Logarithmic extensions of local scale-invariance

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Overview :

1. Critical phenomena
2. Ageing phenomena
3. Critical contact process (DP universality class)
4. Surface growth (KPZ universality class)
5. Form of the scaling functions & LSI
6. Logarithmic conformal invariance
7. Logarithmic ageing invariance
8. Numerical experiments (DP and KPZ classes in $1D$)
9. Conclusions

1. Critical phenomena

Equilibrium critical phenomena : **scale-invariance**
For sufficiently **local** interactions : extend to conformal invariance
space-dependent re-scaling (angles conserved) $\mathbf{r} \mapsto \mathbf{r}/b(\mathbf{r})$

BATEMAN & CUNNINGHAM 1909/10, POLYAKOV 70

In **two** dimensions : ∞ many conformal transformations
$(w \mapsto f(w)$ analytic)
$\Rightarrow$ exact predictions for critical exponents, correlators, ... BPZ 84

What about **time**-dependent critical phenomena ? CARDY 85
Characterised by **dynamical exponent** z : $t \mapsto tb^{-z}$, $\mathbf{r} \mapsto \mathbf{r}b^{-1}$

Can one extend to **local** dynamical scaling, with $z \neq 1$?
If $z = 2$, the **Schrödinger group** is an example : JACOBI 1842, LIE 1881

$$t \mapsto \frac{\alpha t + \beta}{\gamma t + \delta} \ , \ \ \mathbf{r} \mapsto \frac{\mathcal{D}\mathbf{r} + \mathbf{v}t + \mathbf{a}}{\gamma t + \delta} \ ; \ \ \alpha\delta - \beta\gamma = 1$$

$\Rightarrow$ study **ageing** phenomena as paradigmatic example

2. Ageing phenomena

known & practically used since prehistoric times (metals, glasses)
systematically studied in physics since the 1970s   STRUIK '78
occur in widely different systems
(structural glasses, spin glasses, polymers, simple magnets, . . .)

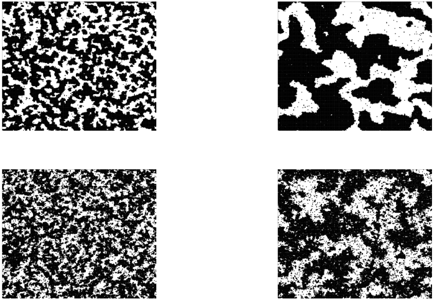
Three defining properties of **ageing** :

1. slow relaxation (non-exponential !)
2. **no** time-translation-invariance (TTI)
3. dynamical scaling   without fine-tuning of parameters

Most existing studies on 'magnets' : relaxation towards equilibrium

Question : what can be learned about intrisically **irreversible** systems by studying their ageing behaviour ?

$t = t_1$ $t = t_2 > t_1$

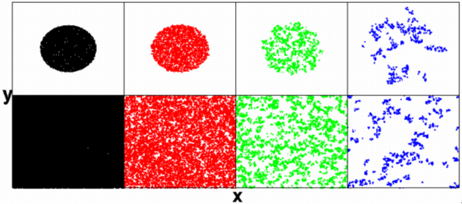


magnet $T < T_c$

$\longrightarrow$ ordered cluster

magnet $T = T_c$

$\longrightarrow$ correlated cluster



critical contact process

$\Longrightarrow$ **cluster dilution**

voter model, contact process,. . .

common feature : growing length scale

z : dynamical exponent

$$L(t) \sim t^{1/z}$$

Two-time observables

time-dependent order-parameter $\phi(t, \mathbf{r})$

two-time **correlator** $C(t,s) := \langle\phi(t,\mathbf{r})\phi(s,\mathbf{r})\rangle - \langle\phi(t,\mathbf{r})\rangle\langle\phi(s,\mathbf{r})\rangle$

two-time **response** $R(t,s) := \left.\frac{\delta\langle\phi(t,\mathbf{r})\rangle}{\delta h(s,\mathbf{r})}\right|_{h=0} = \left\langle\phi(t,\mathbf{r})\widetilde{\phi}(s,\mathbf{r})\right\rangle$

t : observation time, s : waiting time

Scaling regime : $t, s \gg \tau_{\rm micro}$ **and** $t - s \gg \tau_{\rm micro}$

$$C(t,s) = s^{-b} f_C\left(\frac{t}{s}\right) \ , \ R(t,s) = s^{-1-a} f_R\left(\frac{t}{s}\right)$$

asymptotics (for $y \gg 1$) **:** $f_C(y) \sim y^{-\lambda_C/z}, \ \ f_R(y) \sim y^{-\lambda_R/z}$

λ_C : autocorrelation exponent, λ_R : autoresponse exponent,
z : dynamical exponent, a, b : ageing exponents

(a) contact process : $A \xrightarrow{p} 2A$, $A \xrightarrow{1} \emptyset$ + diffusion — HARRIS 74
(b) percolation problem with preferred ($\|$) direction — BROADBENT & HAMMERSLEY 57
(c) Reggeon field theory — CARDY & SUGAR 80

absorbing (= **non**-equilibrium) stationary state

particle density : stat. $a_\infty = \langle A \rangle \sim (p - p_c)^\beta$; critical $a(t) \sim t^{-\beta/\nu_\|}$
relaxation time $\tau = \xi_\| \sim |p - p_c|^{-\nu_\|}$; correlation length $\xi_\perp \sim |p - p_c|^{-\nu_\perp}$
dynamical exponent $z = \nu_\| / \nu_\perp$

Effective action at criticality — JANSSEN, DE DOMINICIS 70s-80s

$$\mathcal{J}[\widetilde{\phi}, \phi] = \int \mathrm{d}t \mathrm{d}\mathbf{r} \, \left[\widetilde{\phi} \left(D\partial_t \phi - \nabla^2 \phi \right) - \kappa \widetilde{\phi} \left(\widetilde{\phi} - \phi \right) \phi \right]$$

