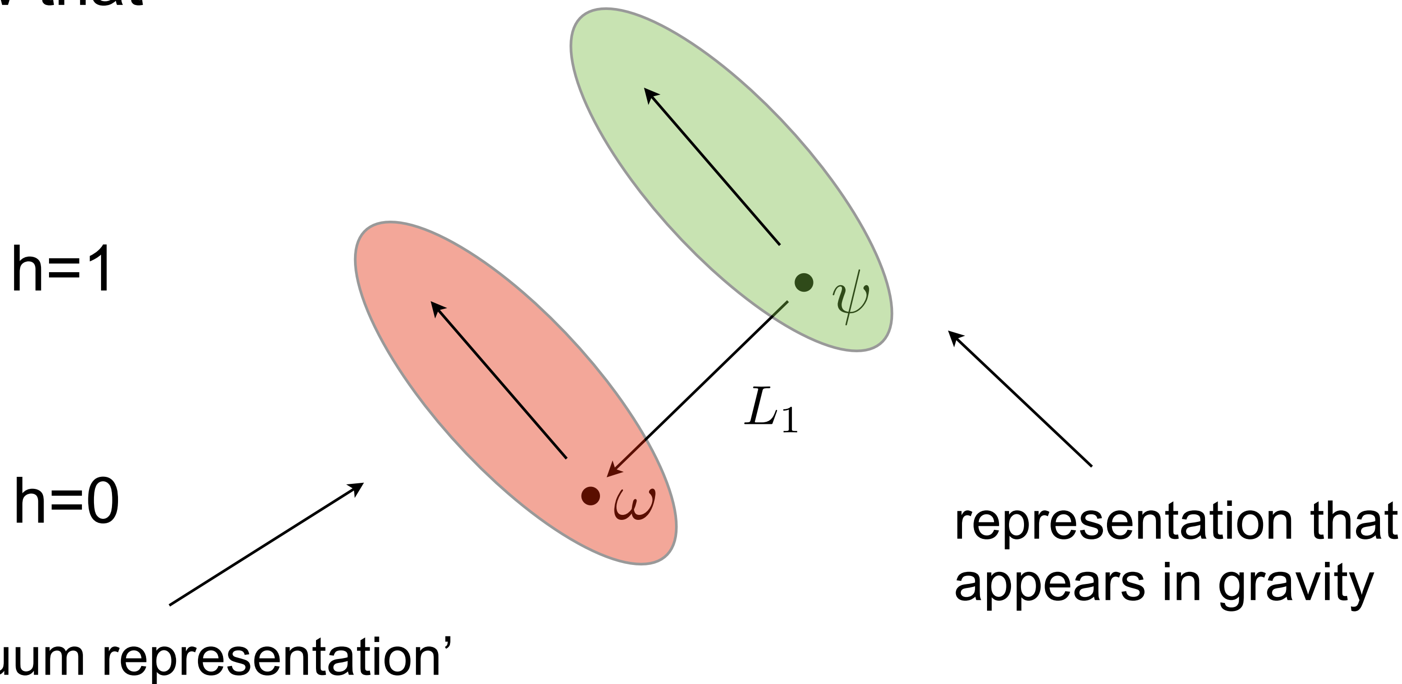
$$= \chi_{(0;0)} + \frac{q^1}{(1-q)^2} \prod_{s=2}^{\infty} \prod_{n=s}^{\infty} \frac{1}{(1-q^n)}$$

vacuum character

only this part is seen by gravity

Resolution

Actual structure of (f;f) representation is more complicated: by writing it as fusion of (f;0) with (0;f) can show that



[MRG, Gopakumar, Hartman, Raju]

Decoupling

The `vacuum representation' is then null, i.e. it decouples in amplitudes:

$$\begin{aligned} \langle \omega | \omega \rangle &= \langle \omega | L_1 \psi \rangle \\ &= \langle L_{-1} \omega | \psi \rangle = 0 \end{aligned}$$

since $L_{-1}\omega = 0$

Thus the actual CFT representation is the one generated from ψ --- perfect agreement with gravity!

Generalisation

We have also tested this for some of the other cases, where we have `cancelling boxes or anti-boxes'.

Each time the same decoupling phenomenon appears. Assuming that this continues to hold for all such representations, we conclude that

$$Z_{\text{gravity}} = Z_{\text{CFT}} \ .$$

strong consistency check!

RG flow

As second consistency condition consider the RG flow in the CFT

$$k \longrightarrow k\text{-}1 \text{ model.}$$

In ‘t Hooft limit RG flow changes coupling by

$$\delta\lambda = \frac{\lambda^2}{N} \ .$$

This leads to a finite change in conformal charge

$$\delta c = -2\lambda^3 \ ,$$

which should be `visible’ in AdS theory.

RG flow

The perturbing field is the double trace operator corresponding to

$$: \mathcal{O}\mathcal{O}^\dagger : \;\longleftrightarrow\; (0;\mathrm{adj}) = (0;\mathrm{f}) \otimes (0;\bar{\mathrm{f}})$$

$$(0;\mathrm{f}) \longleftrightarrow \mathcal{O} \qquad (0;\bar{\mathrm{f}}) \longleftrightarrow \mathcal{O}^\dagger$$

From CFT we know that under this RG flow

$$\begin{array}{ccccc} \mathcal{O} & & (0;\mathrm{f})_k \xrightarrow{\text{RG-flow by } :\mathcal{O}\mathcal{O}^\dagger:} (\mathrm{f};0)_{k-1} & & \mathcal{O}' \\ & \leftrightarrow & & \leftrightarrow & \\ \mathcal{O}^\dagger & & (0;\bar{\mathrm{f}})_k \xrightarrow{\text{RG-flow by } :\mathcal{O}\mathcal{O}^\dagger:} (\bar{\mathrm{f}};0)_{k-1} & & \mathcal{O}'^\dagger \end{array}$$

RG flow

As is familiar from AdS/CFT, perturbation by double trace operator changes boundary condition in bulk.

[Witten]
[Berkooz et.al.]

At the end-point of the RG flow, the fields are quantised in the (+)-way, i.e.

$$(-) \quad \begin{array}{ccc} \mathcal{O} & \xrightarrow{\text{RG-flow}} & \mathcal{O}' \\ \mathcal{O}^\dagger & \xrightarrow{\text{RG-flow}} & \mathcal{O}'^\dagger \end{array} \quad (+)$$

Matches with RG analysis in 2d CFT! [MRG,Gopakumar]

Overview

- The AdS description
- Matching the CFT
- Checks of the proposal
- Conclusions

Conclusions

Given strong evidence for duality between

AdS3:
higher spin theory
with two complex
scalars of mass M

$\longleftrightarrow$

2d CFT:
$\mathcal{W}_N$ minimal models
in large N ‘t Hooft limit
with coupling λ

where $M^2 = -(1-\lambda^2)$

Conclusions

Interesting features of the correspondence:

- Non-trivial non-supersymmetric example
- allows for detailed precision tests
- generalises to other cases, e.g. so(2N)

[Ahn], [MRG, Vollenweider]

Future directions

- Corrections at finite N & k
- Matching of correlation functions [Papadodimas, Raju]
- Supersymmetric version
- Free field theory description of $\lambda = 0$ case

work in progress with
P. Suchanek

closely related to
$c \to 1$ limit of
minimal models

[Runkel, Watts]
[Roggenkamp, Wendland]
[Fredenhagen, Schomerus]