Topological Field Theory for p-wave SCs

with an introduction to topological field theories

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Outline:

- **Topological terms and Topological Field Theories**
  - **Chern-Simons theories for the Laughlin states**
  - **BF theories for Abelian superconductors**
- **TFT for Topological Insulators**
- **Some theorist's facts about p-wave SCs**
- **Majorana fermions and TFT**
  - **p-wave superconductors**
  - **The Moore-Read Pfaffian Quantum Hall state**
  - **Open questions**

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Effective theories for the Laughlin states

A. Microscopic theories with Chern-Simons terms

B. Wen's Topological Field Theory

C. The effective action for the EM field

Recall:

The Laughlin states are incompressible Quantum Hall liquids with filling fraction 1/k (k odd) which support quasiparticles with charge e/k and statistics π/k.

A. Microscopic theory with CS term in action

$$\mathcal{L}(\phi, \vec{A}) = \phi^\dagger(i\partial_t + eA_0)\phi(\vec{r},t) - \frac{1}{2m}|(-i\vec{\nabla} - e\vec{A})\phi(\vec{r},t)|^2 - V(\rho)$$

$\phi(\vec{r},t)$	***Fermionic* field**	m	**Electron mass**
$\rho = \phi^\dagger\phi$	**Density field**	$V(\rho)$	**External potential**

***Rewrite* this as:**

$$\mathcal{L}(\phi, \vec{A}) = \phi^\dagger(i\partial_t + eA_0 - \underline{\tilde{a}_0})\phi(\vec{r},t)$$
$$- \frac{1}{2m}|(-i\vec{\nabla} - e\vec{A} + \underline{\vec{\tilde{a}}})\phi(\vec{r},t)|^2 - \frac{1}{2\pi k}\underline{\tilde{a}_0\tilde{b}} - V(\rho)$$

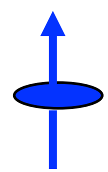
Where we introduced the Chern-Simons term:

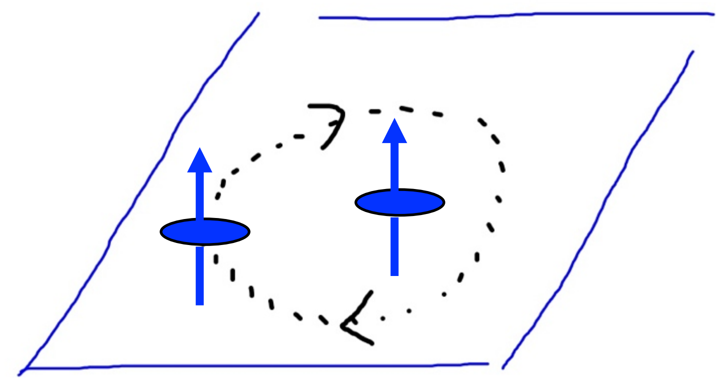
$$\frac{1}{2\pi k}\tilde{a}_0\tilde{b} = \frac{1}{2\pi k}\tilde{a}_0\epsilon^{ij}\partial_i\tilde{a}_j \;\; \text{"} = \text{"} \;\; \frac{1}{4\pi k}\epsilon^{\mu\nu\sigma}\tilde{a}_\mu\partial_\nu\tilde{a}_\sigma$$

$$
\begin{aligned}
\mathcal{L}(\phi,\vec{A}) &= \phi^\dagger(i\partial_t + eA_0 - \underline{\tilde{a}_0})\phi(\vec{r},t) \\
&- \frac{1}{2m^\star}|(-i\vec{\nabla} - e\vec{A} + \underline{\tilde{\vec{a}}})\phi(\vec{r},t)|^2 - \frac{1}{2\pi k}\tilde{a}_0\underline{\tilde{b}} - V(\rho)
\end{aligned}
$$

The Chern-Simons field $\tilde{a}_\mu$ attaches flux to charge:

$$
\rho = \frac{1}{2\pi k}\tilde{b}
$$





and the resulting Aharonov-Bohm phase changes the statistics of the electrons:

For even k :	$\phi(\vec{r},t)$	**is a *Fermionic* field, describing Composite Fermions**
For odd k :	$\phi(\vec{r},t)$	**is a *Bosonic* field, describing Composite Bosons**

B. Wen's effective Chern-Simons theory

Find multi-vortex solutions in the bosonic *mean field approx*, and parametrize:

$$\phi = \sqrt{\rho}\, e^{i\varphi} \qquad j^{\mu} = \frac{1}{2\pi}\epsilon^{\mu\nu\sigma}\partial_{\nu}\partial_{\sigma}\varphi$$

Vortex current

$$J^{\mu}_{em} = \frac{e}{2\pi}\epsilon^{\mu\nu\sigma}\partial_{\nu}a_{\sigma}$$

Dual Chern-Simons field a

Integrate $\rho, \theta, \tilde{a}_{\mu}$ and keep lowest derivative terms,

$$\boxed{\mathcal{L}(a, A; j) = \frac{k}{4\pi}\epsilon^{\mu\nu\sigma}a_{\mu}\partial_{\nu}a_{\sigma} + A_{\mu}\epsilon^{\mu\nu\sigma}\frac{e}{2\pi}\partial_{\nu}a_{\sigma} - a_{\mu}j^{\mu}}$$

Hall current **Quasi particle current**

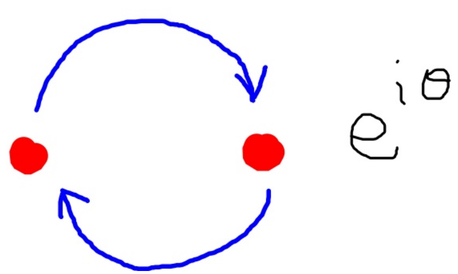
- **Topological Field Theory**
- **Generalizes to a large class of QH states**

C. Effective EM action & quasiparticle current

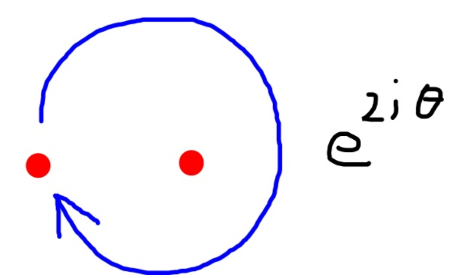
Integrate the CS field a_μ to get:

$$\mathcal{L}(A;j) \quad = \quad -\frac{e^2}{4\pi k}\epsilon^{\mu\nu\sigma}A_\mu\partial_\nu A_\sigma + \frac{e}{k}A_\mu j^\mu - \frac{\pi}{k}j^\mu\left(\frac{1}{d}\right)_{\mu\nu} j^\nu$$

- **Electromagnetic response of the QH liquid** $\dfrac{e^2}{4\pi k\hbar} = \dfrac{1}{2}\dfrac{1}{k}\sigma_0 = \dfrac{\sigma_H}{2}$
- **Fractional quasiparticle charge** $\dfrac{e}{k} = e^\star$
- **Fractional quasiparticle statistics** $\theta = \pi/k$



Exchange phase



Braiding phase

The effective Chern Simons theory encodes

- **the electromagnetic response**
- **the charge of the quasi-particles**
- **the statistics of the quasiparticles**
- **the physics of the edge states**
- **the ground state degeneracy on higher genus surfaces**
- **the response to curvature***

this is the proper "topological field theory"

* By including the "spin vector"

Effective theories for s-wave SCs

A. Microscopic theories

B. Topological Field Theory of BF type

C. The London action for the EM field

Recall:

An s-wave SC is a fully gapped state. The elementary excitations are neutral quasiparticles ("spinons") and quantized vortices carrying flux $\phi_0/2$.

