Quantum Hall transitions and conformal restriction

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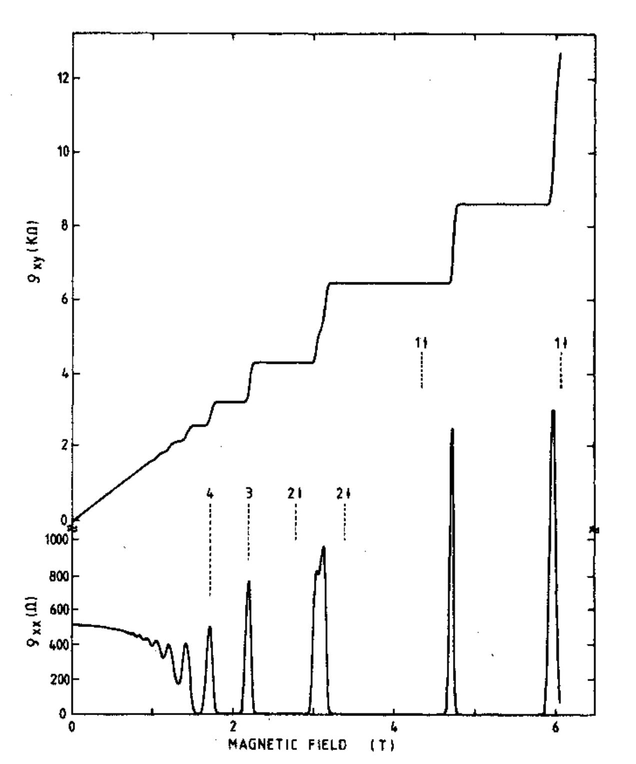
IHP, Paris, November 2nd, 2011

Integer quantum Hall effect

- Two-dimensional electron gas
- Strong magnetic field, low temperature
- Hall resistance shows plateaus

$$R_H = \frac{1}{n}\frac{h}{e^2}$$

- Longitudinal resistance shows peaks separated by insulating valleys
- Transitions between the plateaus – unsolved problem



K. v. Klitzing, Rev. Mod. Phys. 56 (1986)

IQH and localization in strong magnetic field

- Single electron in a magnetic field and a random potential

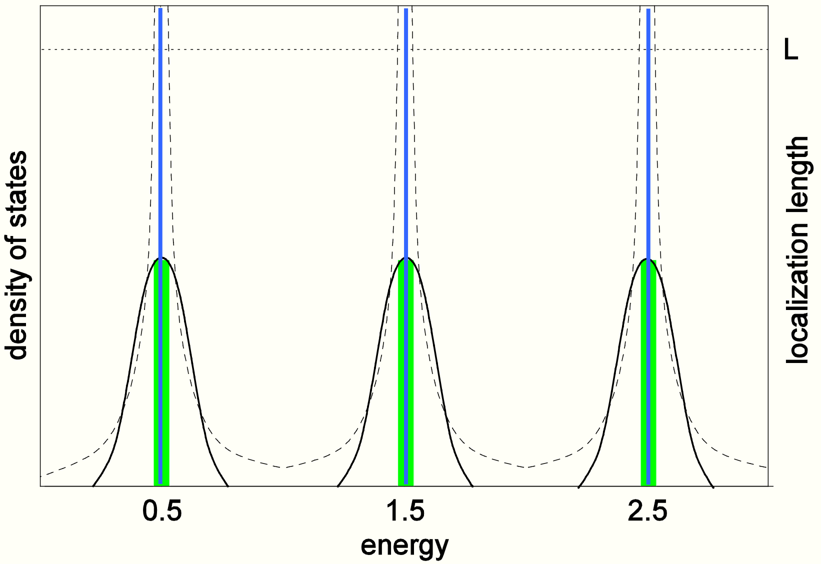
$$H = \frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2 + U(\boldsymbol{r})$$

- Without disorder: Landau levels

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_c, \quad \omega_c = \frac{eB}{mc}$$

- Disorder broadens the levels and localizes most states

- Extended states near E_n (green)

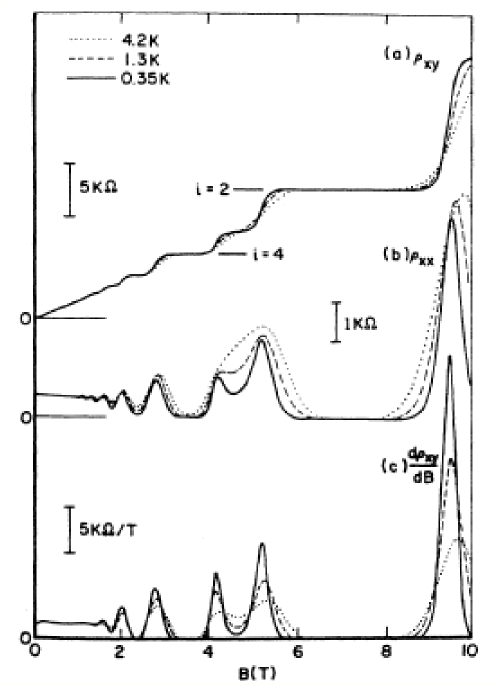


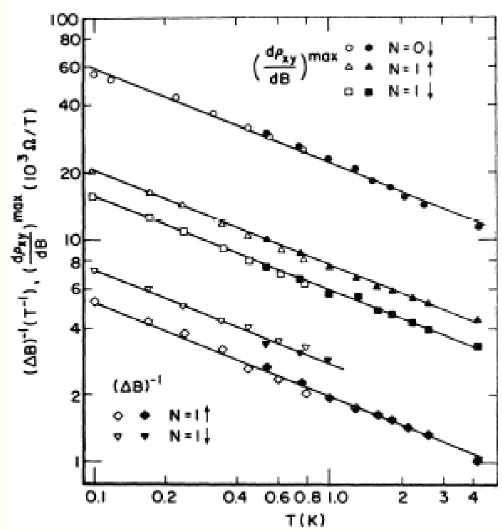
- Transition between QH plateaus upon varying E_F or B

Critical scaling near IQH plateau transition

- Localization length diverges $\xi(E) \propto |E - E_n|^{-\nu}$
- Critical phenomenon, example of Anderson transition – quantum phase transition driven by disorder
- Universal scaling with temperature, current, frequency, and system size

$$\sigma_{ij}(B) = \frac{e^2}{h} S_{ij}\left[L_{\text{eff}}^{1/\nu}(B - B_c)\right]$$





H. P. Wei et al., PRL 61 (1988)

Theory of IQH plateau transition

- Goals for a theory of the transition:

    - Critical exponents
    - Scaling functions
    - Correlation functions at the transition

