Periodic spin-chains, affine Temperley-Lieb, and bulk Logarithmic CFT

AM Gainutdinov

Institute de Physique Théorique, CEA-Saclay

in collaboration with H. Saleur, N. Read, R. Vasseur, and J. Jacobsen

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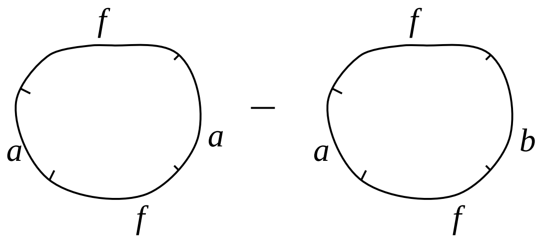
Some physical motivation

Why we need logarithmic models of CFT?

2D percolation

Cardy, 1992



Crossing probability $P = Z_{aa} - Z_{ab}$ of percolation cluster formation between two boundaries (horizontal crossing)

$$Z_{aa} \sim \langle \phi_{f|a}(z_1)\phi_{a|f}(z_2)\phi_{f|a}(z_3)\phi_{a|f}(z_4)\rangle$$

$$Z_{ab} \sim \langle \phi_{f|a}(z_1)\phi_{a|f}(z_2)\phi_{f|b}(z_3)\phi_{b|f}(z_4)\rangle$$

2D percolation and field $\phi_{1,2}(z)$

Cardy, 1992

Crossing probability $P = Z_{aa} - Z_{ab}$ of percolation

$$Z_{aa} \;\sim\; \langle \phi_{f|a}(z_1)\phi_{a|f}(z_2)\phi_{f|a}(z_3)\phi_{a|f}(z_4)\rangle$$

$$Z_{ab} \;\sim\; \langle \phi_{f|a}(z_1)\phi_{a|f}(z_2)\phi_{f|b}(z_3)\phi_{b|f}(z_4)\rangle$$

Conformal dimension of the boundary field $\phi_{f|a}(z)$ should be zero:

$$L_0\,\phi_{f|a}(z) = 0 \quad \& \quad c = 0 \qquad \longrightarrow \qquad \phi_{f|a}(z) = \phi_{1,2}(z)$$

$$(L_{-2} - \tfrac{3}{2}L_{-1}^2)\phi_{1,2}(z) = 0 \qquad \longrightarrow \qquad \text{Cardy formula}$$

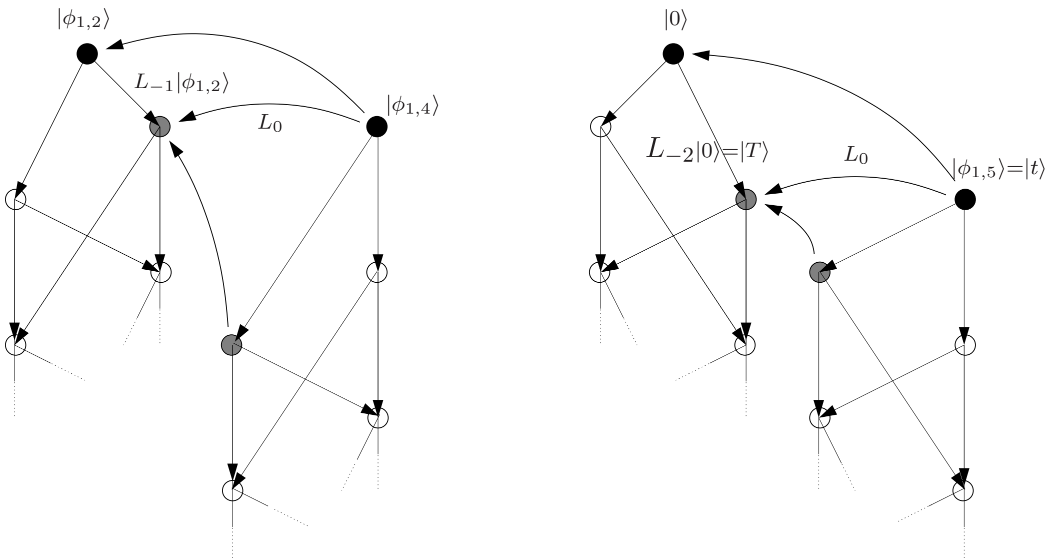
2D percolation and Staggered modules

Mathieu–Ridout, 2007

Boundary fields $\phi_{1,2}(z)$ + operator algebra to be closed

$\Longrightarrow$ appearance of nontrivial Jordan cells in spectrum of the hamiltonian L_0

characterized by indecomposibility parameter b: $\langle t|T\rangle = b \;\; (= -5/8)$



In a log CFT model, we encounter (at least) two **problems**:

(1) The space of states (indefinite inner-product space) is decomposed into complicated **indecomposables** over Virasoro but their structure is not known apriori – a problem in constructing even a consistent chiral theory.

(2) a problem in combining chiral and antichiral parts to construct the full space of states of a local theory (**non-chiral** theory) in order to describe, say, 2D percolation on a torus.

It might be better to begin studying logarithmic behaviour on a finite **lattice** model where algebraic part is under better control and to get then some intuition for the **continuum** (LCFT) in the scaling limit.

Logarithmic lattice models

There are few different approaches:

- 1+1D (super-symmetric) spin-chain models =

  = non-degenerate indef. inner product spaces   Read–Saleur 2001

- 2D (integrable) loop models   Pearce–Rasmussen–Zuber 2006,

  Dubail–Jacobsen–Saleur 2006-2009

Both approaches show presence of Jordan cells for
the hamiltonian $\boldsymbol{H} = \sum_j \boldsymbol{e}_j$ and
are based on "hamiltonian densities", $\boldsymbol{e}_j$, algebra
— the Temperley–Lieb (TL) algebra —
(the hamiltonian densities are representations of TL algebra)

Logarithmic lattice models

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Morally,

- the lattice models are discretizations of LCFTs and
- TL algebra gives a regularization of the energy–momentum tensor $T(z)$:

  its modes L_n are obtained in a scaling limit from the hamiltonian densities e_j keeping "higher" hamiltonians $H(n) = \sum_j \exp(i\pi nj/N)\boldsymbol{e_j}$ —

  — due to somehow underestimated old result of Koo–Saleur, 1994

Logarithmic lattice models (SUSY spin-chains)

We consider $gl(1|1)$ and $sl(2|1)$ SUSY spin-chains

with open and closed boundary conditions (b.c.)

- open b.c. give in the scaling limit

  chiral (or boundary) LCFTs with $c = -2$ and $c = 0$, resp.

- closed b.c. give in the scaling limit

  non-chiral (or bulk) LCFTs with $c = -2$ and $c = 0$, resp.

The space of states is the tensor product space $\mathcal{H} = (V \otimes V^*)^{\otimes L}$ of $N = 2L$ tensorands labelled $j = 1, \ldots, 2L$ with the fundamental representation $V = \mathbb{C}^{1|1}$ for $gl(1|1)$ and $V = \mathbb{C}^{2|1}$ for $sl(2|1)$ on even sites and the dual V^* on odd sites.

Nearest-neighbour interaction is given by e_j's – projectors on the $gl(1|1)$- or $sl(2|1)$-invariant in the product $V \otimes V^*$ of two neighbour tensorands.

Logarithmic lattice models (open SUSY spin-chain)

- The open $gl(1|1)$ spin-chain has a free fermion representation based on operators f_j and $f_j^\dagger$ acting non-trivially only on jth tensorand and obeying

$$\{f_j, f_{j'}\} = 0, \quad \{f_j, f_{j'}^\dagger\} = (-1)^j \delta_{jj'},$$

where the '$-$' sign for an odd j is due to the dual representations of $gl(1|1)$.