A. Microscopic theories

I. BCS Theory

II. Ginzburg-Landau Theory (Abelian Higgs toy model in 2+1 d)

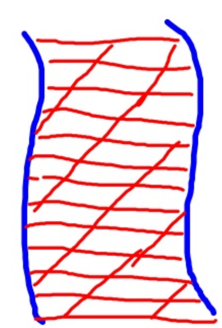
$$\mathcal{L}(\phi, \vec{A}) = \phi^\dagger (i\partial_t + 2eA_0)\phi(\vec{r},t) - \frac{1}{2m^\star}|(-i\vec{\nabla} - 2e\vec{A})\phi(\vec{r},t)|^2 - V(\rho) - \mathcal{L}_{Max}$$

$\phi(\vec{r},t)$ **Charge 2e *Boson* field** $m^\star$ **Effective Cooper pair mass**

Consider a configuration with a number vortices and quasiparticles described by the currents: j

d = 2+1 $j_{\rm v}^{\mu} = \frac{1}{\pi}\epsilon^{\mu\nu\sigma}\partial_\nu a_\sigma$

d = 3+1 $j_{\rm v}^{\mu\nu} = \frac{1}{\pi}\epsilon^{\mu\nu\sigma\lambda}\partial_\sigma a_\lambda$



B. Effective BF theory (d=2+1)

Instead of integrating, use physics reasoning:

- **In the QH case charge and flux come together, so there is one conserved current, and a single topological gauge potential.**
- **In the SC, charge and vorticity are separately conserved, so we expect two topological gauge potentials.**
- **There is a single non-trivial braiding phase:**
  Moving a quasiparticle around a vortex gives -1

$$\mathcal{L}_{BF}(a,b;j_q,j_v) = \frac{1}{\pi}\epsilon^{\mu\nu\rho}b_\mu\partial_\nu a_\rho - j^\mu_{\rm q} a_\mu - j^\mu_{\rm v} b_\mu$$

"BF" lagrangian

qp current

Vortex current

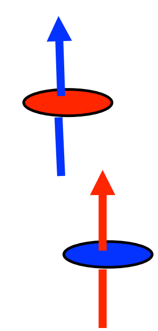
Constraints:

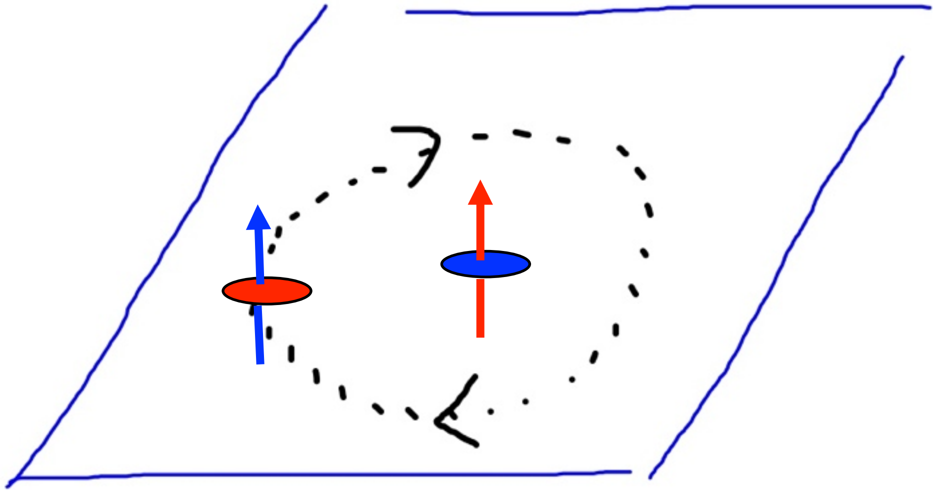
$$a_0 \ : \ \epsilon^{ij}\partial_i b_j = -\pi j^0_q = \pi\rho_q$$

Attaches b-flux to qp

$$b_0 \ : \ \epsilon^{ij}\partial_i a_j = \pi j^0_v = \pi\rho_v$$

Attaches a-flux to vort.





The AB phase gives the sign corresponding to braiding a qp around a vortex!

The BF theory is a TFT that encodes:

- Screening of charges and currents
- Correct braiding phases
- Preserves P and T symmetries
- The physics of edge modes
- Ground state degeneracy on a torus

C. Effective EM action

The characteristic EM response of a superconductor is the Meissner effect that signals that the U(1) gauge symmetry related to electro-magnetism is spontaneously broken. This effect is coded in the London term, which is essentially a mass term for the photon.

Since the TFT does not involve any length scale, we must go beyond the "topological scaling" limit, in order to see the London penetration length. Starting from the Abelian Higgs model we can keep higher derivative terms, and after integrating the b-field we get:

$$\mathcal{L}_{em} = -\frac{1}{4e^2}(f^{(a)}_{\mu\nu})^2 - \frac{1}{2e^2}m_s^2 a_\mu a^\mu \qquad m_s^2 = \lambda_L^{-2} = 4e^2\rho/m^\star$$

The a-field is now not any longer topological, but describes the long-distance behavior of the electromagnetic field A.

Effective theories for Top. Insulators

A. Microscopic theories:

Band theory + strong SO effects

Top. invariants: Chern #s and Z_2 invariants

B. BF Topological Field Theory (J.E Moore & G.Y. Cho)

Ann. Phys. **326**, 1515 2011, 1515-1535

C. Effective topological action for the EM field

$$\mathcal{L}_{EM} = \frac{\theta e^2}{2\pi h}\mathbf{E}\cdot\mathbf{B} = \frac{\theta e^2}{16\pi h}\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}.$$

"Axion Electrodynamics" giving magneto-electric response

Some theorist's facts about p-wave SCs

A (2+1) dim spinless ($p_x + i\, p_y$) paired superconductor:

- Is fully gapped, i.e. no nodal particles.
- Breaks P and T symmetry.
- Supports neutral E=0 states, "Majoranas", at the core of vortices, which obey Non-Abelian statistics of Ising type.
- Supports chiral fermion edge modes.
- For odd number of vortices in the bulk the edge supports an E=0 edge mode.

Warning: These are theoretical results experiments are still not conclusive

Effective theories for p-wave SCs

Work with A. Karlhede and M. Sato

The aim of this section is to find a TFT of type B, that encodes the properties on the previous slide.

Rather than starting from a microscopic theory, we first guess the Lagrangian based on general principles/prejudices, and then check if it has the desired properties:

- **Include fundamental Majorana fields $\gamma(x)$**
- **Break P & T symmetry**
- **Be a TFT of BF type**
- **Have no bulk d.o.f. for the field $\gamma(x)$**

After some thinking & trying:

$$\begin{aligned}\mathcal{L}_{pw} &= \frac{1}{\pi}\epsilon^{\mu\nu\rho}(b_\mu + \frac{1}{4}\tilde{b}_\mu)\partial_\nu a_\rho - j_{\rm q}^\mu a_\mu - j_{\rm v}^\mu b_\mu + \mathcal{L}_{\rm ed} + \mathcal{L}_{\rm gf} \\ &= \frac{1}{\pi}\epsilon^{\mu\nu\rho}(b_\mu + \frac{1}{4}\gamma\, i\partial_\mu\gamma)\partial_\nu a_\rho - j_{\rm q}^\mu a_\mu - j_{\rm v}^\mu b_\mu + \mathcal{L}_{\rm ed} + \mathcal{L}_{\rm gf}\end{aligned}$$

Ordinary BF term

BF term formed from Majoranas $\sim \epsilon^{\mu\nu\rho} a_\mu \partial_\nu\gamma\partial_\rho\gamma$