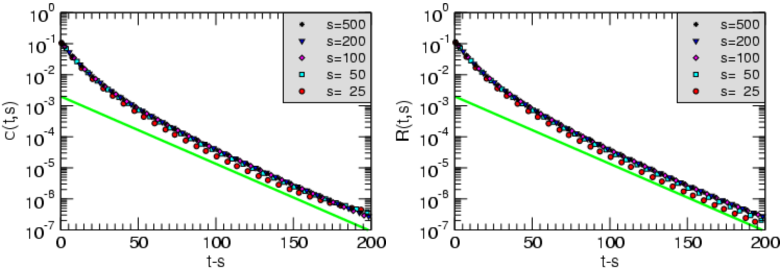
rapidity-reversal symmetry : $\mathcal{J}$ is invariant under — GRASSBERGER 79, JANSSEN 81

$$t \mapsto -t \ , \ \phi(t, \mathbf{r}) \mapsto -\widetilde{\phi}(-t, \mathbf{r}) \ , \ \widetilde{\phi}(t, \mathbf{r}) \mapsto -\phi(-t, \mathbf{r})$$

active (ordered) phase : $\lim_{t\to\infty} a(t) = \rho_\infty > 0$

absorbing (disordered) phase : $\lim_{t\to\infty} a(t) = 0$

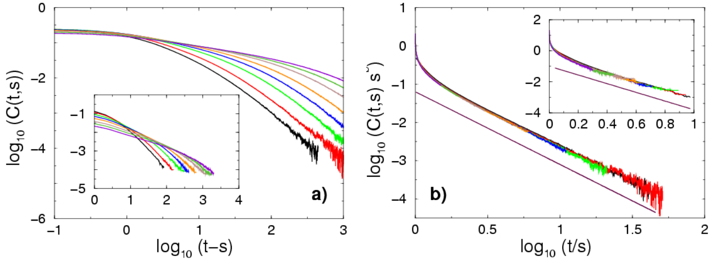
long-time behaviour in the active phase of the $1D$ contact process



contrast to magnets : $\left\{ \begin{array}{l} \textbf{fast} \text{ relaxation} \\ \textbf{no} \text{ scaling} \\ \text{time-translation-}\textbf{inv}. \end{array} \right\}$ in active phase

reason : single non-critical stationary state   Enss, mh, Picone, Schollwöck 04

ageing and scaling for $C(t,s)$: **critical** contact process



main figures : $1D$, insets : $2D$ — Ramasco, mh, Santos, da Silva Santos 04 ; Enss *et al.* 04

observe all **3** properties of **ageing** : $\left\{ \begin{array}{l} \text{slow dynamics} \\ \text{no TTI} \\ \text{dynamical scaling} \end{array} \right.$

<u>contrast</u> to critical magnets : $\boxed{a \neq b}$ $\Longrightarrow$ **no** finite FDR !

autocorrelation exponent : $\lambda_C = d + z + \beta/\nu_\perp$

Calabrese, Gambassi, Krzakala 06/07, Baumann & Gambassi 07

numerical values of some non-equilibrium exponents

contact process (CP) $A \to 2A$, $A \to \emptyset$, parity-conserved model (PC) $A \leftrightarrow 3A$, $2A \to \emptyset$, diffusion-coagulation (DC) $2A \to A$

	d	a	b	λ_C/z	λ_R/z		
CP	1	$-0.68(5)$	0.32(5)	1.85(10)	1.85(10)	TMRG	[1]
		$-0.57(10)$	0.3189	1.9(1)	1.9(1)	MC	[2]
		-0.6810			1.76(5)	MC	[3]
		-0.6810	0.3189	1.7921	1.7921	scal	[5]
	2	0.3(1)	0.901(2)	2.8(3)	2.75(10)	MC	[2]
		$-0.198(2)$	0.901(2)	2.58(2)	2.58(2)	scal	[5]
			0.9(1)	2.5(1)		**exp**	[6]
	> 4	$d/2-1$	$d/2$		$d/2+2$	MF	[2]
PC	1	$-0.430(4)$	0.570(4)	1.9(1)	1.9(2)	MC	[4]
		$-0.430(4)$	0.570(4)	1.86(1)	1.86(1)	scal	
DC	1	$-1/2$	1	2	2	**exact**	[7]

[1] ENSS *et. al.* 04 ; [2] RAMASCO *et. al.* 04 ; [3] HINRICHSEN 06 ; [4] ÓDOR 06 ;

[5] BAUMANN & GAMBASSI 07 ; [6] **Takeuchi** *et. al.* **09** ; [7] DURANG, FORTIN, MH 11

in the contact process $\boxed{1+a=b}$: $\Longleftarrow$ **rapidity-reversal symmetry** of stationary state of CP $\Rightarrow$ **specific property** !

why does $1+a=b$ also hold in the PC class **?**

$\Longrightarrow$ try **new form of FDR** ! ENSS *et. al.* 04 ; BAUMANN & GAMBASSI 07

$$\Xi(t,s) := \frac{R(t,s)}{C(t,s)} = \frac{f_R(t/s)}{f_C(t/s)} \quad , \quad \Xi_\infty := \lim_{s\to\infty}\left(\lim_{t\to\infty}\Xi(t,s)\right)$$

universal function, $\frac{1}{\Xi} \neq 0$ measures distance to stationary state

in $d = 4-\varepsilon$ dimensions, from an one-loop calculation B & G 07

$$\Xi_\infty = 2\left[1 - \varepsilon\left(\frac{119}{480} - \frac{\pi^2}{120}\right)\right] + \mathrm{O}(\varepsilon^2)$$

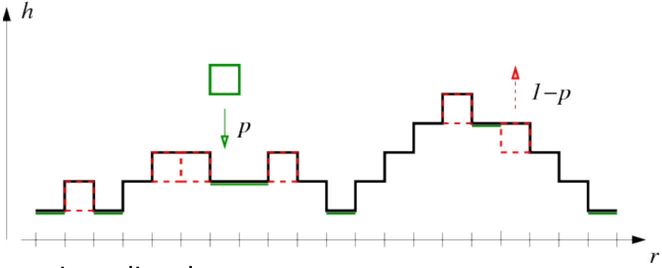
quantitatively consistent with TMRG estimate $\Xi_\infty = 1.15(5)$ in $1D$.