- Nearest-neighbour interaction is then

$$e_j = (f_j + f_{j+1})(f_j^\dagger + f_{j+1}^\dagger), \qquad 1 \leqslant j \leqslant 2L-1,$$

- The critical hamiltonian $H = \sum_{j=1}^{2L-1} e_j$ is hermitian but acts on an indefinite inner product space $\mathcal{H} = (V \otimes V^*)^{\otimes L}$ because of the sign factor.

Logarithmic lattice models (open SUSY spin-chain)

Nearest-neighbour interaction e_j for $sl(2|1)$ spin-chains is quartic in bosonic and fermionic operators 'sitting' at sites j and $j+1$.

- in both cases, they satisfy TL algebra $TL_N(m)$ relations:

$$e_j^2 = m e_j, \qquad e_j e_{j\pm 1} e_j = e_j,$$
$$e_j e_k = e_k e_j, \qquad (j \neq k,\ k \pm 1),$$

- with $m = 0$ for $gl(1|1)$ and with $m = 1$ for $sl(2|1)$ SUSY spin-shains –

  – the algebra is **non semi-simple**

- These open chains provide a **faithful** representation of $TL_N(m)$.

$\longrightarrow$ How to get a decomposition or (indecomposable) module structure over $TL_N(m)$?

Logarithmic lattice models (SUSY spin-chains) Read–Saleur, 2007

SUSY spin-chain approach uses an important concept

— **the full symmetry algebra** Z_{TL} —

the centralizer of the "hamiltonian densities" algebra TL (the centralizer is a largest algebra that commutes with TL, i.e. technically is defined as $\mathrm{End}_{TL}(\mathcal{H})$

In the open $gl(1|1)$ spin-chain, Z_{TL} is generated by the identity and

$$F_{(1)} = \sum_{1 \leqslant j \leqslant N} f_j, \qquad F^{\dagger}_{(1)} = \sum_{1 \leqslant j \leqslant N} f^{\dagger}_j,$$

$$F_{(2)} = \sum_{1 \leqslant j < j' \leqslant N} f_j f_{j'}, \qquad F^{\dagger}_{(2)} = \sum_{1 \leqslant j < j' \leqslant N} f^{\dagger}_{j'} f^{\dagger}_j,$$

$$S^z = \sum_{1 \leqslant j \leqslant N} (-1)^j f^{\dagger}_j f_j - L.$$

Logarithmic lattice models (the centralizer for $gl(1|1)$ spin-chain)

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$$S^z = \sum_{1 \leqslant j \leqslant N} (-1)^j f^{\dagger}_j f_j - L.$$

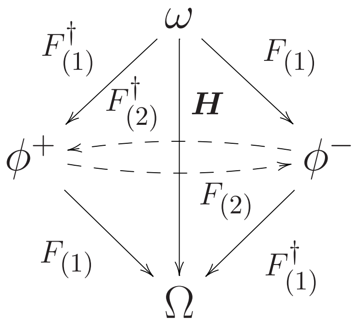
Why this algebra generates the full symmetry algebra?

- Note that the formulas give a representation of the quantum group $U_{\mathfrak{q}} s\ell(2)$ with $\mathfrak{q} = i$. The fermionic generators $F_{(1)}$ and $F^{\dagger}_{(1)}$ are from the nilpotent part and the bosonic ones form the $s\ell(2)$ subalgebra in $U_{\mathfrak{q}} s\ell(2)$.

Logarithmic behaviour of the hamiltonian H and XX spin-chain

- Jordan–Wigner transformation gives an isomorphism

between the open $gl(1|1)$ and XX spin-chains and between Z_{TL} and $U_i s\ell(2)$



$$\mathfrak{q}^{S^z} S^{\pm} \mathfrak{q}^{-S^z} = \mathfrak{q} S^{\pm},$$
$$[S^+, S^-] = \frac{\mathfrak{q}^{S^z} - \mathfrak{q}^{-S^z}}{\mathfrak{q} - \mathfrak{q}^{-1}},$$
$$F^{\dagger}_{(1)} = S^+, \quad F_{(1)} = S^-,$$
$$F^{\dagger}_{(2)} = \frac{(S^+)^2}{[2]!}, \quad F_{(2)} = \frac{(S^-)^2}{[2]!}.$$

the vacuum Ω and the state ω form a 2-dim Jordan cell of the lowest eigenvalue for $\boldsymbol{H}$

Logarithmic lattice models (the centralizer) Read–Saleur, 2007

We have to exploit the symmetry algebra.

• Decomposition over the full symmetry algebra Z_{TL} is usually easier to study than for the "hamiltonian densities" algebra $TL_N(m)$.

Our strategy is then:

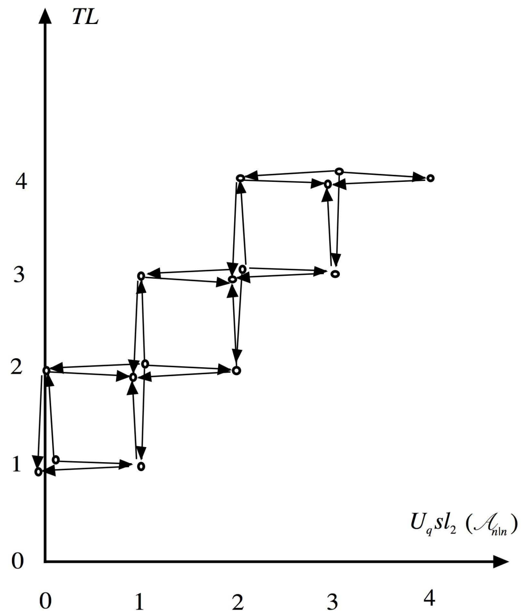
(1) to start with a decomposition of spin-chains over Z_{TL} on indecomposable direct summands which are technically **tilting** modules;

(2) then, studying all homomorphisms (intertw. operators) between the tilting modules gives module structure over the algebra TL (a direct sum of its projective modules).

(3) multiplicities in front of tilting Z_{TL}-modules give dimensions of simple TL-modules.

Read–Saleur, 2007

Open case for the $gl(1|1)$ model (or XX model) — **finite chain with** $N = 8$



TL_8 acts in the vertical direction,

$U_i s\ell(2)$ acts in the horizontal way.

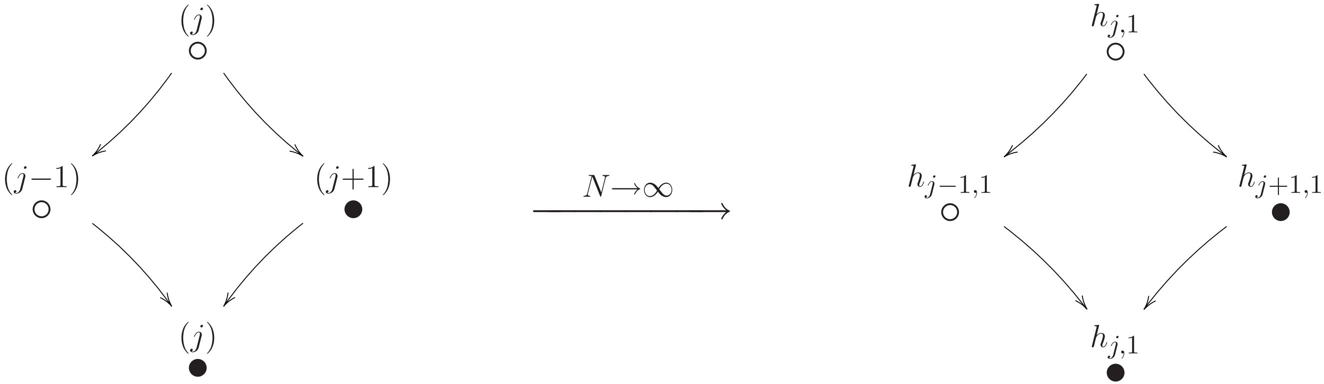
The full space of states is a **bimodule** over $TL_N(0) \otimes U_i s\ell(2)$

- the spin-states are organized into indecomposables for $TL_N(0)$ (or $U_i s\ell(2)$)
  - with nondiag. action of the hamiltonian H (or the Casimir operator)

What is going on with such a structure when the scaling limit is taken?