BF term is non zero because: $\{\gamma(\vec{r},t),\gamma(\vec{r}\,',t)\} = 0$

BF term is hermitan because: $\gamma(\vec{r},t)^\dagger = \gamma(\vec{r},t)$

Global symmetries: **The new term breaks P and T**

Local (gauge) symmetries:

Bosonic as before

$$\begin{aligned} a_\mu &\rightarrow a_\mu + \partial_\mu\Lambda_a \\ b_\mu &\rightarrow b_\mu + \partial_\mu\Lambda_b \end{aligned}$$

Fermionic New

$$\begin{aligned} \gamma &\rightarrow \gamma + 2\theta, \\ b_\mu &\rightarrow b_\mu - i\theta\partial_\mu\gamma/2 - i\gamma\partial_\mu\theta/2 - i\theta\partial_\mu\theta \end{aligned}$$

The Majorana action

E.o.m. for b, $a = \frac{\pi}{d} j_{\mathrm{v}} + d\Lambda_a$**, and substitution in the Lagrangian gives:**

$$\mathcal{L}_{pw} = \frac{1}{4} j_{\mathrm{v}}^{\mu} \gamma i \partial_\mu \gamma - j_{\mathrm{v}} \frac{\pi}{d} j_{\mathrm{q}} = \mathcal{L}_{\mathrm{maj}} - j_{\mathrm{v}} \frac{\pi}{d} j_{\mathrm{q}}$$

Take an external vortex current of N Wilson loops: $W_{C_a} = \exp(i \oint_{C_a} dx^\mu b_\mu)$

equivalent to:

$$j_{\mathrm{v}}^{\mu}(x^\mu) = \sum_{a=1}^{N} \int_0^1 d\tau\, \delta^3(x^\mu - x_a^\mu(\tau)) \frac{dx_a^\mu(\tau)}{d\tau}$$

$$\mathcal{S}_{\mathrm{maj}} = \frac{1}{4} \sum_{a=1}^{N} \int_0^1 d\tau\, \gamma_a(\tau) i \partial_\tau \gamma_a(\tau)$$

$$= \frac{1}{4} \sum_{a=1}^{2N} \int_{-\infty}^{\infty} dt\, \gamma_a(t) i \partial_t \gamma_a(t)$$

$$\gamma_a(t) \equiv \gamma(\vec{x}_a(t), t)$$

Non-Abelian fractional statistics

Canonical quantization: $$\{\hat{\gamma}_a(t), \hat{\gamma}_b(t)\} = 2\delta_{ab}$$

$$H = 0$$

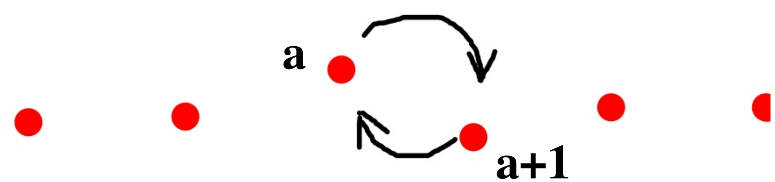
Define Dirac fermions:

$$\hat{\psi}_A = \frac{1}{2}(\hat{\gamma}_{2A-1} + i\hat{\gamma}_{2A}) \qquad A = 1, 2 \ldots N/2 \qquad \{\hat{\psi}_A^\dagger, \hat{\psi}_B\} = \delta_{AB}$$

With $\hat{\psi}_A|1\rangle = |0\rangle$ **, the state vectors for N vortices become**

$$|\vec{x}_1 \ldots \vec{x}_{2N}; \alpha_1 \cdots \alpha_{N/2}\rangle = |\vec{x}_1 \ldots \vec{x}_{2N}\rangle \otimes |\alpha_1\rangle \otimes \cdots \otimes |\alpha_{N/2}\rangle$$

Consider the braiding:



There are only two unitary and bosonic operators that respect the Majorana reality condition:

$$U_\pm = e^{\pm\frac{\pi}{4}\hat{\gamma}_a\hat{\gamma}_{a+1}}$$

Ivanov has shown that both cases imply Non-Abelian Ising statistics!

GS degeneracy on the torus

Solve the constraints without any sources:

$$a_i = \partial_i \Lambda_a + \overline{a}_i / L_i$$

$$\beta_i = b_i + \tilde{b}_i = \partial_i \Lambda_\beta + \overline{\beta}_i / L_i$$

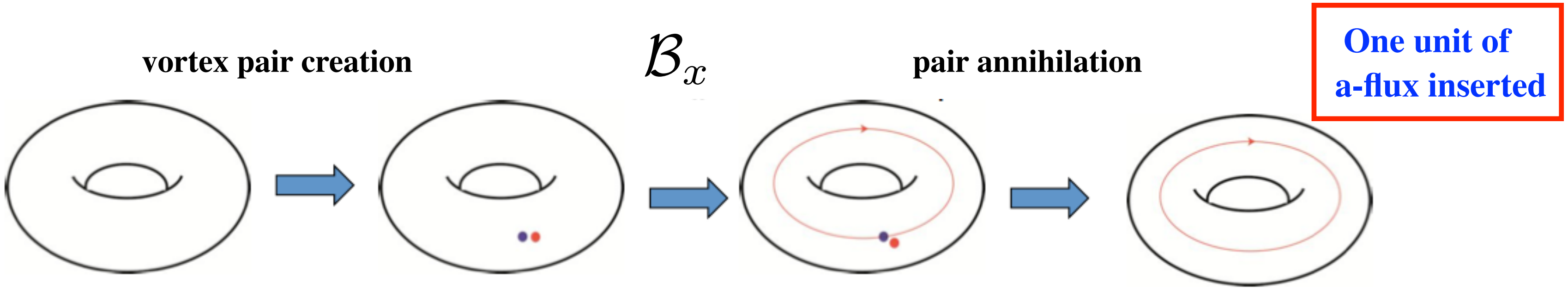
$$\overline{a}_i = \int_{C_i} d\vec{x} \cdot \vec{a}$$

$$\overline{\beta}_i = \int_{C_i} d\vec{x} \cdot \vec{\beta}$$

QM Lagrangian for the constant modes: $$L = \frac{1}{\pi} \epsilon^{ij} \dot{\overline{a}}_i \overline{\beta}_j$$

"Flux insertion operators" corresponding to "dragging" quasiparticles and vortices around the cycles of the torus:

$$\mathcal{A}_i = e^{i\,\overline{a}_i} \quad \text{and} \quad \mathcal{B}_i = \int_{\gamma \in C_i} \mathcal{D}\gamma \, e^{i[\overline{\beta}_i - \frac{i}{4}\int_{C_i} dx^i \gamma \partial_i \gamma]}$$



GS degeneracy on the torus (cont.)