- No analytical description of the critical region so far

- Conformal invariance at the transition in 2D should help

- Plenty of numerical results (confirming conformal invariance)

A. Furusaki, H. Obuse, et al

Our approach

E. Bettelheim, IAG, A. W. W. Ludwig, 2010

- New approach using ideas of stochastic conformal geometry
    - Conformal restriction and Schramm-Loewner evolution (SLE): spectacular and powerful recent mathematical tools

- These ideas were applied to non-random classical statistical mechanics but seemed useless for disordered and/or quantum systems

- We show how to apply them to Anderson transitions including IQH plateau transition

Our approach

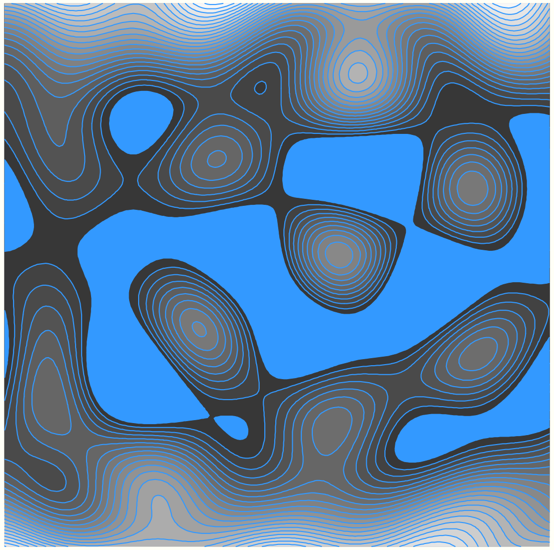
E. Bettelheim, IAG, A. W. W. Ludwig, 2010

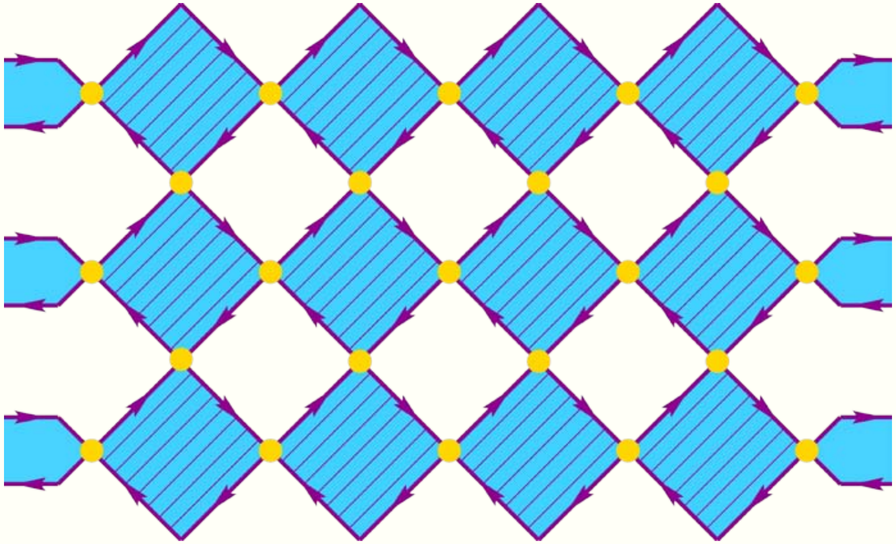
- We consider point contact conductances (PCC) within the Chalker-Coddington network model
- Map average PCC to a *classical* problem
- Establish crucial restriction property
- Assume conformal invariance in the continuum limit and obtain PCC in systems with various boundary conditions