NB : $1+a=b$ **invalid** in other non-equilibrium universality classes $\Rightarrow$ need different forms of FDR ! BAUMANN *et. al.* 05 ; DURANG & MH 09, DURANG *et al.* 11

4. Surface growth

deposition (evaporation) of particles on a surface → height profile $h(t, \mathbf{r})$

generic situation : RSOS (**r**estricted **s**olid-**o**n-**s**olid) model Kim & Kosterlitz 89



$p =$ deposition prob.

$1 - p =$ evap. prob.

here $p = 0.98$

some universality classes :

(a) KPZ $\partial_t h = \nu \nabla^2 h + \frac{\mu}{2} (\nabla h)^2 + \eta$ Kardar, Parisi, Zhang 86

(b) EW $\partial_t h = \nu \nabla^2 h + \eta$ Edwards, Wilkinson 82

(c) MH $\partial_t h = -\nu \nabla^4 h + \eta$ Mullins, Herring 63 ; Wolf & Villain 80

η is a gaussian white noise with $\langle \eta(t, \mathbf{r}) \eta(t', \mathbf{r}') \rangle = 2\nu T \delta(t - t') \delta(\mathbf{r} - \mathbf{r}')$

Family-Viscek scaling on a spatial lattice of extent L^d : $\overline{h}(t) = L^{-d} \sum_j h_j(t)$

$$w^2(t;L) = \frac{1}{L^d} \sum_{j=1}^{L^d} \left\langle \left(h_j(t) - \overline{h}(t) \right)^2 \right\rangle = L^{2\zeta} f\left(tL^{-z}\right) \sim \begin{cases} L^{2\zeta} & ; \text{ if } tL^{-z} \gg 1 \\ t^{2\beta} & ; \text{ if } tL^{-z} \ll 1 \end{cases}$$

β : growth exponent, ζ : roughness exponent, $\boxed{\zeta = \beta z}$

two-time correlator :

$$C(t,s;\mathbf{r}) = \langle h(t,\mathbf{r}) h(s,\mathbf{0}) \rangle - \left\langle \overline{h}(t) \right\rangle \left\langle \overline{h}(s) \right\rangle = s^{-b} F_C \left(\frac{t}{s}, \frac{\mathbf{r}}{s^{1/z}} \right)$$

with ageing exponent : $\boxed{b = -2\beta}$ Kallabis & Krug 96

two-time integrated response :

* sample **A** with deposition rates $p_i = p \pm \epsilon_i$, up to time s,

* sample **B** with $p_i = p$ up to time s ;

then switch to common dynamics $p_i = p$ for all times $t > s$

$$\chi(t,s;\mathbf{r}) = \int_0^s \mathrm{d}u \, R(t,u;\mathbf{r}) = \frac{1}{L} \sum_{j=1}^{L} \left\langle \frac{h_{j+r}^{(\mathbf{A})}(t;s) - h_{j+r}^{(\mathbf{B})}(t)}{\epsilon_j} \right\rangle = s^{-a} F_\chi \left(\frac{t}{s}, \frac{|\mathbf{r}|^z}{s} \right)$$

Effective action of the KPZ equation :

$$\mathcal{J}[\phi,\widetilde{\phi}] = \int \mathrm{d}t\mathrm{d}\mathbf{r}\ \left[\widetilde{\phi}\left(\partial_t\phi - \nu\boldsymbol{\nabla}^2\phi - \frac{\mu}{2}(\boldsymbol{\nabla}\phi)^2\right) - \nu T\widetilde{\phi}^{\,2}\right]$$

$\Longrightarrow$ Very special properties of KPZ in $d=1$ spatial dimension !

Exact critical exponents $\beta=1/3,\ \zeta=1/2,\ z=3/2,\ \lambda_C=1$ KPZ 86 ; Krech 97

Special KPZ symmetry in $1D$: let $v=\frac{\partial\phi}{\partial r}$, $\widetilde{\phi}=\frac{\partial}{\partial r}\left(\widetilde{p}+\frac{v}{2T}\right)$

$$\mathcal{J} = \int \mathrm{d}t\mathrm{d}r\ \left[\widetilde{p}\partial_t v - \frac{\nu}{4T}(\partial_r v)^2 - \frac{\mu}{2}v^2\partial_r\widetilde{p} + \nu T(\partial_r\widetilde{p})^2\right]$$

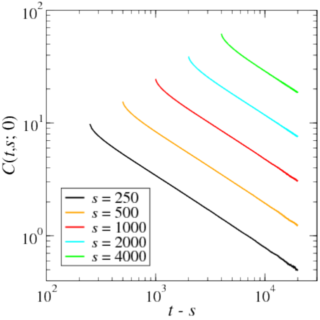
is invariant under **time-reversal** Lvov, Lebedev, Paton, Procaccia 93 ; Frey, Täuber, Hwa 96

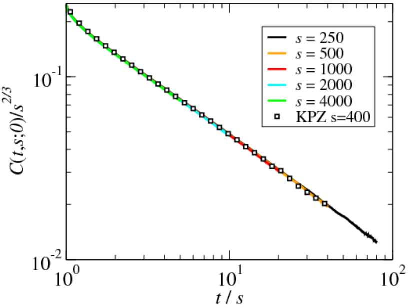
$$t\mapsto -t\ \ ,\ \ v(t,r)\mapsto -v(-t,r)\ \ ,\ \ \widetilde{p}\mapsto +\widetilde{p}(-t,r)$$