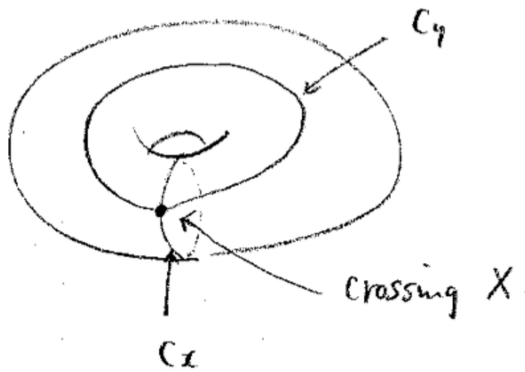
Algebra: $\{\mathcal{A}_x, \mathcal{B}_y\} = \{\mathcal{A}_y, \mathcal{B}_x\} = 0$

so using the a-representation: $\mathcal{A}_i |s_x, s_y\rangle = s_i |s_x, s_y\rangle$

as for the s-wave case, we expect four degenerate ground states:

$$|+,+\rangle \quad , \quad |+,-\rangle = \mathcal{B}_y|+,+\rangle \quad , \quad |-,+\rangle = \mathcal{B}_x|+,+\rangle \quad \text{and} \quad |-,-\rangle = \mathcal{B}_x\mathcal{B}_y|+,+\rangle$$

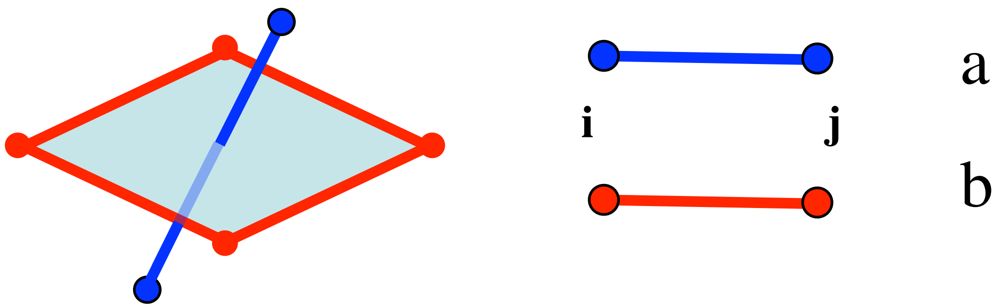
But the operator $\mathcal{B}_x\mathcal{B}_y$ **vanish because of the fermionic path integral!**



This looks like the "blocking mechanism" of Oshikawa, Kim, Sthengel, Nayak and Tewari.
(Ann. Phys. 322, 1477 (2007).)

But to see this in a more convincing way, we must be a bit more careful in defining the operators....

Discrete formulation & gauge fixing



1-form variables on the links

Z_2 version by Fisher & Senthil
PRB **62,** 7850 (**2000**)

D.H. Adams:

PRL **78**, 4155 (1997)

$$S_{BF} = \frac{1}{\pi}\int d^3x\, b \star^K d^K a$$

Vortex loops

$$\langle W_{C_1} \dots W_{C_n} \rangle = \sum_{\{a_{ij}, b_{ij}\}} e^{iS_{BF}[a,b]}\, e^{i\sum_{\langle ij\rangle \in \cup_a C_a} b_{ij}}$$

Add Majorana variables γ_i as Grassmann 0-forms living on the sites of the blue lattice

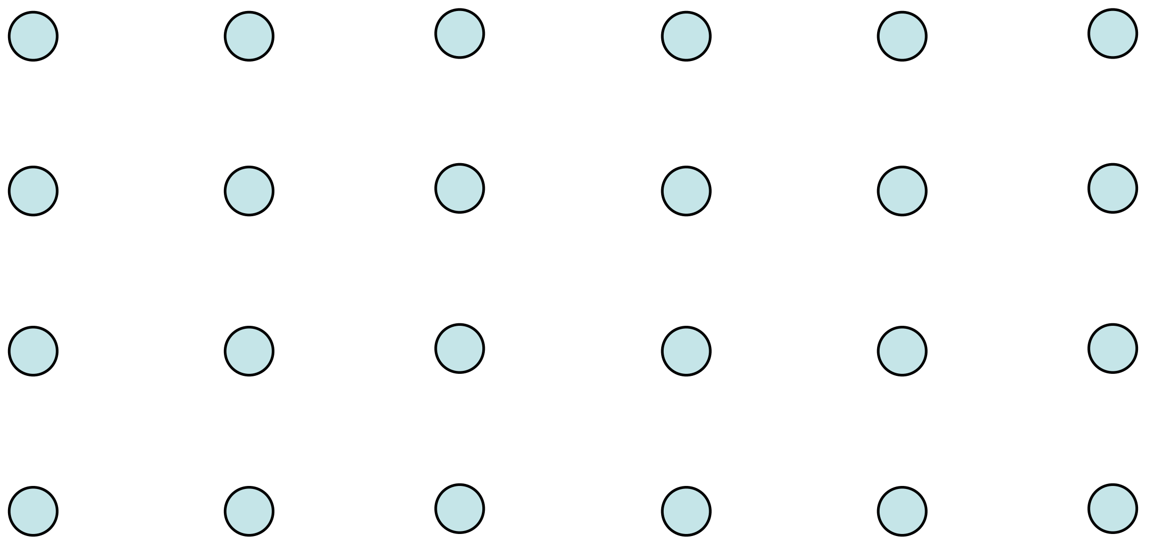
Discrete fermionic gauge fixing function: $$\mathcal{G}(\gamma_i) = \prod_{i \notin \cup_a C_a} \delta(\gamma_i)$$

Discrete form of $\mathcal{B}_x \mathcal{B}_y$:

$$(\mathcal{B}_x\mathcal{B}_y)' = \int \prod_{\gamma_k \in C_x \cup C_y} d\gamma_k\, e^{i\sum_{\langle ij\rangle \in C_x\cup C_y}\left(\beta_{ij} - \frac{i}{4}\gamma_i\gamma_j\right)}\,.$$

How the blocking works:

Even - Even lattice



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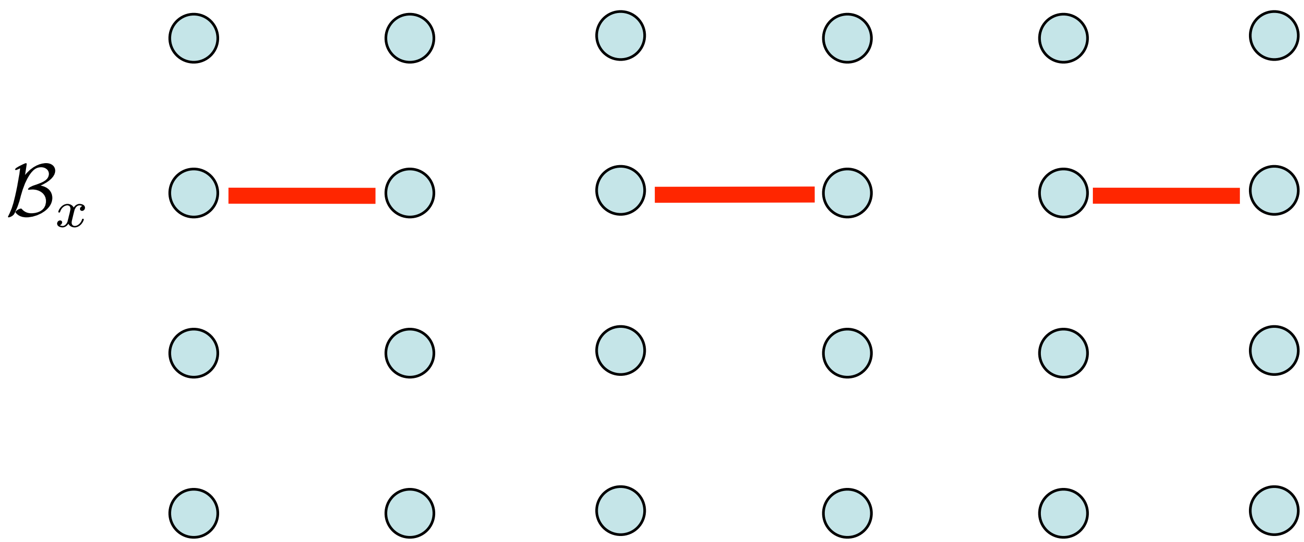