Read–Saleur, 2007

Projective TL-modules and staggered Vir-modules

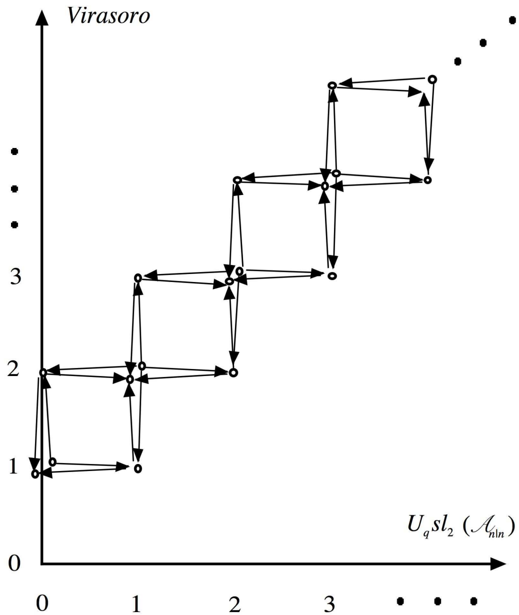


TL-projectives go over to

staggered modules for chiral Virasoro

Read–Saleur, 2007

Open case for the $gl(1|1)$ model ($\mathfrak{q} = i$, $c = -2$) — **scaling limit** $N \to \infty$

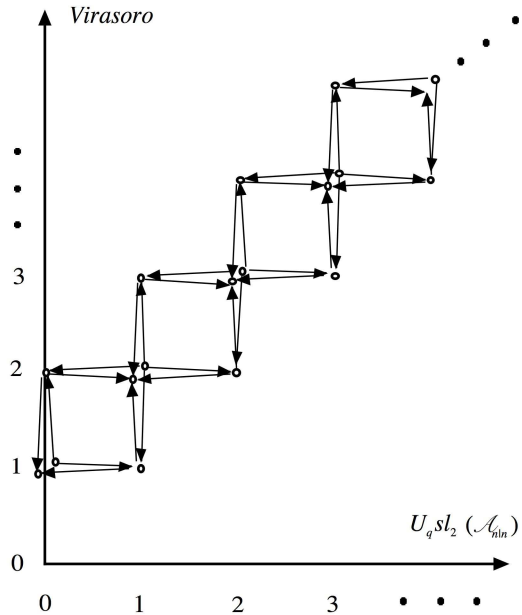


Symplectic fermions theory with $c = -2$

$$S = \int d^2 z J_{\alpha\beta} \partial_\mu \Phi^\alpha \partial^\mu \Phi^\beta$$

Read–Saleur, 2007

Open case for the $gl(1|1)$ model ($\mathfrak{q} = i$, $c = -2$) — **scaling limit** $N \to \infty$

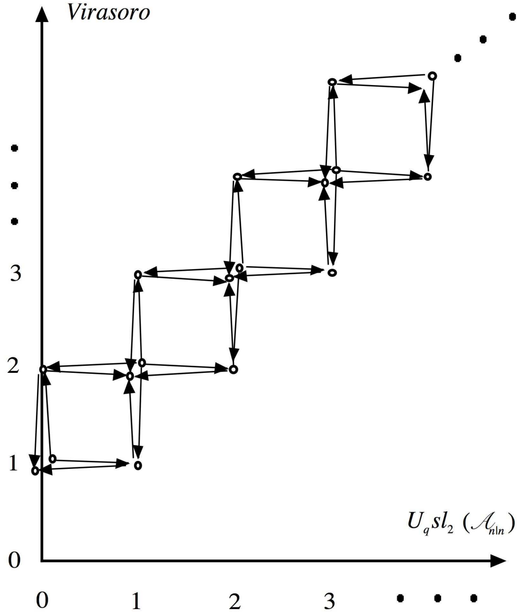


Virasoro $\mathrm{Vir}_{1,2}$ acts in vertical direction,

$U_i s\ell(2)$ acts in the horizontal way.

Read–Saleur, 2007

Open case for the $gl(1|1)$ model ($\mathfrak{q} = i$, $c = -2$) — **scaling limit** $N \to \infty$

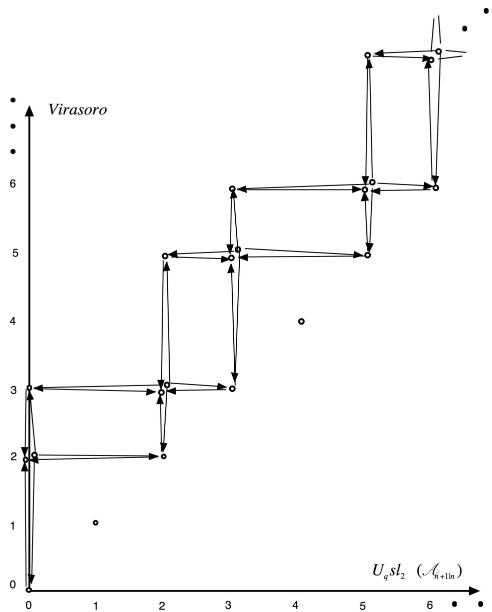


Screening construction gives the **centralizer** for chiral Virasoro

$[\mathrm{Vir}_{1,p}, U_{\mathfrak{q}}s\ell(2)] = 0$ (BFGT, 2009).

Open case for the $sl(2|1)$ model ($\mathfrak{q} = e^{i\pi/3}$, $c = 0$) — **scaling limit** $N \to \infty$

- The full symmetry algebra $Z_{TL(1)}$ is Morita-equivalent to $U_{\mathfrak{q}}s\ell(2)$ (the same module structure but different dimensions for simples)

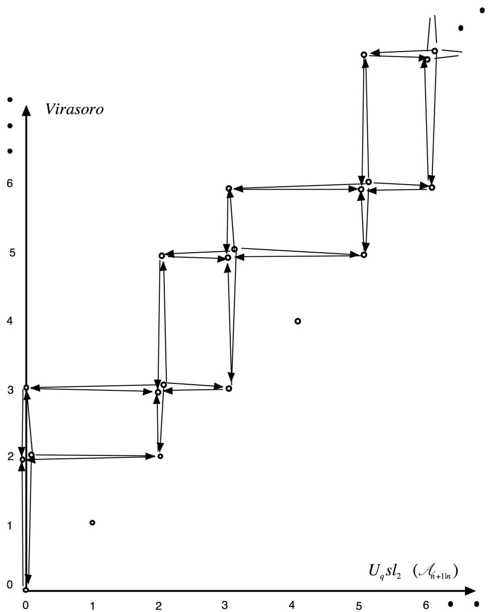


Virasoro $\text{Vir}_{2,3}$ acts in vertical direction,

$Z_{TL(1)}$ acts in the horizontal way.

Open case for the $sl(2|1)$ model ($\mathfrak{q} = e^{i\pi/3}$, $c = 0$) — **scaling limit** $N \to \infty$

- In a paper of Vasseur–Jacobsen–Saleur-2011, the indecomposibility parameter $\langle T|t\rangle = b$ was measured in the $sl(2|1)$ chain using the Koo–Saleur formula with the result $b = -5/8$, as expected for boundary 2D percolation.