Chalker-Coddington network model

J. T. Chalker, P. D. Coddington, 1988

- Obtained from semi-classical drifting orbits in smooth potential

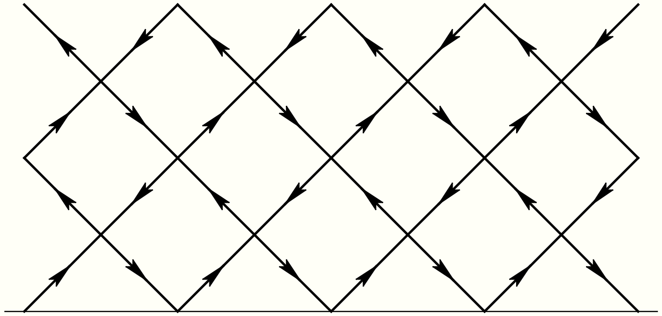


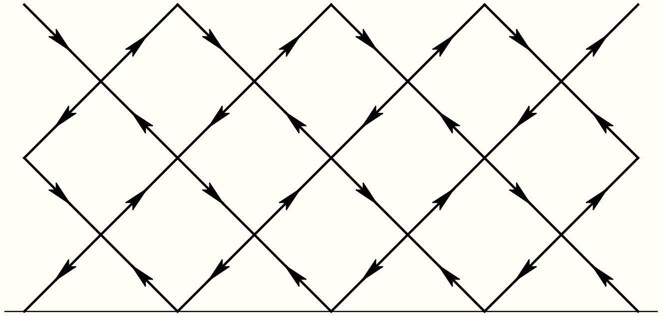


- Fluxes (currents) on links, scattering at nodes

- The model is designed to describe transport properties

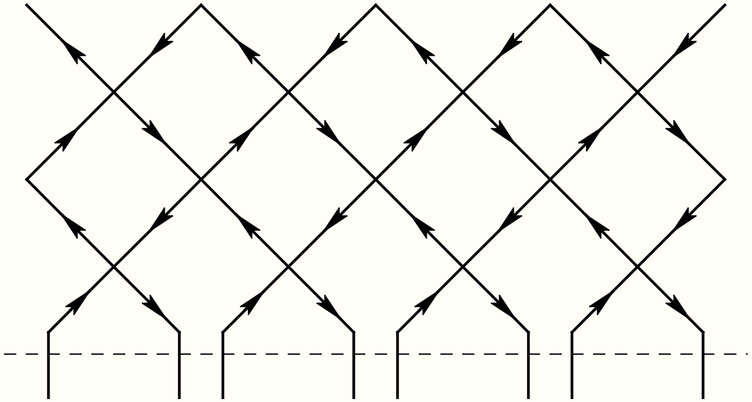
Boundary conditions





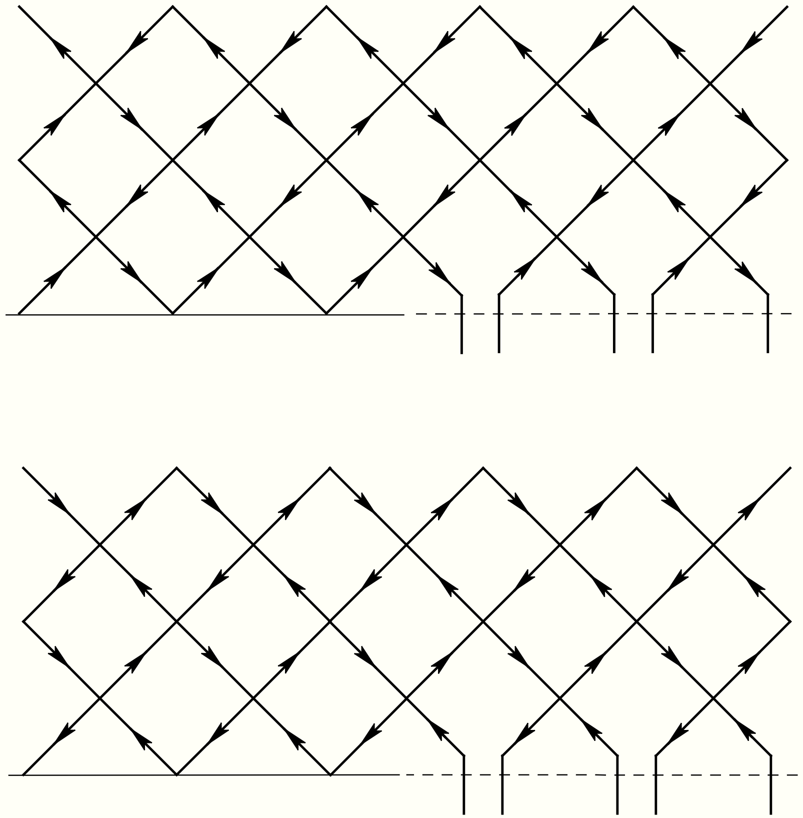
• Reflecting (right)

• Reflecting (left)



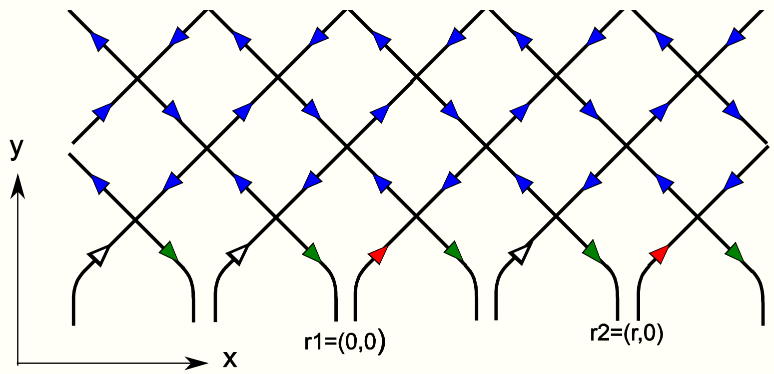
• Absorbing: boundary nodes are the same as in the bulk

Boundary condition changes

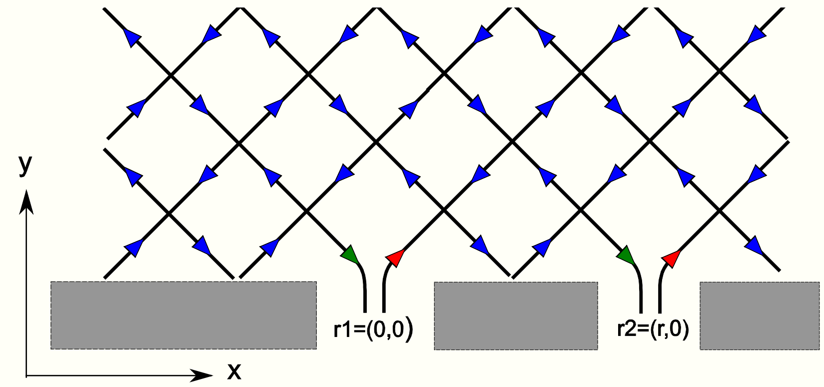


• Reflecting (right or left) to absorbing

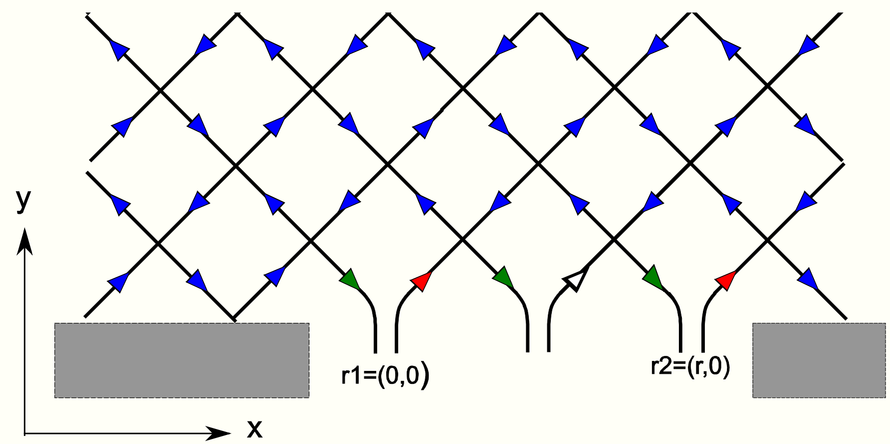
Boundary PCC and boundary conditions



- Absorbing



- Reflecting (left or right)