$\Rightarrow$ **fluctuation-dissipation relation** for $t\gg s$ $TR(t,s;r) = -\partial_r^2 C(t,s;r)$

find ageing exponents : $\lambda_R=\lambda_C=1,\ 1+a=b+\frac{2}{z}$

1D relaxation dynamics, starting from an initially flat interface





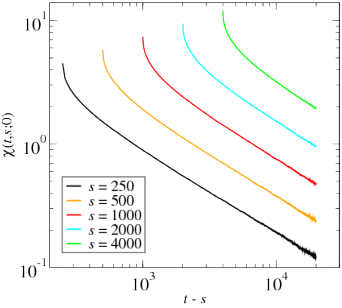
observe all **3** properties of **ageing** : $\left\{ \begin{array}{l} \text{slow dynamics} \\ \text{no TTI} \\ \text{dynamical scaling} \end{array} \right.$

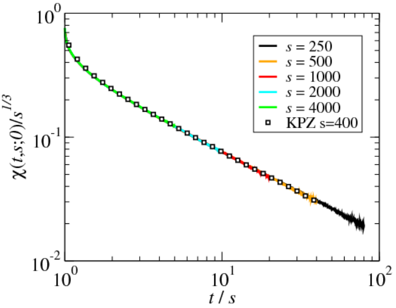
confirm expected exponents $b = -2/3$, $\lambda_C/z = 2/3$

N.B. : this confirmation is out of the stationary state

Kallabis & Krug 96 ; Krech 97 ; Bustingorry *et al.* 07-10 ; Chou & Pleimling 10 ; D'Aquila & Täuber 11

relaxation of the integrated response,$1D$



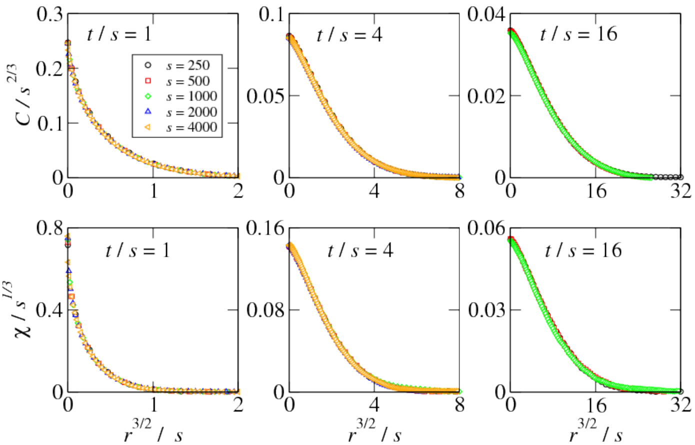


observe all **3** properties of **ageing** : { slow dynamics; no TTI; dynamical scaling }

exponents $a = -1/3$, $\lambda_R/z = 2/3$, as expected from FDR

N.B. : numerical tests for 2 models in KPZ class

Simple ageing is also seen in space-time observables



correlator $C(t,s;r) = s^{2/3} F_C\left(\frac{t}{s}, \frac{r^{3/2}}{s}\right)$

integrated response $\chi(t,s;r) = s^{1/3} F_\chi\left(\frac{t}{s}, \frac{r^{3/2}}{s}\right)$

confirm expected value of dynamical exponent $z = 3/2$

Values of some growth and ageing exponents in $1D$

model	z	a	b	$\lambda_R = \lambda_C$	β	ζ
KPZ	3/2	$-1/3$	$-2/3$	1	1/3	1/2
exp					0.336(11)	0.50(5)
EW	2	$-1/2$	$-1/2$	1	1/4	1/2
MH	4	$-3/4$	$-3/4$	1	3/8	3/2

Takeuchi, Sano, Sasamoto, Spohn 10/11

Two-time space-time responses and correlators consistent with simple ageing for $1D$ KPZ

Similar results known for EW and MH universality classes

Roethlein, Baumann, Pleimling 06

5. Form of the scaling functions & LSI

Observation : dynamical scaling generic property of non-equilibrium criticality
Question : can one extend non-equilibrium dynamical scaling ?

analogy : conformal invariance at equilibrium phase transitions, but $z = 1$ there $\Rightarrow$ **other important differences** ?

Schrödinger group, $z = 2$ JACOBI 1842, LIE 1881

$$t \mapsto t' = \frac{\alpha t + \beta}{\gamma t + \delta} \;\; , \;\; \mathbf{r} \mapsto \mathbf{r}' = \frac{\mathcal{D}\mathbf{r} + \mathbf{v}t + \mathbf{a}}{\gamma t + \delta} \;\; , \;\; \alpha\delta - \beta\gamma = 1$$

dynamical symmetry of free Schrödinger/diffusion equation

$$\mathcal{S}\phi(t,\mathbf{r}) = 0 \;\; , \;\; \mathcal{S} = 2\mathcal{M}\frac{\partial}{\partial t} - \frac{\partial}{\partial \mathbf{r}} \cdot \frac{\partial}{\partial \mathbf{r}}$$

consider infinitesimal transformations $(t', \mathbf{r}') = (t, \mathbf{r}) + \epsilon X(t, \mathbf{r})$

$$
\begin{aligned}
X_{-1} &= -\partial_t && \text{time translation} \\
X_0 &= -t\partial_t - \frac{1}{2}\mathbf{r}\cdot\partial_{\mathbf{r}} - \frac{x}{2} && \text{dilatation} \\
X_1 &= -t^2\partial_t - t\mathbf{r}\cdot\partial_{\mathbf{r}} - \frac{\mathcal{M}}{2}\mathbf{r}^2 - xt && \text{special transformation} \\
\mathbf{Y}_{-1/2} &= -\partial_{\mathbf{r}} && \text{spatial translations} \\
\mathbf{Y}_{+1/2} &= -t\partial_{\mathbf{r}} - \mathcal{M}\mathbf{r} && \text{Galilei transformations} \\
M_0 &= -\mathcal{M} && \text{phase shift}
\end{aligned}
$$

close into **Schrödinger Lie algebra** $\mathfrak{sch}(d) = \langle X_{\pm 1,0}, \mathbf{Y}_{\pm 1/2}, M_0, \mathcal{D} \rangle$

$$
\begin{aligned}
[X_n, X_{n'}] &= (n-n')X_{n+n'} \\
[X_n, \mathbf{Y}_m] &= \left(\frac{n}{2} - m\right)\mathbf{Y}_{n+m} \quad , \quad [\mathcal{D}, \mathbf{Y}_m] \subset \mathbf{Y}_m \\
\left[Y_m^{(j)}, Y_{m'}^{(j')}\right] &= \delta^{j,j'}(m-m')M_0
\end{aligned}
$$