Virasoro $Vir_{2,3}$ acts in vertical direction, $Z_{TL(1)}$ acts in the horizontal way.

How to extend this approach for description of **bulk** LCFTs?

How to extend this approach for description of **bulk** LCFTs?

Consider **periodic** $gl(1|1)$ and $sl(2|1)$ spin-chains and their scaling limit.

The periodic $gl(1|1)$ spin-chain

The closed (periodic) spin-chain is obtained simply by adding a coupling between the sites with $j = 2L$ and $j = 1$, that is by adding a generator

$$e_{2L} = (f_{2L} + f_1)(f_{2L}^\dagger + f_1^\dagger),$$

which corresponds to the periodic boundary condition $f_{2L+1}^{(\dagger)} = f_1^{(\dagger)}$ on the fermions.

The critical hamiltonian for our model is then expressed as

$$H = \sum_{j=1}^{2L} e_j$$

We have now a translation symmetry for H given by operator $u : j \to j + 2$.

The periodic $gl(1|1)$ model and affine TL

The set of $2L$ generators $e_j, j = 1, \ldots, 2L$, with the translation operator u, satisfy affine TL algebra $\widehat{TL}_N(m)$ relations:

$$e_j^2 = m e_j, \qquad e_j e_{j\pm1} e_j = e_j$$

$$e_j e_k = e_k e_j, \qquad (j \neq k,\ k \pm 1),$$

$$u e_j u^{-1} = e_{j+2}$$

(with $m = 0$) where now the indices have to be interpreted **cyclically**.

- The periodic gl$(1|1)$ spin chain provides a **non-faithful** representation of the affine TL algbera $\widehat{TL}_N(0) \equiv \widehat{TL}_N$.

The periodic $gl(1|1)$ and a twisted closed XX-model

The $gl(1|1)$ model is equivalent to a twisted XX spin chain. The expression of the TL generators in this case is well known

$$e_j^{XX} = \frac{1}{2}\left[\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + i(\sigma_j^z - \sigma_{j+1}^z)\right], \qquad 1 \leqslant j \leqslant N,$$

with twisted boundary conditions

$$\sigma_{N+1}^{\pm} = -(-1)^{S^z}\sigma_1^{\pm}, \qquad S^z = \frac{1}{2}\sum_{j=1}^{N}\sigma_j^z$$

Then

$$e_j = i(-1)^j e_j^{XX}$$

The periodic $gl(1|1)$ model and the centralizer $Z_{\widehat{TL}(0)}$ GRS, 2011

While the $gl(1|1)$ symmetry remains, being generated by $F_{(1)}$, $F^{\dagger}_{(1)}$, and S^z, the 'bosonic' $s\ell(2)$ generators $F_{(2)}$ and $F^{\dagger}_{(2)}$ **do not commute** with the action of $\widehat{TL}_N(0)$. We have only an **odd** (or fermionic) subalgebra of $Z_{TL(0)}$ generates the centralizer of $\widehat{TL}_N(0)$, with $n \geqslant 0$,

$$\begin{aligned}
F_{(2n+1)} &= \sum_{\substack{1 \leqslant j_1 < j_2 < \dots \\ \dots < j_{2n+1} \leqslant 2L}} f_{j_1} f_{j_2} \dots f_{j_{2n+1}}, \\
F^{\dagger}_{(2n+1)} &= \sum_{\substack{1 \leqslant j_1 < j_2 < \dots \\ \dots < j_{2n+1} \leqslant 2L}} f^{\dagger}_{j_1} f^{\dagger}_{j_2} \dots f^{\dagger}_{j_{2n+1}}, \\
S^z &= \sum_{1 \leqslant j \leqslant 2L} (-1)^j f^{\dagger}_j f_j - L,
\end{aligned}$$

Definition of $U_{\mathfrak{q}}^{\mathsf{odd}} s\ell(2)$ (with $\mathfrak{q} = i$)

GRS, 2011

$U_{\mathfrak{q}}^{\mathsf{odd}} s\ell(2)$ is generated by $S_n^{\pm}$ $(n \geqslant 0)$ and S^z with the def. relations

$$\mathfrak{q}^{S^z} S_n^{\pm} \mathfrak{q}^{-S^z} = \mathfrak{q} S_n^{\pm}, \qquad [S^z, S_n^{\pm}] = \pm(2n+1) S_n^{\pm},$$

$$[S_m^+, S_n^-] = \sum_{r=1}^{\mathsf{min}(n,m)} P_r(S^z) S_{n-r}^- S_{m-r}^+,$$

$$[S_m^{\pm}, S_n^{\pm}] = 0, \quad (S_m^{\pm})^2 = 0.$$

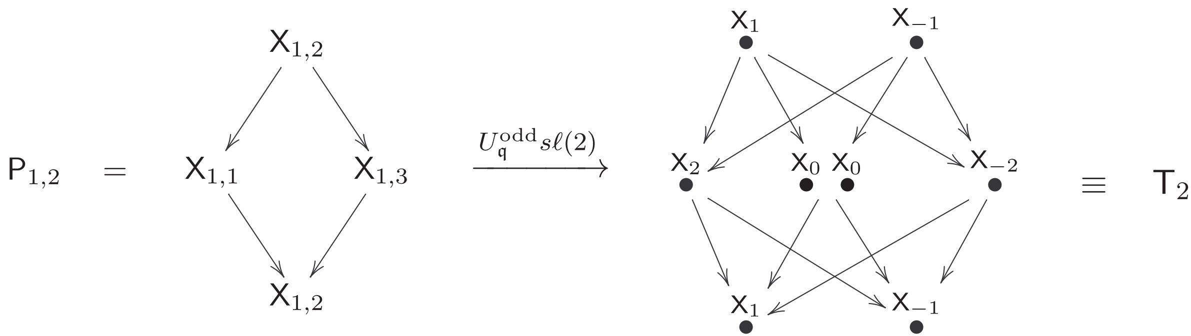
- $U_{\mathfrak{q}}^{\mathsf{odd}} s\ell(2)$ is a subalgebra in $U_{\mathfrak{q}} s\ell(2)$ and acts as