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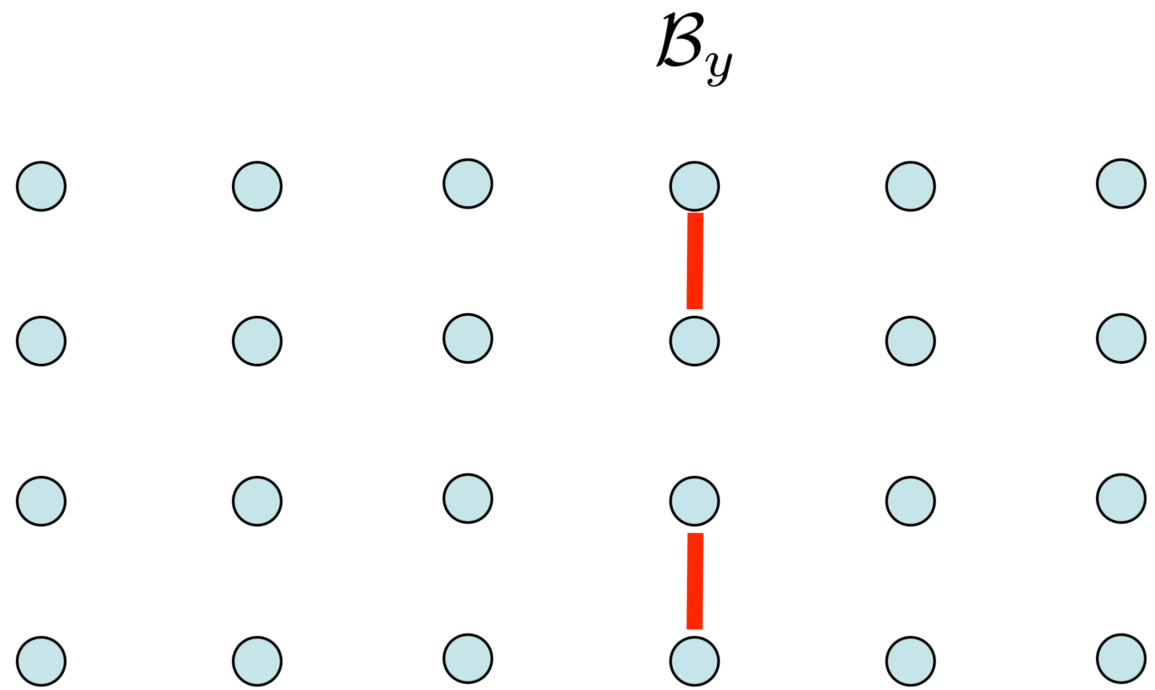

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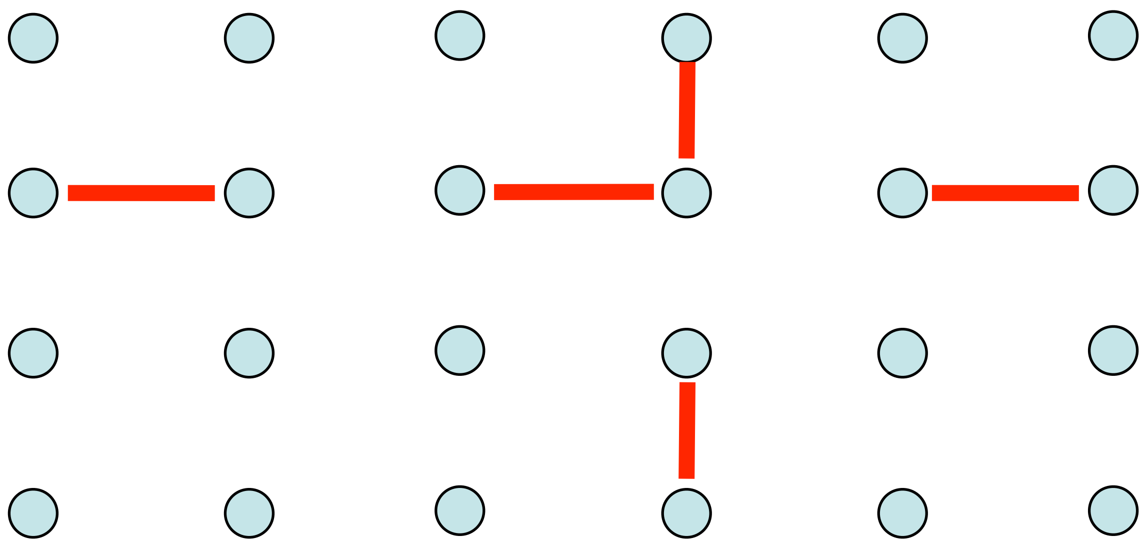
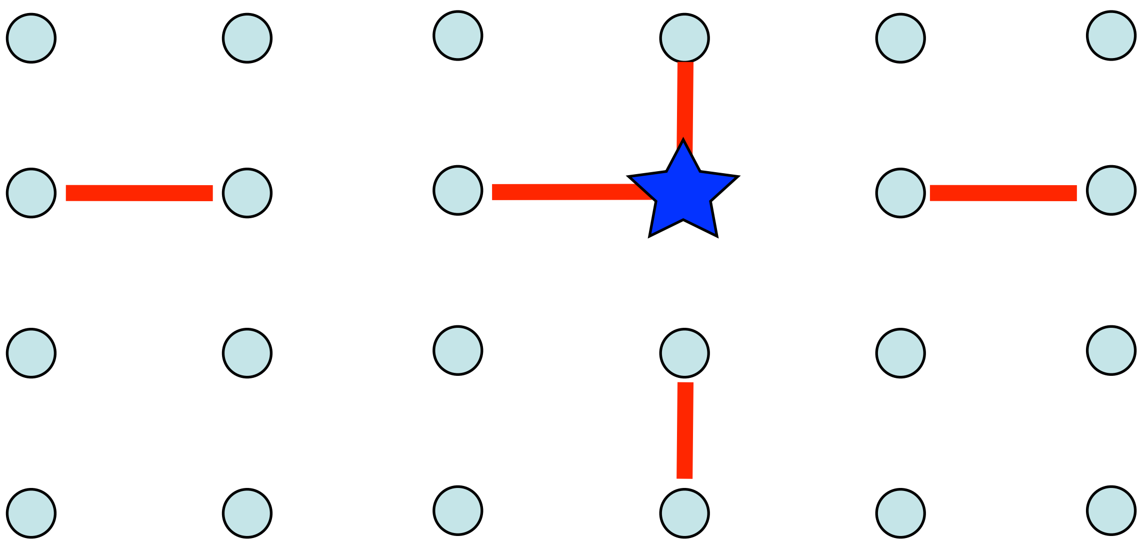


$$(\mathcal{B}_x \mathcal{B}_y)'$$

How the blocking works:

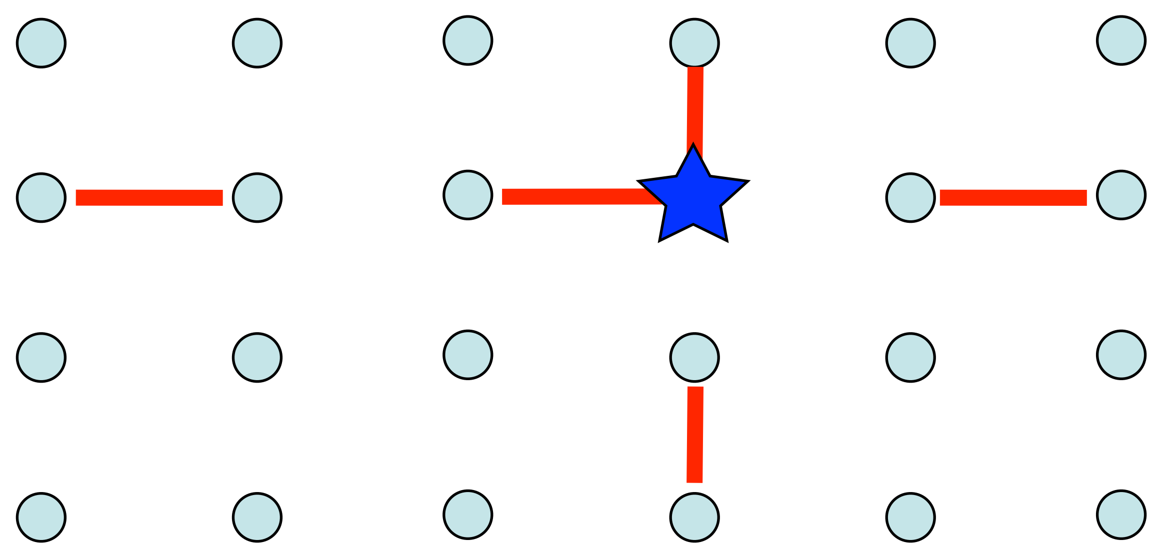
Even - Even lattice



$$\cancel{(\mathcal{B}_x\mathcal{B}_y)'}$$

How the blocking works:

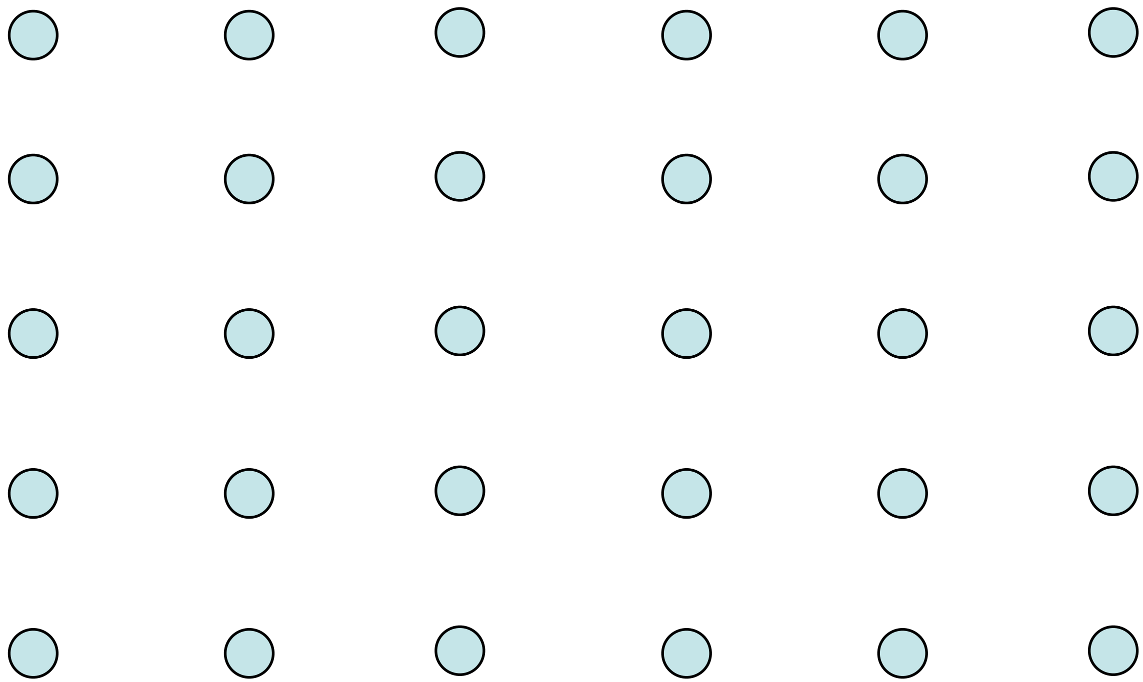
Even - Even lattice



$$\cancel{(\mathcal{B}_x\mathcal{B}_y)'}$$

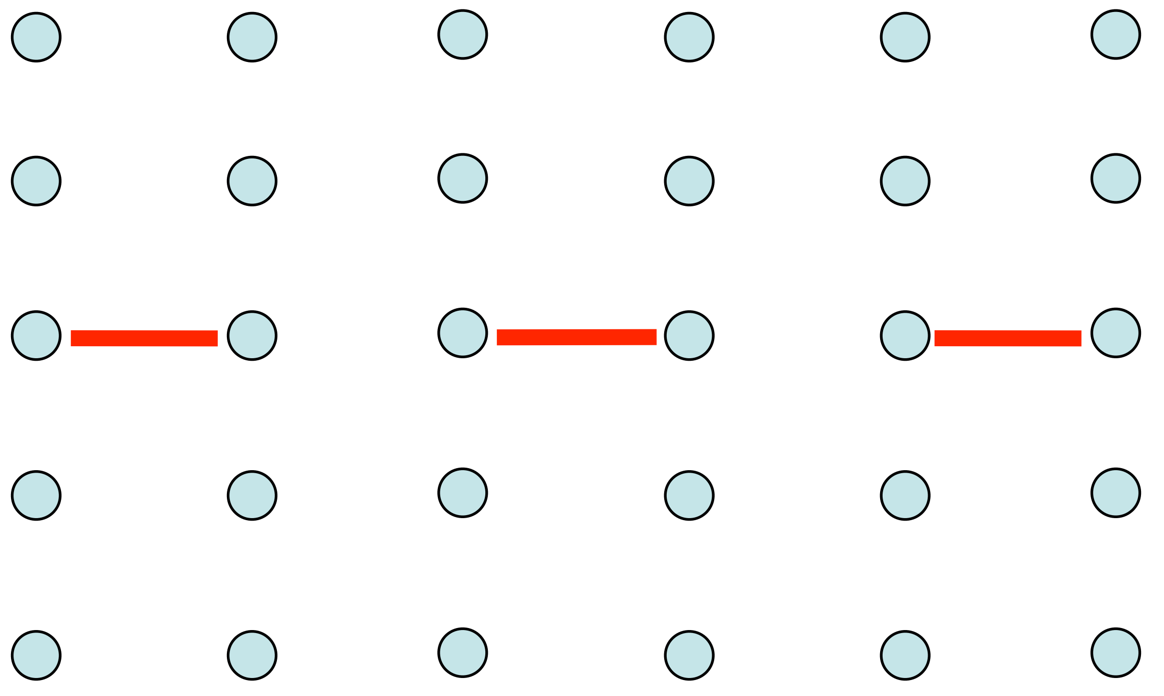
$$|+,+\rangle \quad |-,+\rangle \quad |+,-\rangle \quad \cancel{|-,-\rangle}$$