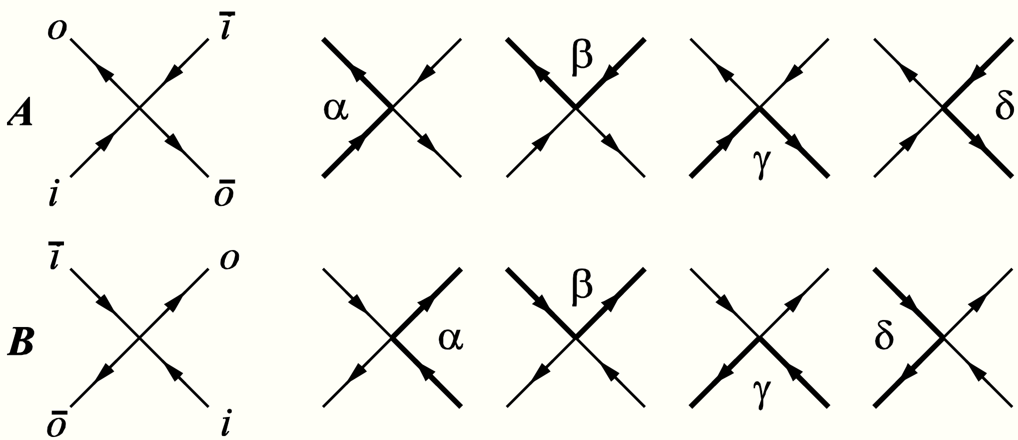
- Mixed

Chalker-Coddington network model

- States of the system specified by $Z \in \mathbb{C}^{N_l}$

  N_l the number of links

- Evolution (discrete time) specified by a random $U \in \mathrm{U}(N_l)$



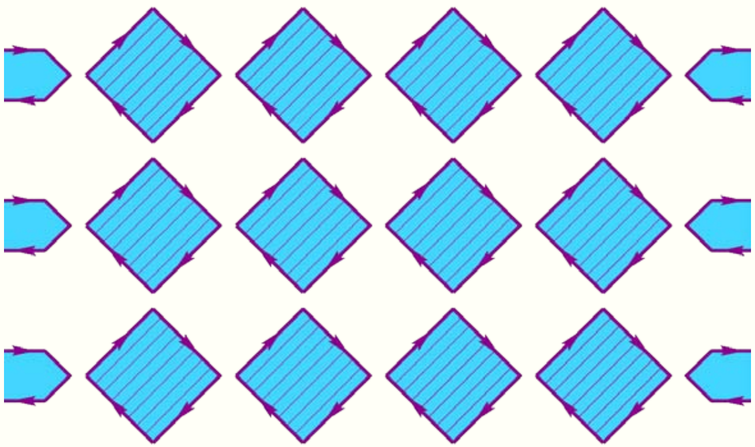
$$\begin{pmatrix} z_o \\ z_{\bar{o}} \end{pmatrix} = \mathcal{S} \begin{pmatrix} z_i \\ z_{\bar{\imath}} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} z_i \\ z_{\bar{\imath}} \end{pmatrix}$$

Chalker-Coddington network model

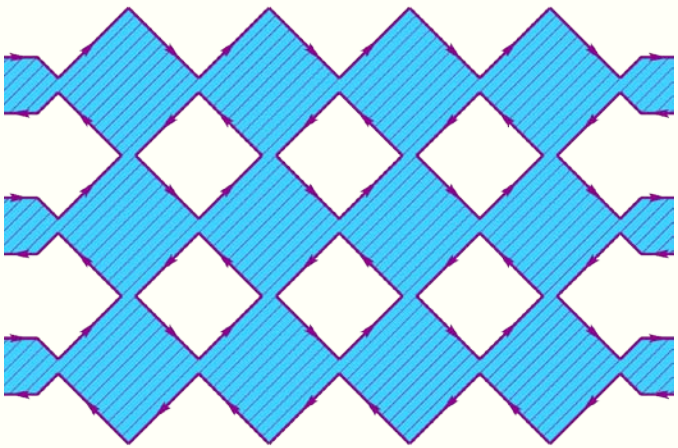
- Choice of disorder

$$\mathcal{S} = \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{i\phi_2} \end{pmatrix} \begin{pmatrix} \sqrt{1-t^2} & t \\ -t & \sqrt{1-t^2} \end{pmatrix} \begin{pmatrix} e^{i\phi_3} & 0 \\ 0 & e^{i\phi_4} \end{pmatrix}$$

- Extreme limits



$t = 0$ Insulator



$t = 1$ Quantum Hall

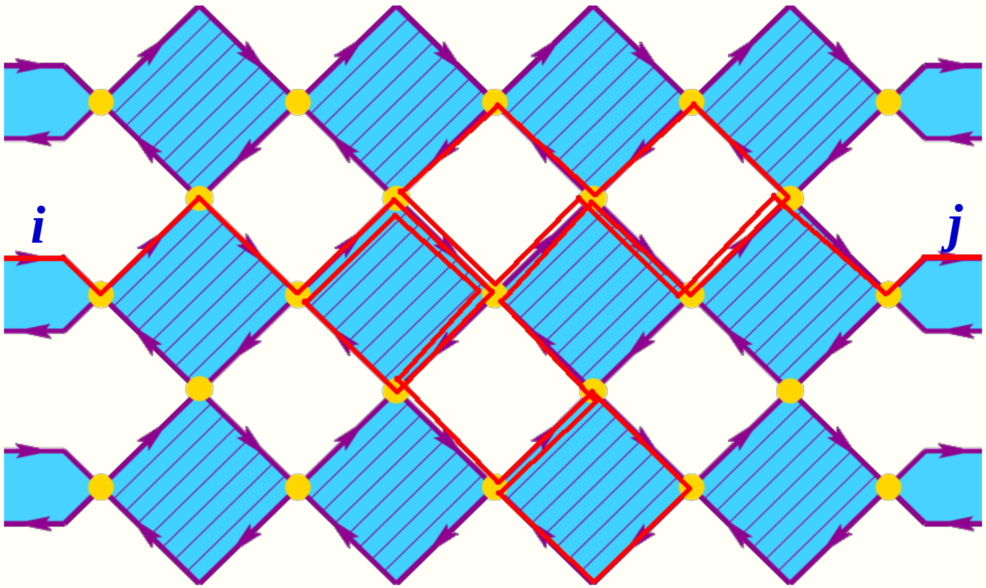
- Critical point at $t = 1/\sqrt{2}$

Chalker-Coddington network model

- Graphical representation of propagator as a sum over (Feynman) paths

$$A(i,j) = \sum_{f:i\to j} A_f(i,j), \qquad A_f(i,j) = \prod_{\text{nodes}\in f} S_{ab}$$

$$A_p(i,j) =$$



PCC and mapping to a classical problem

E. Bettelheim, IAG, A. W. W. Ludwig, 2010

- Average point contact conductance (PCC)

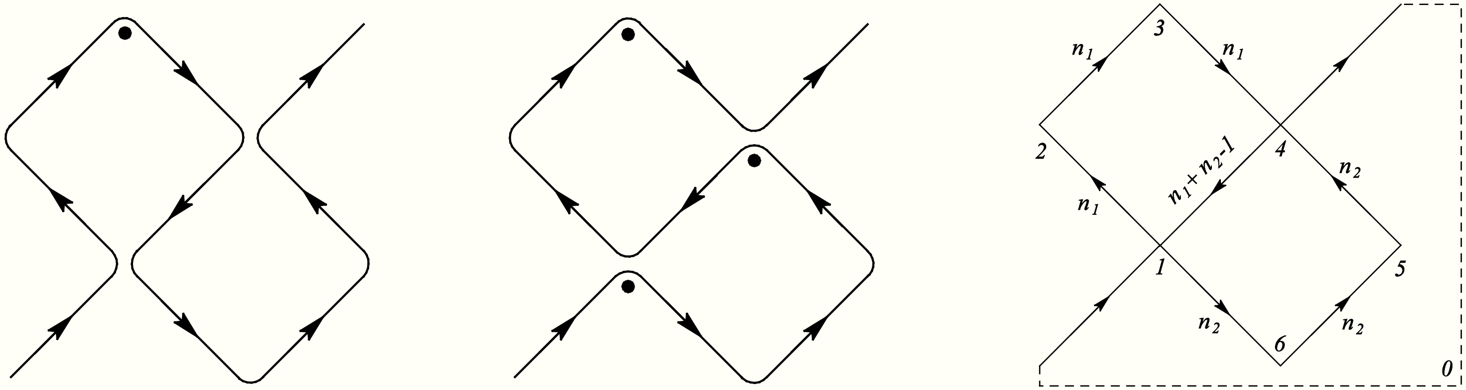
$$\overline{g_{ij}} = \overline{|A(i,j)|^2} = \sum_p W(p)$$