$$S_n^+ = F_{(2n+1)}^{\dagger}, \qquad S_n^- = F_{(2n+1)}.$$

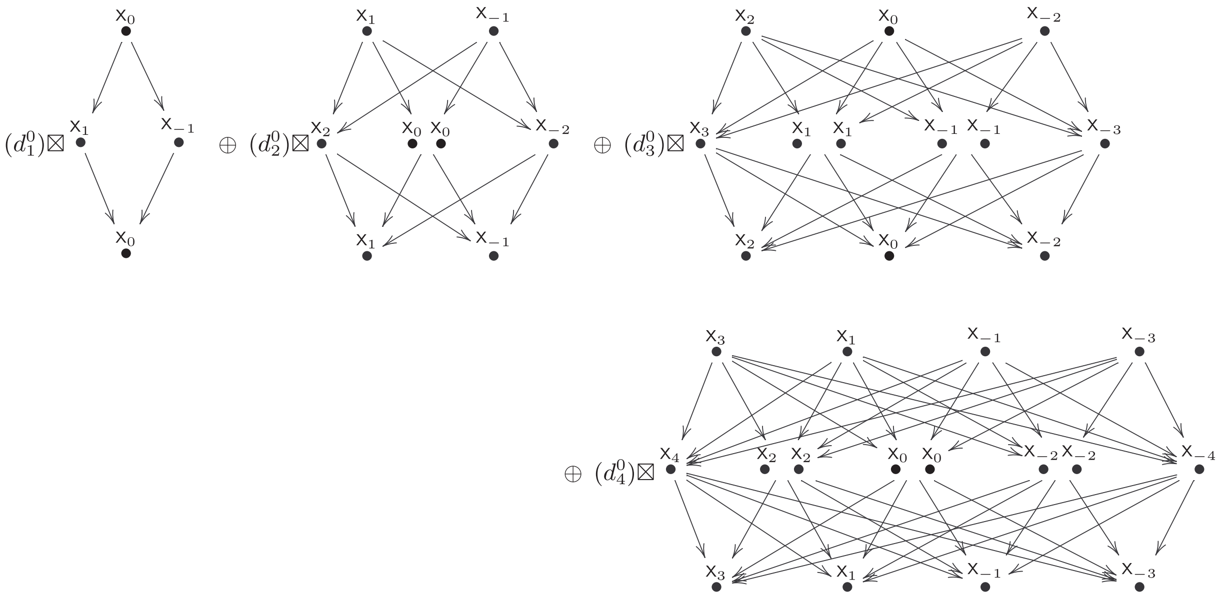
Theorem. *On the periodic $gl(1|1)$ spin-chain, the associative algebra $U_{\mathfrak{q}}^{\mathsf{odd}} s\ell(2)$ is the* **centralizer** *of the affine Temperley–Lieb algebra $\widehat{TL}_N$.*

Representation theory of $U_{\mathfrak{q}}^{\mathbf{odd}} s\ell(2)$

- All the irreps for $U_{\mathfrak{q}}^{\mathrm{odd}} s\ell(2)$ which appear as subfactors in the spin-chain are **one**-dimensional and parametrized by the weights with respect to S^z and depicted in diagrams by X_n, where $-\frac{N}{2} \leqslant n \leqslant \frac{N}{2}$.
- The indecomposable $U_{\mathfrak{q}}^{\mathrm{odd}} s\ell(2)$-modules T_n are **restrictions** of the tilting $U_{\mathfrak{q}} s\ell(2)$-modules $\mathsf{P}_{1,n}$ that appear in open spin-chains.



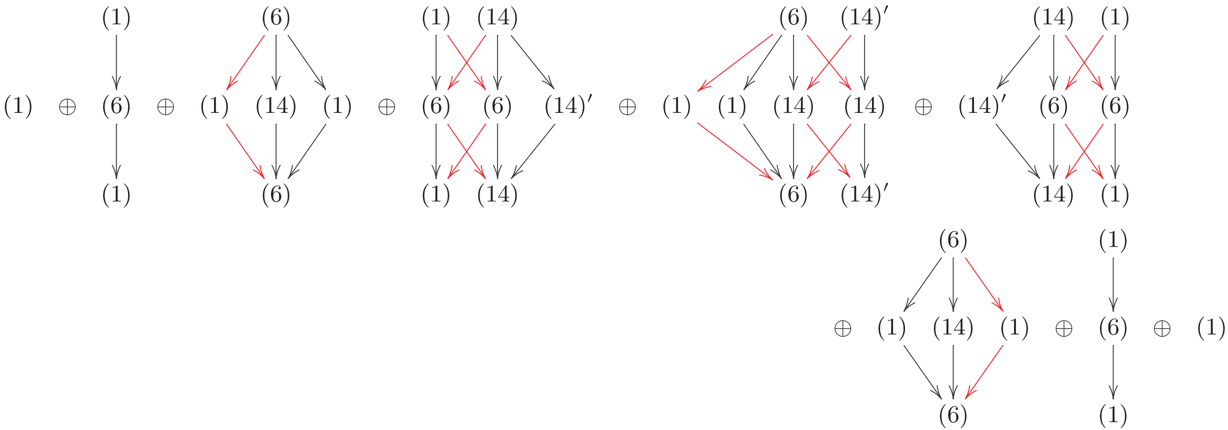
Example. Decomposition over $U_{\mathfrak{q}}^{\mathrm{odd}} s\ell(2)$ for $N = 8$



- the multiplicities $(d_1^0) = (14)'$, $(d_2^0) = (14)$, $(d_3^0) = (6)$, and $(d_4^0) = (1)$ are dimensions of simples for $\widehat{TL}_N(0)$.

Example. Decomposition over $\widehat{TL}_N$ for $N = 8$

The decomposition of the full spin-chain on 8 sites with respect to $\widehat{TL}_N$.



where the **black** arrows represent the action of the open TL and the **red** ones – of the last generator e_N.

Decomposition over $U_{\mathfrak{q}}^{\mathbf{odd}} s\ell(2)$ for $N = 2L$

Decomposition w.r.t. the odd quantum group $U_{\mathfrak{q}}^{\mathsf{odd}}(s\ell(2))$

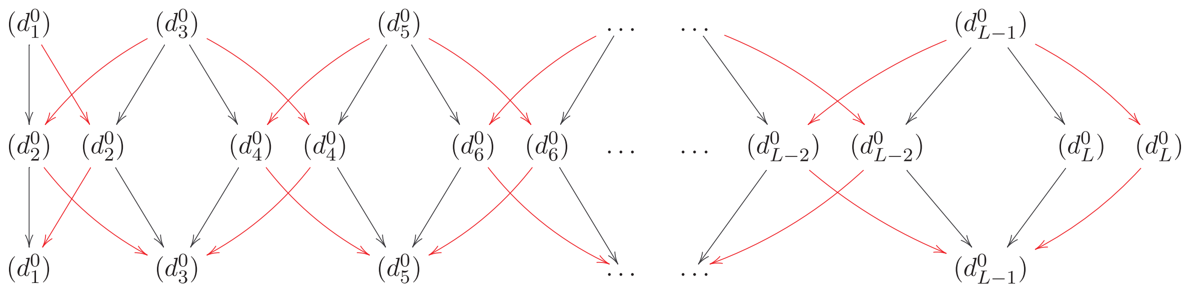
$$\mathcal{H}\big|_{U_{\mathfrak{q}}^{\mathsf{odd}} s\ell(2)} = \bigoplus_{n=1}^{N/2} (d_n^0) \boxtimes \mathsf{T}_n$$

the multiplicities (d_n^0) are dimensions of irreducibles for $\widehat{TL}_N$.

Decomposition over $\widehat{TL}_N$ GRS, 2011

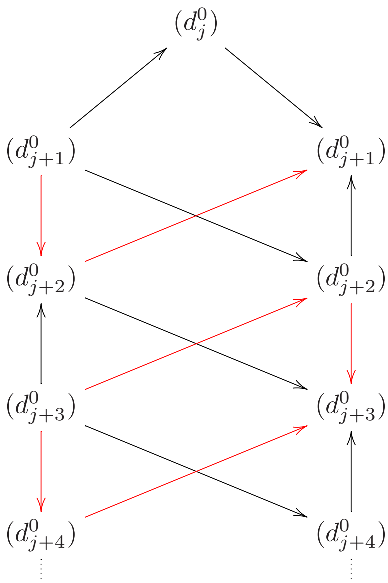
We compute $\mathrm{End}_{U^{\mathsf{odd}}_{\mathfrak{q}} s\ell(2)}(\mathcal{H})$ to decompose the periodic $gl(1|1)$ spin-chain over $\widehat{TL}_N$ for any even N.