Even - Odd lattice

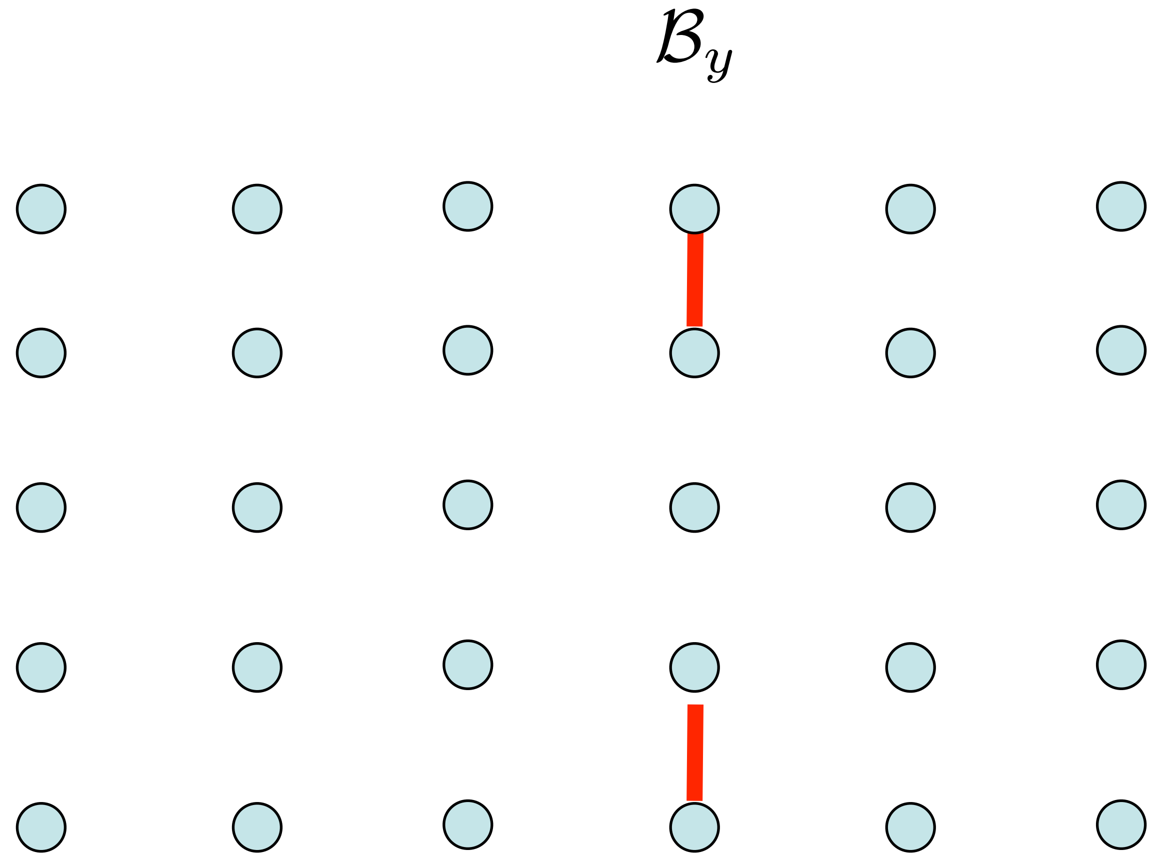


Even - Odd lattice

$\mathcal{B}_x$



Even - Odd lattice



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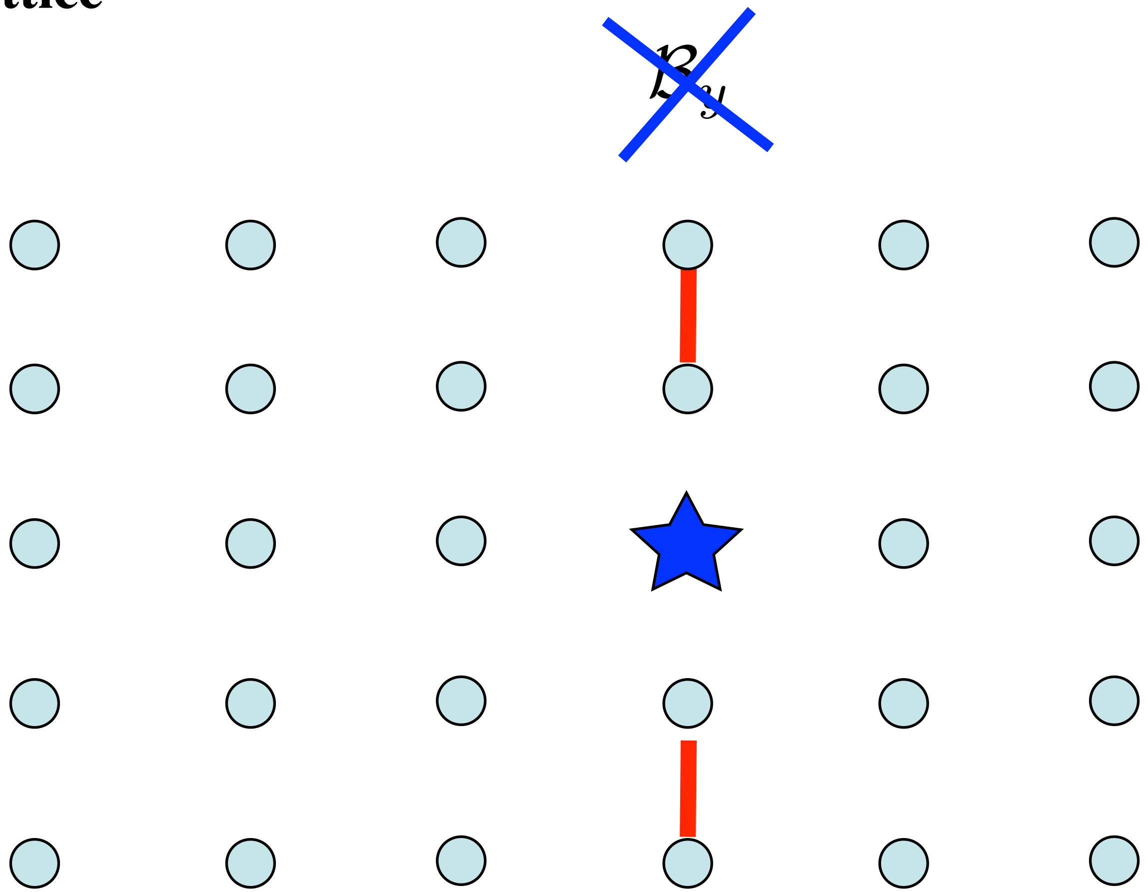
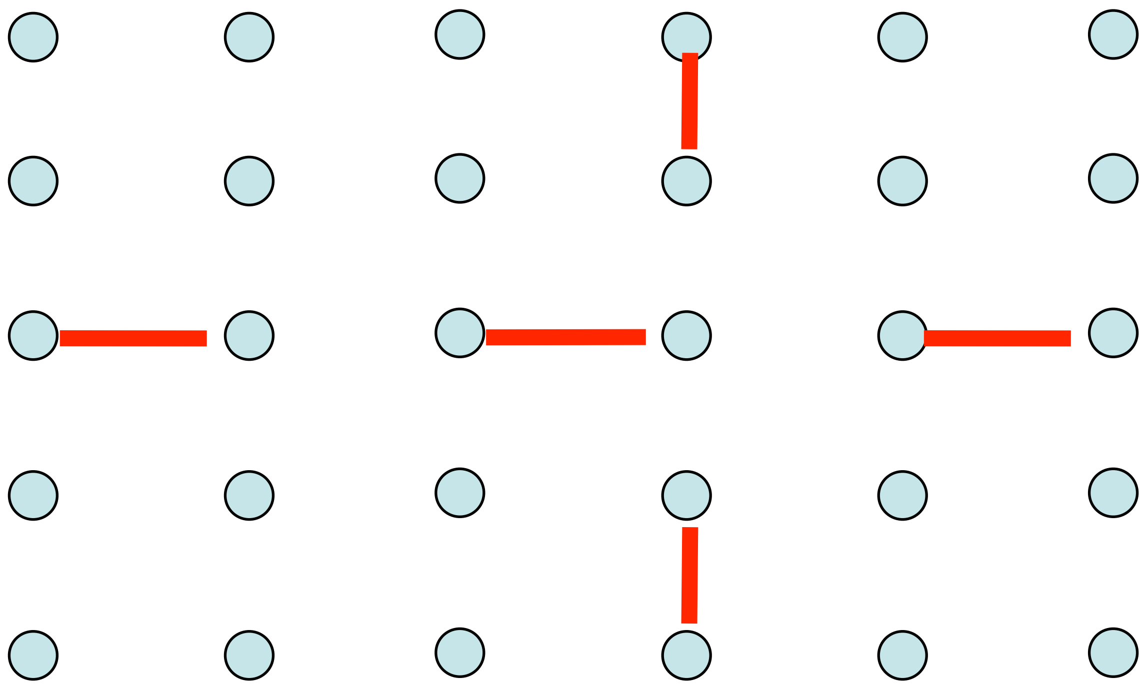
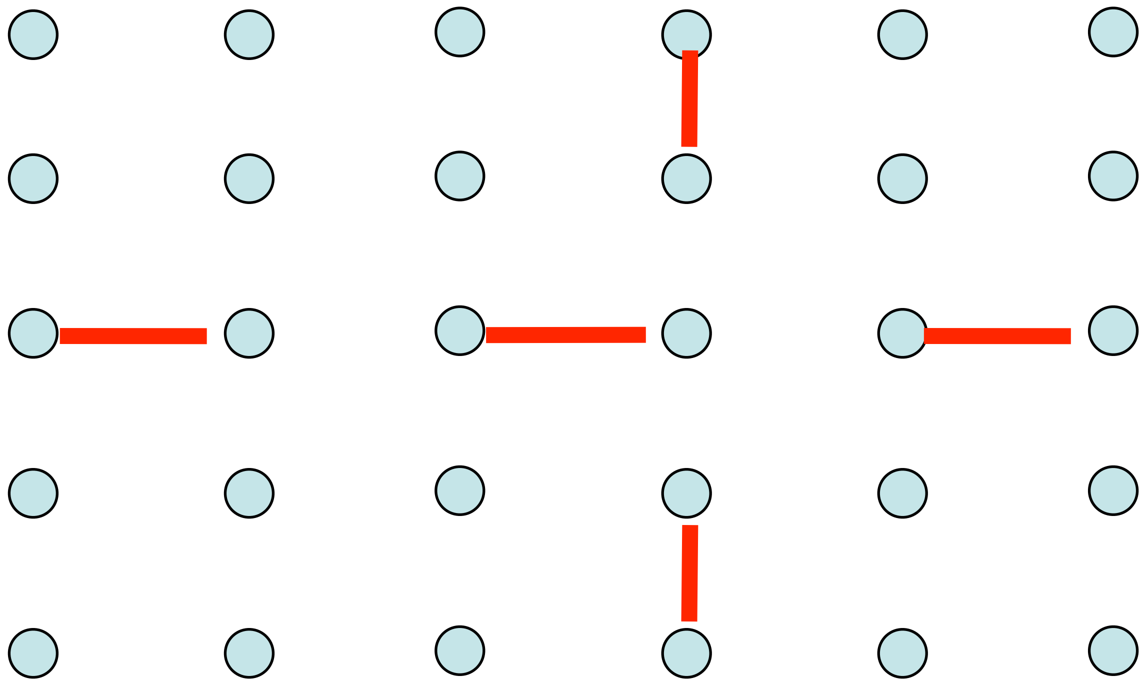