- $W(p)$ are intrinsic positive weights of "pictures" p

- This representation is valid at and away from the critical point

Pictures and paths

- Picture is obtained by “forgetting” the order in which links are traversed



$$W(p) = S^2(p), \qquad S(p) = 2^{-N(p)/2} \sum_{f \in F(p)} (-1)^{N_-(f)}$$

- We know how to enumerate paths giving rise to a picture
- Detailed analysis of the weights $W(p)$ may lead to a complete solution
- We try to go to continuum directly using restriction property

Pictures in a truncated model

Y. Ikhlef, P. Fendley, J. Cardy, 2011

- Truncated version of the CC model
- Equivalent to keeping pictures with $n_j = 1$
- In this case the weights simplify $W(p) = S^2(p) = 2^{-N(p)}|F(p)|^2$

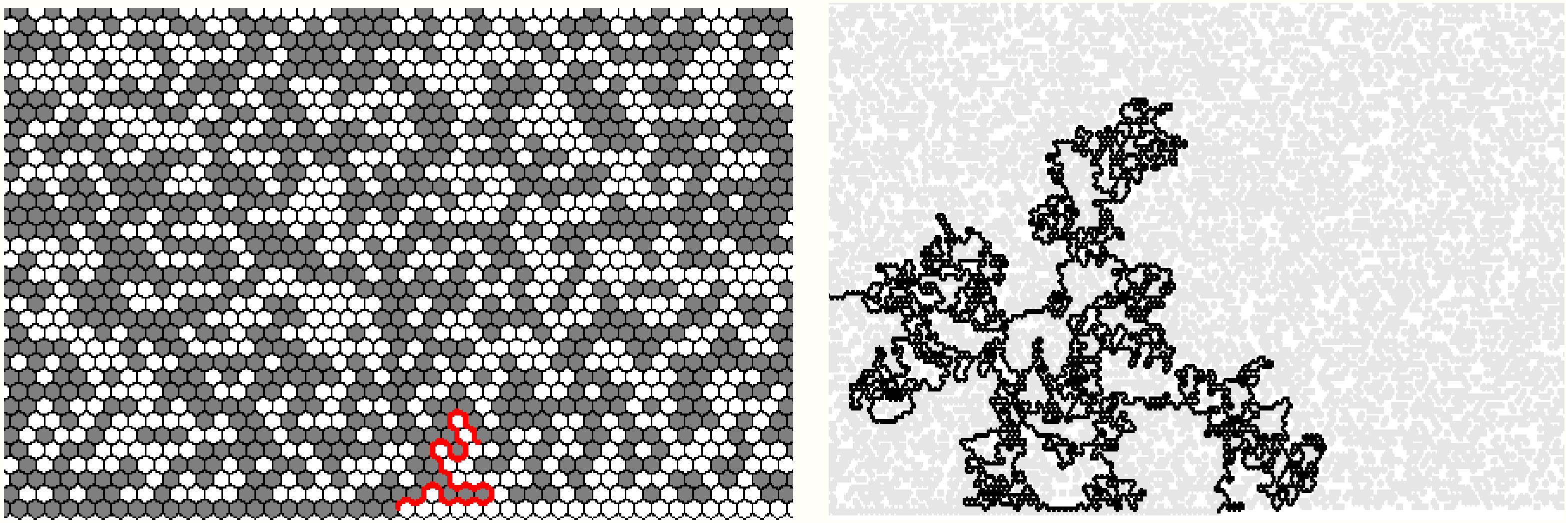
  where $N(p)$ is the number of links in the picture p and

$|F(p)| = \det L$ is the number of distinct paths leading to the picture p

- This model, as well as "higher" truncations with $n_j \leqslant k$ satisfy the restriction property
- O(1) model in class D can be treated in the same way.
  Its minimal truncation leads to the same model as Ikhlef et al.

Stochastic geometry and conformal invariance

- Schramm-Loewner evolution (SLE_κ) — O. Schramm, 1999
- Precise geometric description of classical conformally-invariant 2D systems
- Complementary to conformal field theory (CFT)
- Focuses on extended random geometric objects: cluster boundaries



- Powerful analytic and computational tool

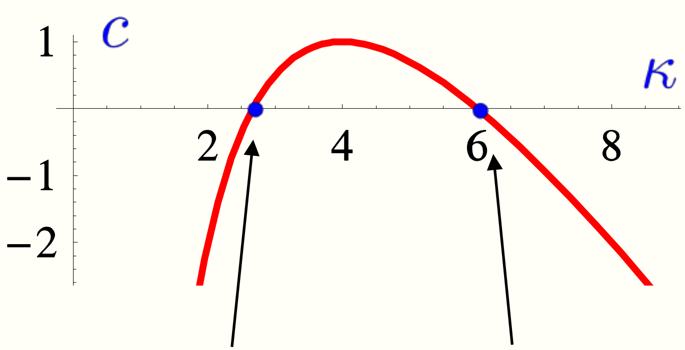
Stochastic geometry and conformal invariance

- SLE does not seem to apply to our case

- Pictures are neither lines nor clusters in a local model

- SLE_κ corresponds to CFT with

$$c_\kappa = \frac{(8-3\kappa)(\kappa-6)}{2\kappa}$$



- CFTs for Anderson transitions in 2D should have $c = 0$ $\Rightarrow$ $\kappa = 8/3$ $\kappa = 6$

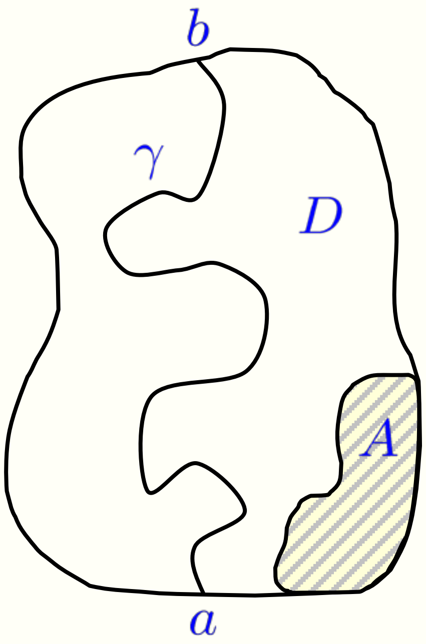
- Not enough for all 2D Anderson transitions and other $c = 0$ theories