$\Longrightarrow$ **not** semi-simple $\Longrightarrow$ projective representations ('mass' $\mathcal{M}$!)

dynamical symmetry of Schrödinger equation :

$$[\mathcal{S}, X_0] = -\mathcal{S} \ \ , \ \ [\mathcal{S}, X_1] = -2t\mathcal{S} - \left(x - \frac{d}{2}\right) M_0$$

$\Longrightarrow$ fixes scaling dimension of solution of $\mathcal{S}\phi = 0$, $x = x_\phi = d/2$

co-variant two-point (response) function : MH 92/94, MH & UNTERBERGER 03

$$\langle \phi_1(t_1, \mathbf{r}_1)\phi_2^*(t_2, \mathbf{r}_2)\rangle \sim \overbrace{\Theta(t_1 - t_2)}^{\text{Causality}} \cdot \underbrace{\delta(\mathcal{M}_1 - \mathcal{M}_2)}_{\text{Bargman superselection rule}} \cdot \delta_{x_1, x_2}$$

$$\times \quad (t_1 - t_2)^{-x_1} \exp\left[-\frac{\mathcal{M}_1}{2}\frac{(\mathbf{r}_1 - \mathbf{r}_2)^2}{t_1 - t_2}\right]$$

non-relativistic AdS/CFT correspondence, cold atoms, ... SINCE 2008
$\longrightarrow$ **classification** of non-relativistic conformal symmetries,
but for pure vector fields only DUVAL & HORVÁTHY 2009

infinite-dimensional extension $\mathfrak{sv}(d)$ Schrödinger-Virasoro algebra MH 94

! **but** Schrödinger-invariance **cannot** be applied to ageing, since it contains time-translations !

<u>essential</u> : **absence** of TTI in ageing phenomena !

Transformation $t \mapsto t'$ with

$$t = \beta(t') \quad , \quad \phi(t) = \left(\frac{\mathrm{d}\beta(t')}{\mathrm{d}t'}\right)^{-x/z} \left(\frac{\mathrm{d}\ln\beta(t')}{\mathrm{d}t'}\right)^{-2\xi/z} \phi'(t')$$

with $\beta(0) = 0$ and $\dot{\beta}(t') \geq 0$.

out of equilibrium, have **2 distinct** scaling dimensions, x and ξ.

mean-field for **magnets** : expect $\left\{ \begin{array}{l} \xi = 0 \text{ in ordered phase } T < T_c \\ \xi \neq 0 \text{ at criticality } T = T_c \end{array} \right.$

<u>**NB :**</u> if TTI (equilibrium criticality), then $\xi = 0$.

physical requirement :

co-variance of **response functions** under local scaling !

$\Rightarrow$ set of linear differential equations for $R(t,s)$

most simple case !

$$R(t,s) = \left\langle \phi(t)\widetilde{\phi}(s) \right\rangle = s^{-1-a} f_R\left(\frac{t}{s}\right)$$

$$f_R(y) = f_0 y^{1+a'-\lambda_R/z}(y-1)^{-1-a'} \underbrace{\Theta(y-1)}_{\text{causality}}$$

$$a = \frac{1}{z}(x+\widetilde{x}) - 1 \; , \; a' - a = \frac{2}{z}\left(\xi + \widetilde{\xi}\right) \; , \; \frac{\lambda_R}{z} = x + \xi$$

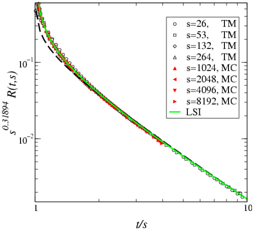
magnetic example : $1D$ Glauber-Ising model at $T = T_c = 0$:

$$a = 0 \; , \; a' - a = -\frac{1}{2} \; , \; \lambda_R = 1 \; , \; z = 2$$

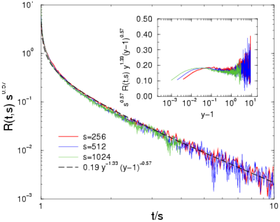
NB : $\Phi(t,\mathbf{r}) := t^{-\xi}\phi(t,\mathbf{r})$ has standard local scaling,
with $x_\Phi = x_\phi + 2\xi_\phi$, $\xi_\Phi = 0$

Particle models : comparison of $R(t,s)$ with LSI-prediction :

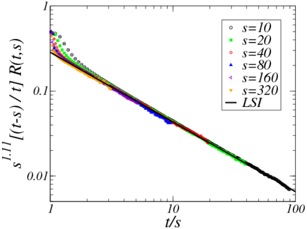
contact process (CP)



nonequil. kinetic Ising (PC)



voter Potts-3 (VP3)



CP : $a' - a \simeq 0.27$ — MH, ENSS, PLEIMLING 06; ENSS 06 ; HINRICHSEN 06

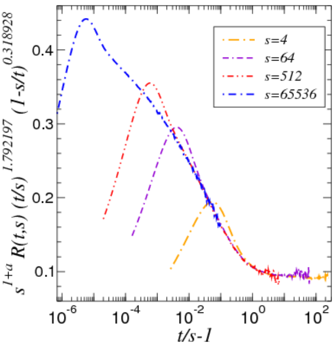
PC : $a' - a \simeq 0.00(1)$ — ÓDOR 06

VP3 : $a' - a \simeq -0.1$ — CHATELAIN, TOMÉ, DE OLIVEIRA 11

? is this good general agreement already conclusive ?

Observation : the **hidden assumption** $a = a'$, uncritically taken over from equilibrium, is often **invalid** out of equilibrium. Observables **cannot** always be identified with scaling operators.