Even - Odd lattice



$$(\mathcal{B}_x\mathcal{B}_y)'$$

Even - Odd lattice



$$(\mathcal{B}_x\mathcal{B}_y)'$$

$$|+,+\rangle \quad |-,+\rangle \quad \cancel{|+,-\rangle} \quad |-,-\rangle$$

The Pfaffian connection

From the work of Moore and Read, Greiter, Wen and Wilczek, and Read and Green, we know that the NA Pfaffian state can be viewed as a p-wave paired state of composite fermions. Thus we expect that our effective BF theory should, *mutatis mutandis*, also describe the Pfaffian QH state.

In the QH system, charge and vorticity come together, and since the MR quasiparticle is a Laughlin hole split in half with charge e/4, we take:

$$\mathcal{L}_{\mathrm{MR}} = \frac{1}{\pi}\epsilon^{\mu\nu\rho}(b_\mu + \tilde{b}_\mu)\partial_\nu(a_\rho + \frac{e}{2}A_\rho) + \frac{e}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu A_\rho - j^\mu(\frac{1}{4}a_\mu + \frac{1}{2}b_\mu)$$

Proceeding as before we get:

$$\mathcal{L}_{\mathrm{MR}} = \frac{1}{2}\mathcal{L}_{\mathrm{maj}} - \frac{e^2}{8\pi}AdA + \frac{e}{4}jA - \frac{1}{8}j\frac{\pi}{d}j$$

QH effect

e/4 charge

Provides the Abelian phase needed to properly describe the e/4 NA quasiparticles

Generalization to 3+1 dimensions

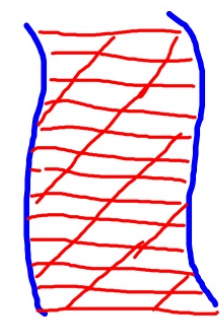
s-wave:

$$\mathcal{L}_{sw} = \frac{1}{\pi}\epsilon^{\mu\nu\sigma\lambda} b_{\mu\nu}\partial_\sigma a_\lambda - j_q^\mu a_\mu - j_v^{\mu\nu} b_{\mu\nu}$$

Antisymm. tensor gauge potential

Source describing the worldline of the quasiparticle

Source describing the worldsheet of the vortex



p-wave:

$$\mathcal{L}_{pw} = \frac{1}{\pi}\epsilon^{\mu\nu\sigma\lambda}(b_{\mu\nu} + \tilde{b}_{\mu\nu})\partial_\sigma a_\lambda - j_q^\mu a_\mu - j_v^{\mu\nu} b_{\mu\nu}$$

$$\tilde{b}_{\mu\nu} = \frac{i}{2}\gamma^T \sigma_\mu \partial_\nu \gamma \qquad \gamma^T = (\gamma_1, \gamma_2), \qquad \sigma_\mu = (\mathbf{1}, \sigma_i)$$

The edge action

$$\mathcal{L}_{ed} = \frac{i}{4\pi}\epsilon^{\mu\nu\rho}\partial_\mu\left[a_\nu\gamma\,\partial_\rho\gamma\right]$$

Proceeding as before we get:

$$S_{\rm ed} = \frac{i}{4\pi}\int dxdt\,\gamma(x,t)[a_x\partial_0 - a_0\partial_x]\gamma(x,t)$$

Depends on the edge vorticity

Depends on the edge potential

The boundary condition must be chosen as periodic or antiperiodic depending on whether the number of vortices in bulk is odd or even.

Summary:

- TFTs of BF type for chiral SCs in d=2, 3
- Majorana modes at vortex cores
- Non-Abelian statistics of vortices
- Edge Majorana modes
- Correct GS degeneracy on torus
- Discrete formulation
- Direct connection to the MR QH state

Open problems:

- Explore the d=3 case more systematically
- Quantize the TFT and evaluate the braiding phases both in the continuum and lattice theories
- Derive the correct bcs for the edge modes
- Generalize to TRI, helical, SCs
- Derive action from a microscopic model

Thank you for listening!

A few background references:

Effective theory of the QH effect:

Shou Cheng Zhang, Int. Journ. of Mod. Phys., **B6**, 25 (1992).
X.-G. Wen, Adv. in Phys., **44**, 405 (1995).

Effective BF theory for s and d-wave superconductors:

T.H.H, V. Oganesyan and S. L. Sondhi, Ann. of Phys., **312**, 497 (2004).
M. Hermanns, *Order in 2D nodal superconductors*, master thesis, arXiv:0804.1332 .

Theory of p-wave superconductors and MR QH state:

N. Read and D. Green, Phys. Rev. B 61, 10267 (2000)
P. Fendley, M.P.A. Fisher and C. Nayak, Phys. Rev. B **75**, 045317 (2007)
M. Oshikawa, Y.B. Kim, K. Shtengel and C. Nayak, Ann. of Phys., **322**, 1477 (2007).