- In particular, the subquotient structure for $S^z = 0$ is



Decomposition over $\widehat{TL}_N$

For any sector with $j = |S^z|$ (with fixed number of fermions)



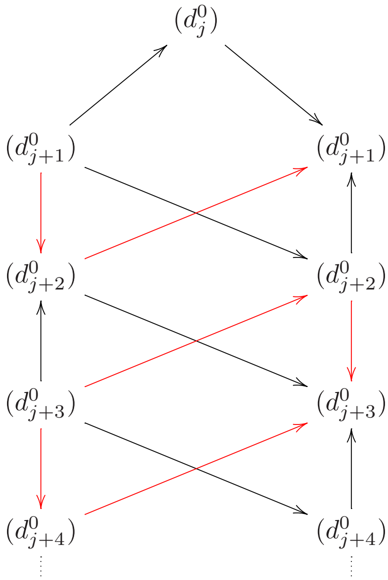
Dimensions of $\widehat{TL}_N$ simples:

$$d^0_{j,(-1)^{j+1}} = \binom{2L-2}{L-j} - \binom{2L-2}{L-j-2}$$

Decomposition over $\widehat{TL}_N$

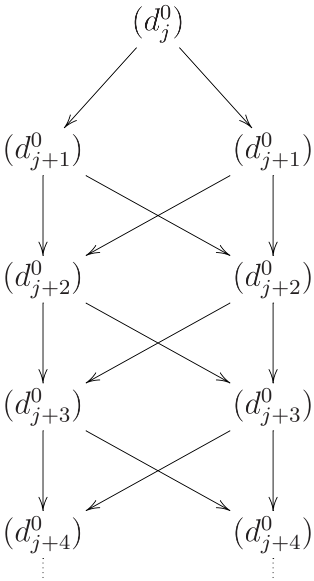
GRS, 2011

For any sector with $j = |S^z|$ (with fixed number of fermions)

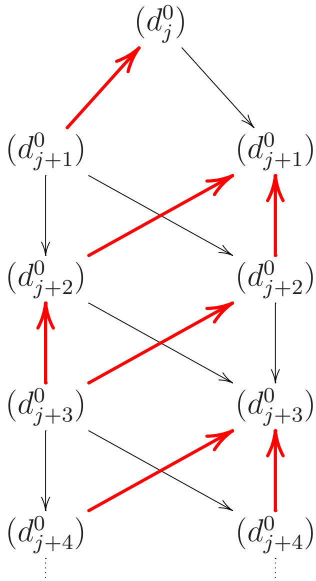


The hamiltonian acts by Jordan blocks of **rank $\mathbf{2}$**, horizontally

Spin-chain vs. Standard modules over $\widehat{TL}_N$



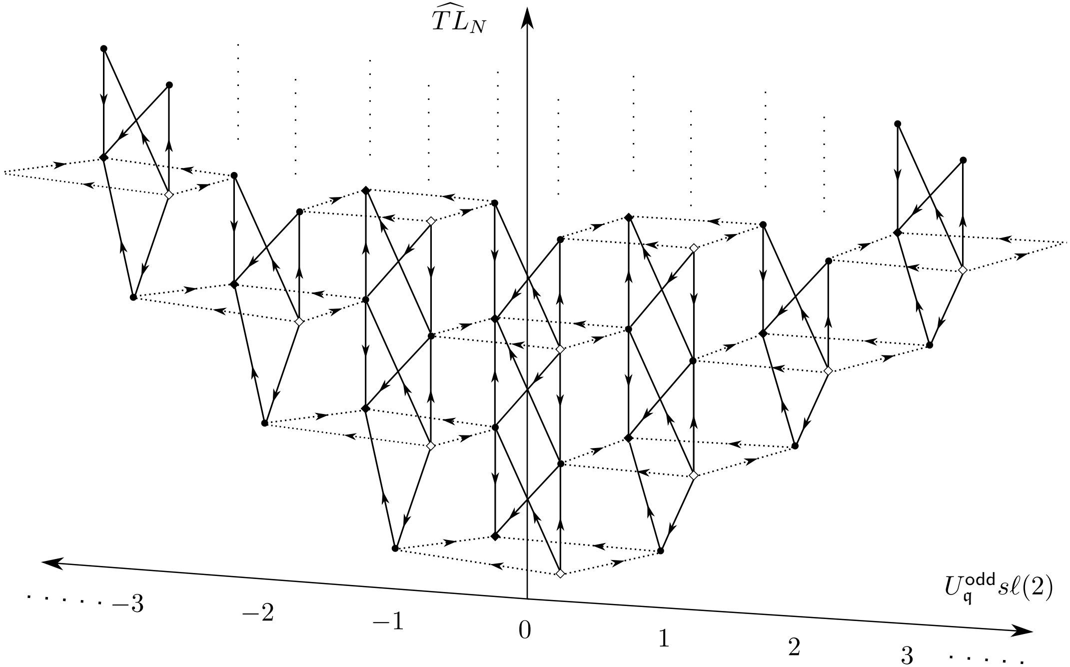
structure of the **cell** (standard) modules at $q = i$



structure of the **spin-chain** modules at $q = i$

The arrows in **red** have been flipped w.r.t. the structure of the standard modules.

Bimodule over the pair $(\widehat{TL}_N, U_{\mathfrak{q}}^{\text{odd}} s\ell(2))$



The action of $\widehat{TL}_N$ is depicted by vertical arrows

The action of $U_{\mathfrak{q}}^{\text{odd}} s\ell(2)$ is shown by dotted horizontal lines.

The full space of states is a **bimodule** over $\widehat{TL}_N(0) \otimes U_{\mathfrak{q}}^{\mathsf{odd}} s\ell(2)$

- the states are organized into indecomposables for $U_{\mathfrak{q}}^{\mathsf{odd}} s\ell(2)$ (or $\widehat{TL}_N(0)$)
  - with nodiag. action of the Casimir operator (or the hamiltonian H)

What is going on with such a structure when the scaling limit is taken?

Scaling limit $N \to \infty$

GRS, 2011

- ground state $\leftrightarrow$ $|0\rangle$;
- low-lying excitations $\leftrightarrow$ many-particles states in Fock spaces;
- correspondence between generating spectrum algebras:

(periodic)Temperley–Lieb $\leftrightarrow$ Virasoro ($+$ $\overline{\text{Virasoro}}$)

Fourier modes (1) $\frac{N}{\pi}\sum_k \mathrm{e}^{ik\mathbf{n}\frac{\pi}{N}} \boldsymbol{e_k} \xrightarrow[\text{asymptotic}]{\text{leading}}$ Virasoro $\left(L_{\mathbf{n}} + \bar{L}_{-\mathbf{n}}\right)$

(2) $\frac{N}{\pi}\sum_k \mathrm{e}^{ik\mathbf{n}\frac{\pi}{N}} [\boldsymbol{e_k}, \boldsymbol{e_{k+1}}] \longrightarrow \left(L_{\mathbf{n}} - \bar{L}_{-\mathbf{n}}\right)$