1D critical contact process (TMRG data)



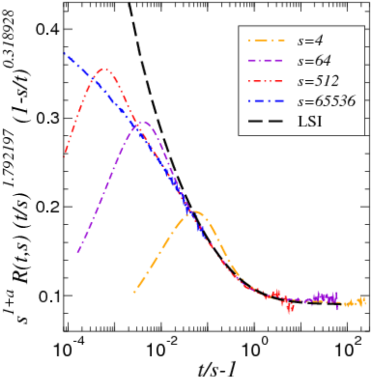
study more closely the limit $t, s \to \infty$, $y = t/s$ fixed ; let $y \to 1$

$$R(t,s) = s^{-1-a} f_R\left(\frac{t}{s}\right) \;,\; h_R(y) := f_R(y) y^{\lambda_R/z} (1-1/y)^{1+a}$$

observe good collapse of data, when $y = t/s$ large enough

LSI with $a = a'$ predicts : $h_R(y) = f_0 = \text{cste.}$

$\Rightarrow$ reproduces TMRG data for $y \gtrsim 3-4$



$$h_R(y) := f_R(y) y^{\lambda_R/z} (1-1/y)^{1+a} \stackrel{\text{LSI}}{=} f_0 (1-1/y)^{a-a'}$$

with the choice $a' - a = 0.26$, LSI works well for $y \gtrsim 1.1$ but **systematic deviations**, still **inside the ageing scaling region**, for smaller values of $y = t/s$ (down to $y \simeq 1.001$) !

Question : improve the prediction of local scale-invariance (LSI) ?

6. Logarithmic conformal invariance

generalise conformal invariance → doubletts $\Psi = \begin{pmatrix} \psi \\ \phi \end{pmatrix}$ ROZANSKY & SALEUR 92, GURARIE 93

generators : $\ell_n = -w^{n+1}\partial_w - (n+1)w^n \begin{pmatrix} \Delta & 1 \\ 0 & \Delta \end{pmatrix}$

two-point functions : have $\Delta_1 = \Delta_2$ GURARIE 93, RAHIMI TABAR *et al.* 97...

$$\begin{aligned} F &= \langle \phi_1(w_1)\phi_2(w_2)\rangle = 0 \\ G &= \langle \phi_1(w_1)\psi_2(w_2)\rangle = G_0|w|^{-2\Delta_1} \\ H &= \langle \psi_1(w_1)\psi_2(w_2)\rangle = (H_0 - 2G_0 \ln|w|)\,|w|^{-2\Delta_1} \\ &= w_2^{-2\Delta_1}(H_0 - 2G_0 \ln|y-1| - 2G_0 \ln|w_2|)\,|y-1|^{-2\Delta_1} \end{aligned}$$

with $w = w_1 - w_2$ and $y = w_1/w_2$.

Simultaneous log corrections to scaling **and** modified scaling function

Logarithmic conformal invariance has been found in, e.g.

- critical $2D$ percolation — CARDY 92, WATTS 96, MATHIEU & RIDOUT 07/08
- disordered systems — CAUX *et al.* 96
- sand-pile models — RUELLE *et al.* 08-10

7. Logarithmic ageing-invariance

Schrödinger-invariance cannot be a dynamical symmetry for ageing, since it contains time-translations X_{-1} !

Go to **ageing algebra** $\mathfrak{age}(d) := \left\langle X_{1,0}, Y^{(j)}_{\pm 1/2}, M_0, R^{(jk)}_0 \right\rangle_{j,k=1,\ldots d}$
Need generalised form of generator

$$X_n = -t^{n+1}\partial_t - \frac{n+1}{2} t^n \mathbf{r}\cdot\nabla_{\mathbf{r}} - \frac{\mathcal{M}}{2}(n+1)nt^{n-1}\mathbf{r}^2 - \frac{n+1}{2} x t^n - n\xi t^n$$

construct **logarithmic ageing-invariance** by the formal changes :

$$x \mapsto \begin{pmatrix} x & x' \\ 0 & x \end{pmatrix} , \ \xi \mapsto \begin{pmatrix} \xi & \xi' \\ \xi'' & \xi \end{pmatrix}$$

concentrate on time-dependence

$$X_0 = -t\partial_t - \frac{1}{2}\begin{pmatrix} x & x' \\ 0 & x \end{pmatrix} \quad , \quad X_1 = -t^2\partial_t - t\begin{pmatrix} x+\xi & x'+\xi' \\ \xi'' & x+\xi \end{pmatrix}$$

and compute commutator

$$[X_1, X_0] = X_1 + \tfrac{1}{2} t\, x'\xi'' \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \stackrel{!}{=} X_1 \implies \boxed{x'\xi'' \stackrel{!}{=} 0}$$

$x' = 0$: either, $\begin{pmatrix} \xi & \xi' \\ \xi'' & \xi \end{pmatrix} \to \begin{pmatrix} \xi_+ & 0 \\ 0 & \xi_- \end{pmatrix}$ is diagonalisable
$\Rightarrow$ non-logarithmic case.
Or else, it reduces to a Jordan form $\Rightarrow$ 2^{nd} case.

$\xi'' = 0$: simultaneous Jordan forms $\Rightarrow$ generic case.
(one can arrange for $x' = 0$ or $x' = 1$).

we can always arrange for $\xi'' = 0$.

invariant Schrödinger equation $\mathcal{S}\Psi = 0$, with :

$$\mathcal{S} := \left(2\mathcal{M}\partial_t - \nabla_{\mathbf{r}}^2 + \frac{2\mathcal{M}}{t}\left(x + \xi - \frac{d}{2} \right) \right) \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right)$$

If $x + \xi = d/2$, have also log-invariance under $\mathfrak{sch}(d)$.