- Appropriate stochastic/geometric notion is conformal restriction (closely related to $\mathrm{SLE}_{8/3}$)

G. Lawler, O. Schramm, W. Werner, 2003

Conformal restriction

G. Lawler, O. Schramm, W. Werner, 2003



- Consider an ensemble of curves γ in a domain D and a subset $A \subset D$ "attached" to boundary of D

- From ensemble of curves γ in D we can get an ensemble in $D \setminus A$ in two ways:

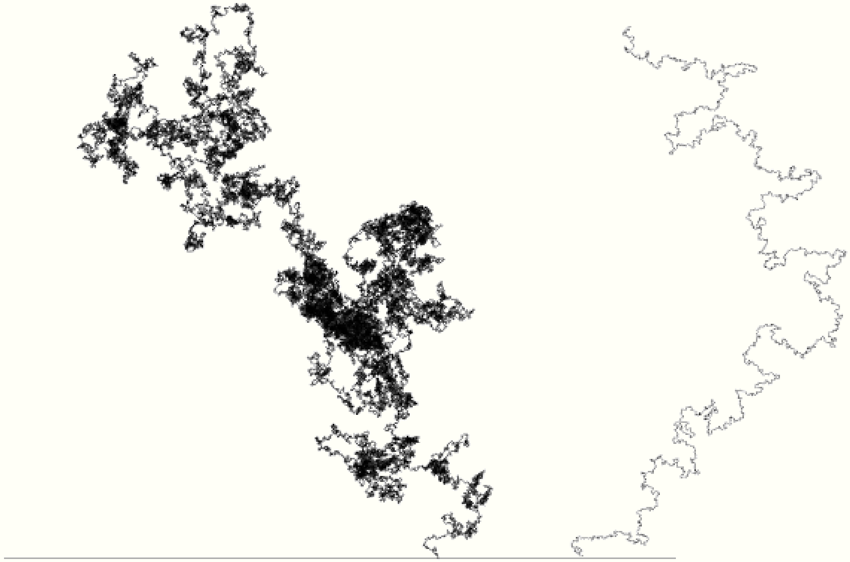
  - conditioning (keep only curves in the subset)
  - conformal transformation $f : D \to D \setminus A$

- If the two ways give the same result, the ensemble is said to satisfy conformal restriction

- Essentially, any *intrinsic* probability measure on curves satisfies restriction

Restriction measures

- More general sets than curves satisfying conformal restriction
- (Filled in) Brownian excursions, self-avoiding random walks, conditioned percolation hulls



Restriction exponent

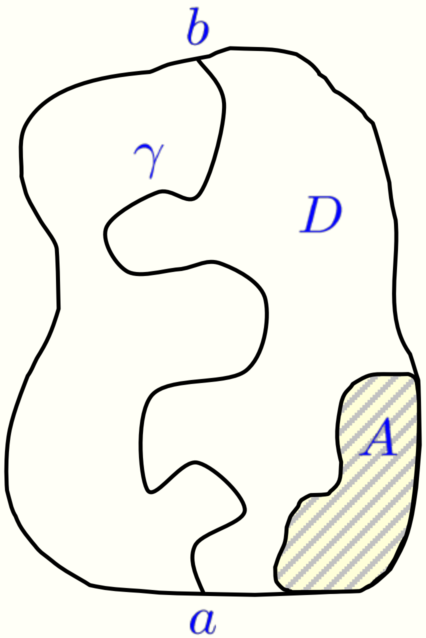
- Consider the probability that a sample from a restriction measure avoids a set A

- Collection of these probabilities for all possible A completely specifies the restriction measure

- Main result (LSW): these probabilities are

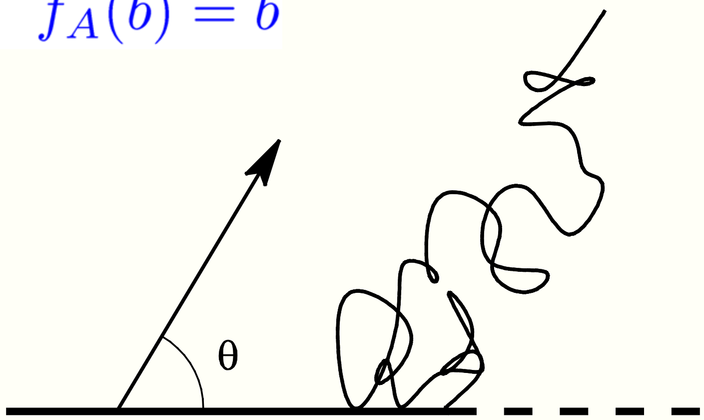
$$\boldsymbol{P}[\gamma \subset D \setminus A] = |f'_A(a)|^h |f'_A(b)|^h$$

$$f_A : D \setminus A \to D, \quad f_A(a) = a, \quad f_A(b) = b$$



- General construction using reflected Brownian motions

$$h = 1 - \frac{\theta}{\pi}$$



Restriction measures and SLE

- Statistics of a restriction measure is fully determined by the statistics of its boundaries

- Boundary of a restriction measure is a variant of SLE: $\mathrm{SLE}(8/3, \rho)$

$$h(\rho) = \frac{(3\rho + 10)(2 + \rho)}{32}, \qquad \rho(h) = \frac{2}{3}\sqrt{24h + 1} - \frac{8}{3}$$

- CFT interpretation: $\mathrm{SLE}(8/3, \rho)$ is obtained by the simple fusion of $\psi_{1,2}$ (that creates an $\mathrm{SLE}_{8/3}$ trace) with an auxiliary operator whose Coulomb charge $\alpha \propto \rho$

- h is the weight of a primary operator with charge $\alpha + \alpha_{1,2}$

IQH transition and restriction

E. Bettelheim, IAG, A. W. W. Ludwig, 2010

- Weights of pictures $W(p)$ are *intrinsic*: their ensemble satisfies restriction property with respect to *absorbing* boundaries