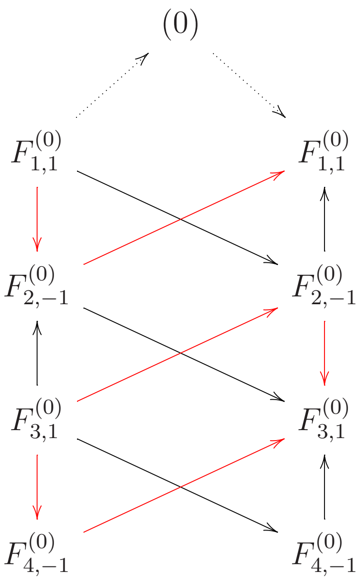
— **non-chiral** Virasoro with the central charge $c = -2$

in the symplectic-fermions representation

Scaling limit $N \to \infty$ GRS, 2011

- Decomposition of bulk sympl. ferm. over Vir and $\overline{\text{Vir}}$

  is **consistent** with the limit of $\widehat{TL}(0)$ spin-chain modules



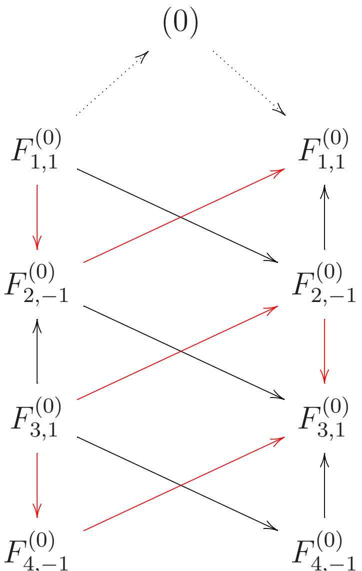
$F^{(0)}_{j,(-1)^{j+1}} = \sum^{*}_{j_1,j_2>0} \chi_{j_1,1}\overline{\chi}_{j_2,1}$, where the sum is done with the constraints:

$$|j_1 - j_2| + 1 \leqslant j, \qquad j_1 + j_2 - 1 \geqslant j,$$

$$j_1 + j_2 - 1 = j \bmod 2$$

Scaling limit $N \to \infty$ GRS, 2011

- Scaling limit of first irrdeucible subquotients $F^{(0)}_{j,(-1)^{j+1}}$



$$F^{(0)}_{0,-1} = 0$$

$$F^{(0)}_{1,1} = \sum_{r=1}^{\infty} \chi_{r1} \bar{\chi}_{r1}$$

$$F^{(0)}_{2,-1} = \sum_{r=1}^{\infty} \chi_{r1} \left(\bar{\chi}_{r-1,1} + \bar{\chi}_{r+1,1}\right)$$

$$F^{(0)}_{3,1} = \chi_{21} \left(\bar{\chi}_{21} + \bar{\chi}_{41}\right) + \sum_{r=3}^{\infty} \chi_{r1} \left(\bar{\chi}_{r-2,1} + \bar{\chi}_{r1} + \bar{\chi}_{r+2,1}\right)$$

...

Symmetry algebra for non-chiral Virasoro

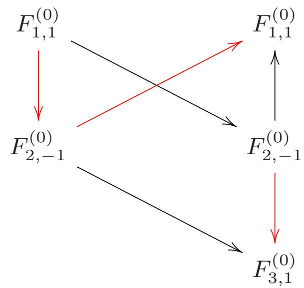
It turns out that centralizer for the **non**-chiral Virasoro is **bigger** than $U_{\mathfrak{q}}^{\mathsf{odd}} s\ell(2)$:

- a lattice analogue of Kausch's **global** $\boldsymbol{s\ell}(2)$ does not commute with $\widehat{TL}_N(0)$.

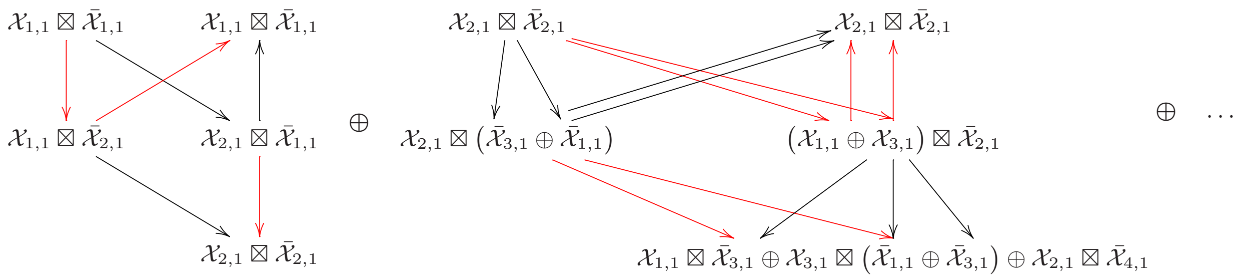
This $sl(2)$ symmetry 'splits' the block $F_{j,(-1)^j}$ into sectors with isospins $k \geqslant \frac{j}{2}$.

How are the indecomposable spin-chain modules decomposed (splitted) onto modules for Vir $\otimes$ $\overline{\text{Vir}}$?

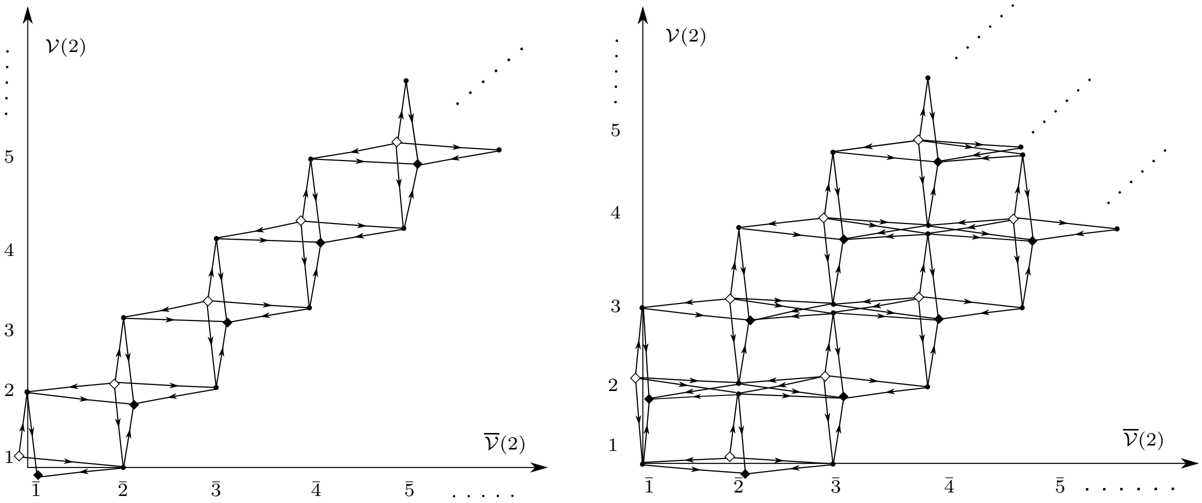
Splitting of $\widehat{TL}$-modules into left-right Virasoro



It is decomposed over $\mathrm{Vir} \otimes \overline{\mathrm{Vir}}$ into the direct sum (over all integer isospins)