Co-variant two-point functions :

$$\begin{aligned} F = F(t_1, t_2) &:= \langle \phi_1(t_1)\phi_2(t_2) \rangle \\ G_{12} = G_{12}(t_1, t_2) &:= \langle \phi_1(t_1)\psi_2(t_2) \rangle \\ G_{21} = G_{21}(t_1, t_2) &:= \langle \psi_1(t_1)\phi_2(t_2) \rangle \\ H = H(t_1, t_2) &:= \langle \psi_1(t_1)\psi_2(t_2) \rangle \end{aligned}$$

co-variance conditions (with $\partial_i = \partial/\partial t_i$) :

$$\begin{aligned}
\left[t_1\partial_1 + t_2\partial_2 + \frac{1}{2}(x_1+x_2)\right] F(t_1,t_2) &= 0 \\
\left[t_1^2\partial_1 + t_2^2\partial_2 + (x_1+\xi_1)t_1 + (x_2+\xi_2)t_2\right] F(t_1,t_2) &= 0 \\
\left[t_1\partial_1 + t_2\partial_2 + \frac{1}{2}(x_1+x_2)\right] G_{12}(t_1,t_2) + \frac{x_2'}{2} F(t_1,t_2) &= 0 \\
\left[t_1^2\partial_1 + t_2^2\partial_2 + (x_1+\xi_1)t_1 + (x_2+\xi_2)t_2\right] G_{12}(t_1,t_2) + (x_2'+\xi_2')t_2 F(t_1,t_2) &= 0 \\
\left[t_1\partial_1 + t_2\partial_2 + \frac{1}{2}(x_1+x_2)\right] G_{21}(t_1,t_2) + \frac{x_1'}{2} F(t_1,t_2) &= 0 \\
\left[t_1^2\partial_1 + t_2^2\partial_2 + (x_1+\xi_1)t_1 + (x_2+\xi_2)t_2\right] G_{21}(t_1,t_2) + (x_1'+\xi_1')t_1 F(t_1,t_2) &= 0 \\
\left[t_1\partial_1 + t_2\partial_2 + \frac{1}{2}(x_1+x_2)\right] H(t_1,t_2) + \frac{x_1'}{2} G_{12}(t_1,t_2) + \frac{x_2'}{2} G_{21}(t_1,t_2) &= 0 \\
\left[t_1^2\partial_1 + t_2^2\partial_2 + (x_1+\xi_1)t_1 + (x_2+\xi_2)t_2\right] H(t_1,t_2) & \\
+(x_1'+\xi_1')t_1 G_{12}(t_1,t_2) + (x_2'+\xi_2')t_2 G_{21}(t_1,t_2) &= 0
\end{aligned}$$

8 eqs. for 4 functions in 2 variables $\Rightarrow$ expect **unique solution**, up to normalisations.

Solve these via the following **ansatz**, with $\boxed{y := t_1/t_2 > 1}$.

Set $\mathcal{F}(y) := y^{\xi_2+(x_2-x_1)/2}(y-1)^{-(x_1+x_2)/2-\xi_1-\xi_2}$. Then

$$\begin{aligned}
F(t_1,t_2) &= t_2^{-(x_1+x_2)/2}\,\mathcal{F}(y)\,f(y)\\
G_{12}(t_1,t_2) &= t_2^{-(x_1+x_2)/2}\,\mathcal{F}(y)\sum_{j\in\mathbb{Z}}\ln^j t_2\cdot g_{12,j}(y)\\
G_{21}(t_1,t_2) &= t_2^{-(x_1+x_2)/2}\,\mathcal{F}(y)\sum_{j\in\mathbb{Z}}\ln^j t_2\cdot g_{21,j}(y)\\
H(t_1,t_2) &= t_2^{-(x_1+x_2)/2}\,\mathcal{F}(y)\sum_{j\in\mathbb{Z}}\ln^j t_2\cdot h_j(y)
\end{aligned}$$

must find the functions $f,\ g_{12,j},\ g_{21,j},\ h_j$; where $j\in\mathbb{Z}$

Results :

(1) : $f(y) = f_0 = \mathrm{cste.}$ standard form of LSI

(2) : consider G_{12}. Dilatation-covariance (X_0) gives

$$\left(g_{12,1}(y) + \frac{1}{2}x_2' f(y)\right) + \sum_{j\neq 0}(j+1)\ln^j t_2 \cdot g_{12,j+1}(y) = 0$$

Must hold true for all times t_2. The only non-vanishing terms are :

$$g_{12}(y) := g_{12,0}(y) \;\; , \;\; \gamma_{12}(y) := g_{12,1}(y) = -\frac{1}{2}x_2' f(y)$$

Co-variance under the special transformations (X_1) gives

$$\sum_{j\in\mathbb{Z}} \ln^j t_2 \left(y(y-1)\frac{\mathrm{d}g_{12,j}(y)}{\mathrm{d}y} + (j+1)g_{12,j+1}(y)\right) + (x_2' + \xi_2')f(y) = 0$$

for all times t_2 and leads to

$$y(y-1)\frac{\mathrm{d}g_{12}(y)}{\mathrm{d}y} + \left(\frac{x_2'}{2} + \xi_2'\right) f(y) = 0$$

(3) : consider G_{21}. We find the only non-vanishing terms

$$g_{21}(y) := g_{21,0}(y) \ , \ \ \gamma_{21}(y) := g_{21,1}(y) = -\frac{1}{2}x_1' f(y)$$

and the differential equation

$$y(y-1)\frac{\mathrm{d}g_{21}(y)}{\mathrm{d}y} + \left(x_1' + \xi_1'\right) y f(y) - \frac{1}{2}x_1' f(y) = 0$$

(4) : consider H. We find the only non-vanishing terms $h_0(y)$ and

$$\begin{aligned} h_1(y) &= -\frac{1}{2}\left(x_1' g_{12}(y) + x_2' g_{21}(y)\right) \\ h_2(y) &= \frac{1}{4}x_1' x_2' f(y) \end{aligned}$$

and the differential equation

$$y(y-1)\frac{\mathrm{d}h_0(y)}{\mathrm{d}y} + \left(\left(x_1' + \xi_1'\right) y - \frac{1}{2}x_1'\right) g_{12}(y) + \left(\frac{1}{2}x_2' + \xi_2'\right) g_{21}(y) = 0$$