- Assume conformal invariance, then can use conformal restriction theory

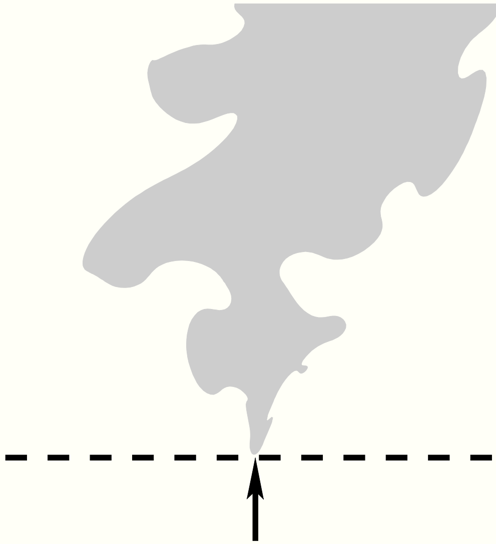
- Current insertions create pictures, and are *primary* CFT operators

- Important to know their scaling dimensions

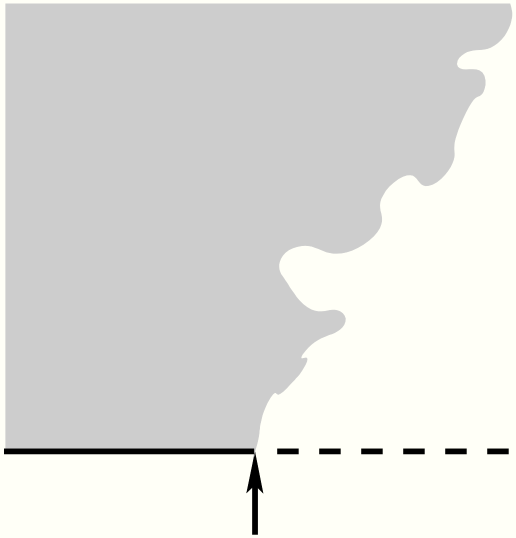
- Explicit analytical results for average PCC with various boundaries

Boundary operators and dimensions

E. Bettelheim, IAG, A. W. W. Ludwig, 2010



$h_A = 1$ (exact ?)



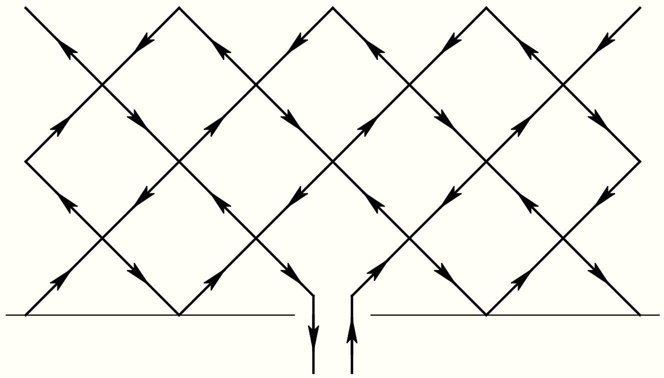
h_{LA}, h_{RA}

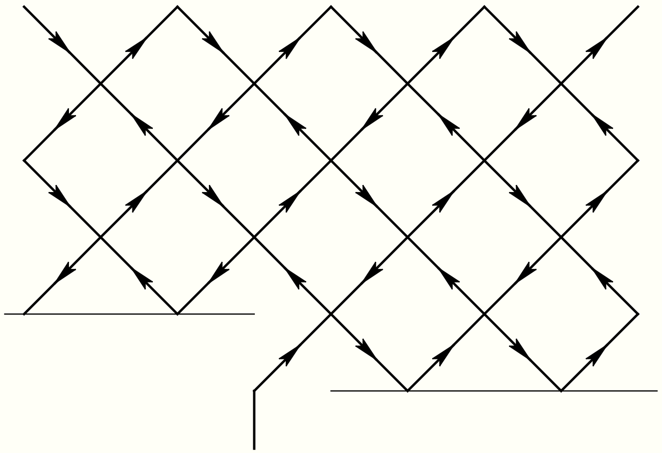
- Dimensions known numerically

S. Bera, F. Evers, H. Obuse, in progress

Boundary operators and dimensions

E. Bettelheim, IAG, A. W. W. Ludwig, 2010

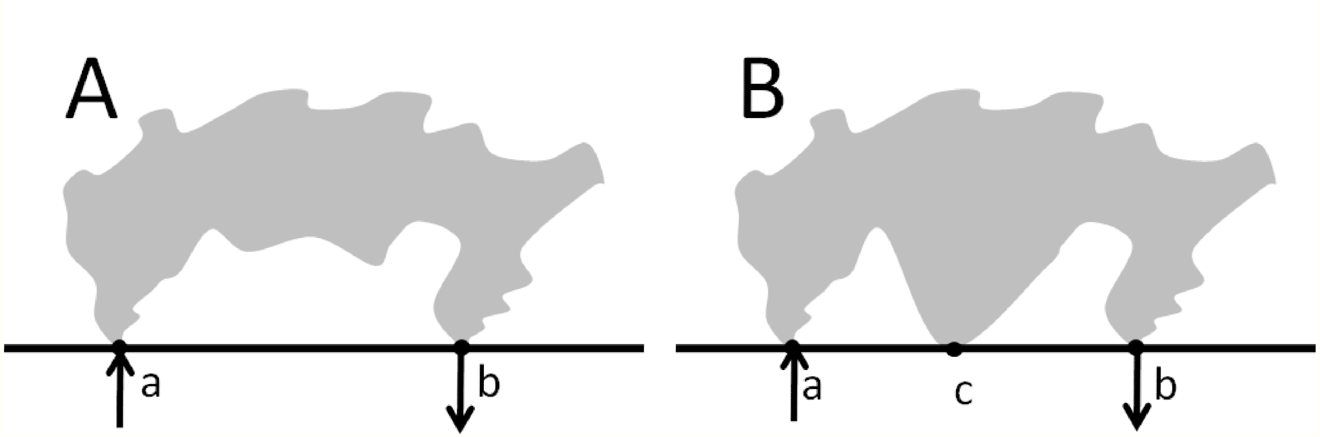




h_R $\quad\quad$ $h_{LR} = 0$ (exact)