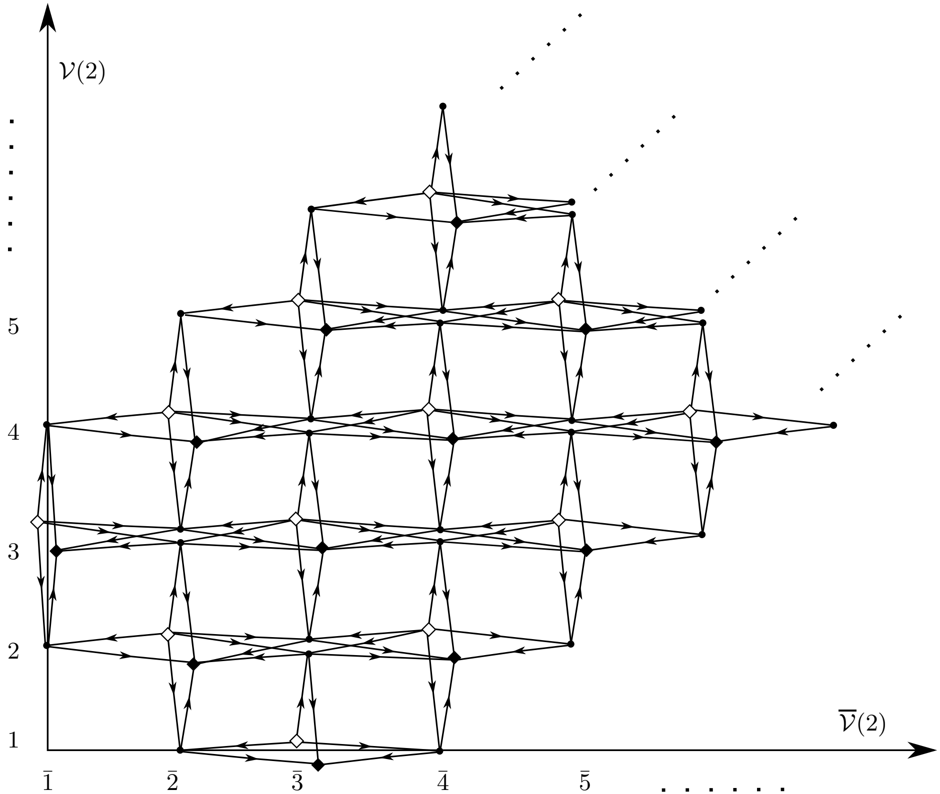


Vacuum sector for left and right Virasoro



Module structure over Vir $\otimes$ $\overline{\text{Vir}}$ for the vacuum sector (with zero $\boldsymbol{s\ell}(2)$-isospin) on the left diagram while the right one is for the doublet-sector $1/2$-isospin.

Isospin-1 **sector** for left and right Virasoro



Node $(\bar{n}, n') = (\Delta_{n',1}, \bar{\Delta}_{\bar{n},1})$.

Conclusion for periodic $gl(1|1)$ spin-chain and the limit of $\widehat{TL}(0)$

In the scaling limit, an indecomposable module for the $\widehat{TL}(0)$ algebra splits into an infinite sum of indecomposable representations under the product of the left and right Virasoro algebra.

The (full) scaling limit of the $\widehat{TL}(0)$ algebra is thus should be bigger than the product $\mathsf{Vir}\otimes\overline{\mathsf{Vir}}$. Indeed, there exist additional fields in the bulk that generate the limit of $\widehat{TL}(0)$! These bulk fields 'link' infinitely many indecomposables for $\mathsf{Vir}\otimes\overline{\mathsf{Vir}}$ into one indecomposable of **the same structure** as we found from the lattice analysis. The theory is still non-rational in the sense we have infinitely-many primaries but the limit of $\widehat{TL}(0)$ gives a good organinzing principle for bulk fields.

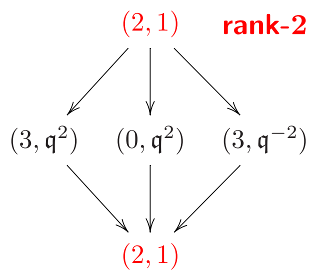
$sl(2|1)$ **periodic spin-chain** ($\mathfrak{q} = e^{i\pi/3}$)

Adding a coupling e_{2L} between $j = 2L$ and $j = 1$ tensorands and introducing the translational operator $u : j \longrightarrow j + 2$, as we did for the $gl(1|1)$ case, we obtain a **faithful** representation of the $\widehat{TL}_N(m = 1)$ and the decomposition

$$\mathcal{H}_N = \mathcal{T}_{0,\mathfrak{q}^2} \oplus 8\mathcal{T}_{1,1} \oplus 22\mathcal{T}_{2,1} \oplus 24\mathcal{T}_{2,-1} \oplus 112\mathcal{T}_{3,1} \oplus 75\mathcal{T}_{3,\mathfrak{q}^{\pm 2}} \oplus \ldots.$$

where the weights (j, P) of indecomposable direct summands correspond to $2j$ strings and the pseudomomentum $P = e^{2i\pi j/l}$, $1 \leqslant l \leqslant j$, of u on the strings.

$\mathcal{T}_{0,\mathfrak{q}^2}$:

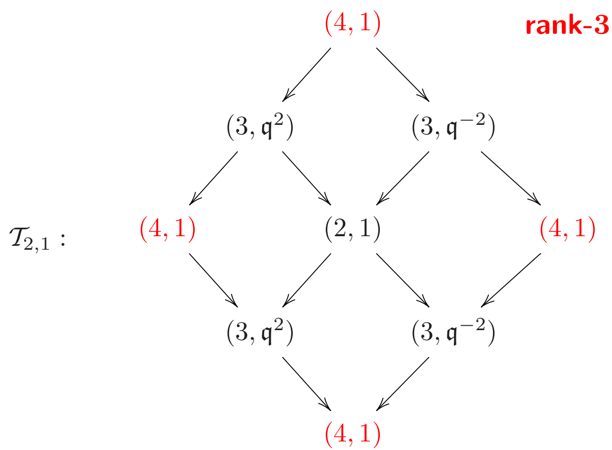


The vacuum sector: the model has unique vacuum living in $(0, \mathfrak{q}^2)$, the 'energy-momentum' state $|T\rangle$ and its logarithmic partner $|t\rangle$ living in $(2, 1)$ subquotients.

In a paper of VGJS-2011, the indecomposibility parameter $\langle T|t\rangle = b$ was measured with the result $b = -5$

$sl(2|1)$ **periodic spin-chain** ($\mathfrak{q} = e^{i\pi/3}$)

There are Jordan cells for H of rank 3

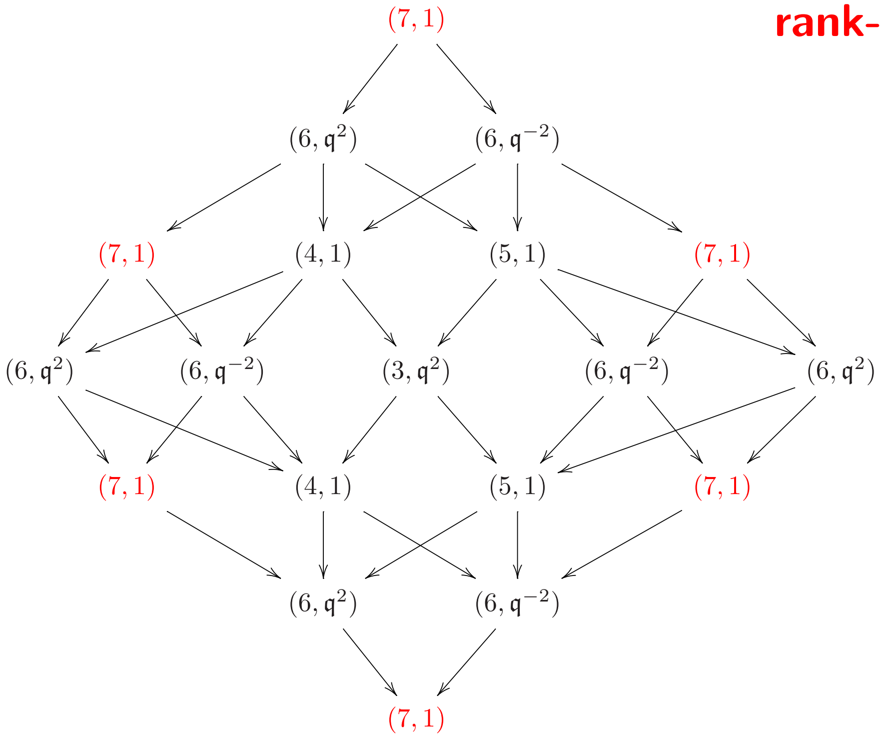


$sl(2|1)$ **periodic spin-chain** ($\mathfrak{q} = e^{i\pi/3}$)

There are Jordan cells for H of rank 4!

rank-4

$\mathcal{T}_{3,\mathfrak{q}^2}$:



Thank You!