- Dimensions known numerically $\quad$ S. Bera, F. Evers, H. Obuse, in progress

Explicit exact results for PCC



- Case A: two-point PCC $\overline{g_{ab}} = \frac{C}{(b-a)^{2h}}$

- Case B: change in the PCC upon perturbing the boundary near c

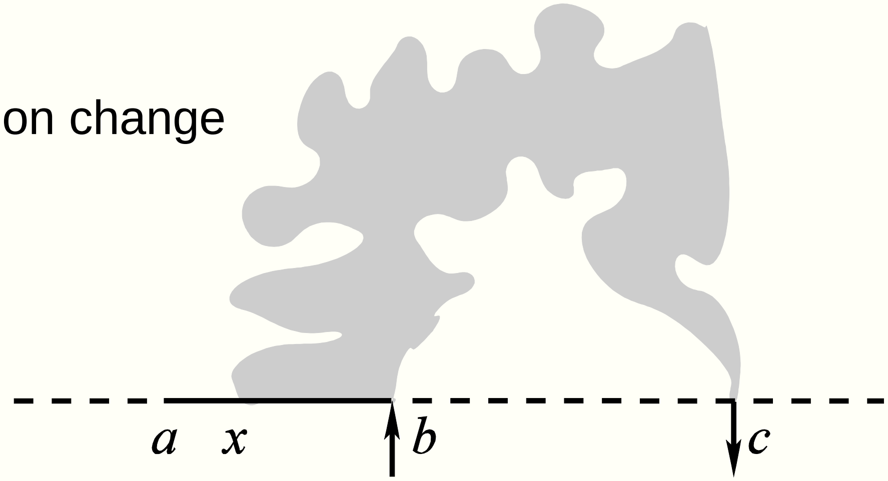
$$\delta\overline{g_{ab}} = \frac{C}{(c-a)^{h_a+h_c-h_b}(b-a)^{h_a+h_b-h_c}(b-c)^{h_b+h_c-h_a}}$$

Superuniversal weights

- Current insertion at absorbing boundary $h_A = 1$
- Forcing the current to pass through a boundary point: stress energy tensor and $\psi_{1,5}$ with $h = 2$
- Current insertion at a twist in reflecting boundary $h_{LR} = 0$
- "Lift-off" points $h_l = 1$
- Consequence: boundary condition change

$$h_c = 0$$

- PCC with reflecting interval

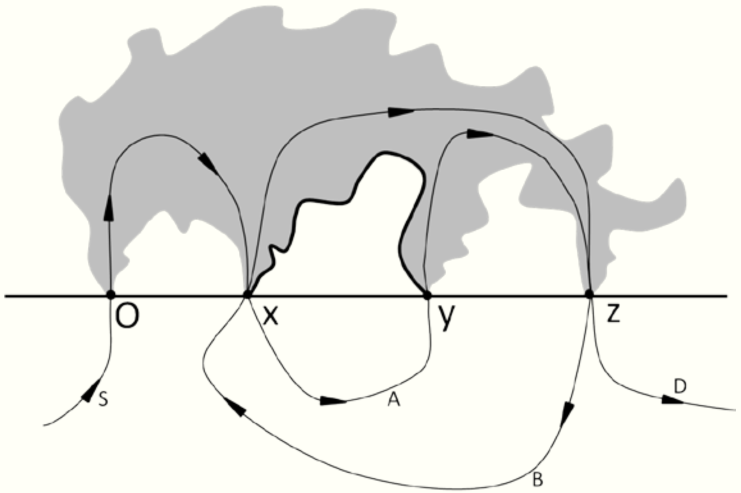


$$\overline{g_{bc}} = \frac{C}{(b-a)^{h_m - h_A}(c-b)^{h_m + h_A}(c-a)^{h_A - h_m}}$$

More complicated PCC

- Use external wires A and B

- Consider a particular combination g_c of conductances with and without these wires, so that the point X corresponds to $\psi_{1,5}$



- This combination satisfies

$$(\mathcal{L}_{-1}^3 - 6\mathcal{L}_{-2}\mathcal{L}_{-1} + 6\mathcal{L}_{-3})(3\mathcal{L}_{-1}^2 - 2\mathcal{L}_{-2})g_c = 0$$

$$\mathcal{L}_{-m} = \sum_i \left[\frac{(m-1)h_i}{(z_i - x)^m} - \frac{1}{(z_i - x)^{m-1}} \partial_{z_i} \right]$$

- Choice of solution depends on the model

Other systems and conformal restriction

E. Bettelheim, IAG, A. W. W. Ludwig, 2010

- Same approach applies to other disordered systems in 2D:

  - Spin QH transition where we have an exact mapping to 2D percolation

    IAG, A. W. W. Ludwig, N. Read, 1999
    E. J. Beamond, J. Cardy, J. T. Chalker, 2002
    A. D. Mirlin, F. Evers, A. Mildenberger, 2003

    In this case all the dimensions are known analytically

  - The classical limit of CC model (diffusion in strong magnetic fields)

    S. Xiong, N. Read, A. D. Stone, 1997

  - Metal in class D

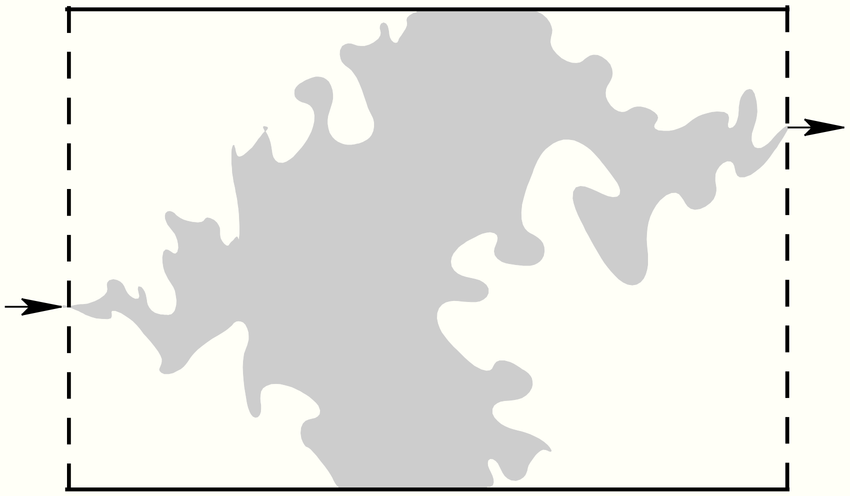
    T. Senthil, M. P. A. Fisher, 2000
    M. Bocquet, D. Serban, M. R. Zirnbauer, 2000
    J. T. Chalker, N. Read, V. Kagalovsky, B. Horovitz, Y. Avishai, A. W. W. Ludwig, 2002
    A. Mildenberger, F. Evers, A. D. Mirlin, J. T. Chalker, 2007

    In these cases all the dimensions are known analytically
    in terms of the Hall angle

Conclusions and future directions

- Conformal restriction: a new exact analytical approach to quantum Hall transitions

- Other boundary conditions and (degenerate) operators



- Numerical studies

S. Bera, F. Evers, H. Obuse, in progress

Conclusions and future directions

- Conformal restriction in the bulk: bulk-boundary and bulk-bulk PCCs
- “Massive” (off-critical) restriction: exponent ν and scaling functions
- Other Anderson transitions and disordered systems