The remaining differential equations have the solutions :

$$
\begin{aligned}
g_{12}(y) &= g_{12,0} + \left(\frac{x_2'}{2} + \xi_2'\right) f_0 \ln\left|\frac{y}{y-1}\right| \\
g_{21}(y) &= g_{21,0} - \left(\frac{x_1'}{2} + \xi_1'\right) f_0 \ln|y-1| - \frac{x_1'}{2} f_0 \ln|y| \\
h_0(y) &= h_0 - \left[\left(\frac{x_1'}{2} + \xi_1'\right) g_{21,0} + \left(\frac{x_2'}{2} + \xi_2'\right) g_{12,0}\right] \ln|y-1| - \left[\frac{x_1'}{2} g_{21,0} - \left(\frac{x_2'}{2} + \xi_2'\right) g_{12,0}\right] \ln|y| \\
&\quad + \frac{1}{2} f_0 \left[\left(\left(\frac{x_1'}{2} + \xi_1'\right) \ln|y-1| + \frac{x_1'}{2} \ln|y|\right)^2 - \left(\frac{x_2'}{2} + \xi_2'\right)^2 \ln^2\left|\frac{y}{y-1}\right|\right]
\end{aligned}
$$

where $f_0, g_{12,0}, g_{21,0}, h_0$ are normalisation constants. **Summary :**

$$
\begin{aligned}
F(t_1,t_2) &= t_2^{-(x_1+x_2)/2}\, \mathcal{F}(y)\, f_0 \\
G_{12}(t_1,t_2) &= t_2^{-(x_1+x_2)/2}\, \mathcal{F}(y) \left(g_{12}(y) - \ln t_2 \cdot \frac{x_2'}{2} f_0 \right) \\
G_{21}(t_1,t_2) &= t_2^{-(x_1+x_2)/2}\, \mathcal{F}(y) \left(g_{21}(y) - \ln t_2 \cdot \frac{x_1'}{2} f_0 \right) \\
H(t_1,t_2) &= t_2^{-(x_1+x_2)/2}\, \mathcal{F}(y) \left(h_0(y) - \ln t_2 \cdot \frac{1}{2}\left(x_1' g_{12}(y) + x_2' g_{21}(y)\right) \right. \\
&\qquad \left. + \ln^2 t_2 \cdot \frac{x_1' x_2'}{4} f_0 \right)
\end{aligned}
$$

add time-translations $\Rightarrow$ logarithmic Schrödinger-invariance

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Retour to ageing phenomena

find co-variant two-point (auto-response) functions (with $y = t/s$) :

$$\begin{aligned}
\left\langle \phi(t)\widetilde{\phi}(s)\right\rangle &= s^{-(x+\widetilde{x})/z}\mathcal{F}(y)f(y) \\
\left\langle \phi(t)\widetilde{\psi}(s)\right\rangle &= s^{-(x+\widetilde{x})/z}\mathcal{F}(y)\left(g_{12}(y)+\gamma_{12}(y)\ln s\right) \\
\left\langle \psi(t)\widetilde{\phi}(s)\right\rangle &= s^{-(x+\widetilde{x})/z}\mathcal{F}(y)\left(g_{21}(y)+\gamma_{21}(y)\ln s\right) \\
\left\langle \psi(t)\widetilde{\psi}(s)\right\rangle &= s^{-(x+\widetilde{x})/z}\mathcal{F}(y)\left(h_0(y)+h_1(y)\ln s+h_2(y)\ln^2 s\right)
\end{aligned}$$

all scaling functions explicitly known

Question 1 : $1D$ **directed percolation described by logarithmic** LSI **?**
as motivated by the applications of logarithmic conformal invariance to $2D$ critical normal percolation MATHIEU & RIDOUT '07-08

Question 2 : what about the $1D$ **Kardar-Parisi-Zhang equation ?**

8. Numerical experiments

(A) directed percolation (**DP**)

(B) Kardar-Parisi-Zhang (**KPZ**)

simple ageing of the correlators and responses, especially

$$C(t,s) = s^{-b} f_C\left(\frac{t}{s}\right) \quad , \quad R(t,s) = s^{-1-a} f_R\left(\frac{t}{s}\right)$$

$$f_C(y) \sim y^{-\lambda_C/z} \quad , \quad f_R(y) \sim y^{-\lambda_R/z} \qquad y \gg 1$$

values of the non-equilibrium exponents & scaling relations

DP : $\lambda_C = \lambda_R = d + z + \frac{\beta}{\nu_\perp}$, $1 + a = b = \frac{2\beta}{\nu_\|}$

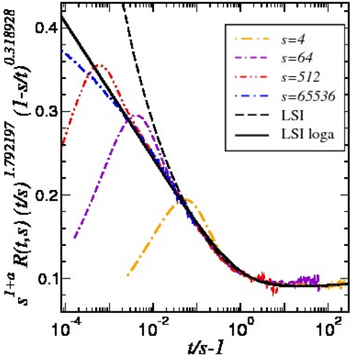
KPZ in $1D$: $\lambda_C = \lambda_R = 1$, $1 + a = b + \frac{2}{z}$, $b = -2\beta = -\frac{2}{3}$, $z = \frac{3}{2}$

what can be said on the form of the scaling function of the auto-response ?

(A) **assumption :** $R(t,s) = \left\langle \psi(t)\widetilde{\psi}(s)\right\rangle$ $1D$ **critical contact process**

good collapse $\Rightarrow$ **no** logarithmic corrections $\Rightarrow$ $\boxed{x' = \widetilde{x}' = 0}$

$$h_R(y) = \left(1-\frac{1}{y}\right)^{a-a'} \left[h_0 - g_{12,0}\widetilde{\xi}' \ln(1-1/y) - g_{21,0}\xi' \ln(y-1) - \frac{1}{2} f_0 \widetilde{\xi}'^2 \ln^2(1-1/y) + \frac{1}{2} f_0 \xi'^2 \ln^2(y-1)\right]$$



find empirically : very small amplitude of $\ln^2$-terms

$\Rightarrow$ $\boxed{f_0 = 0}$

require both $\xi \neq 0$, $\widetilde{\xi}' \neq 0$

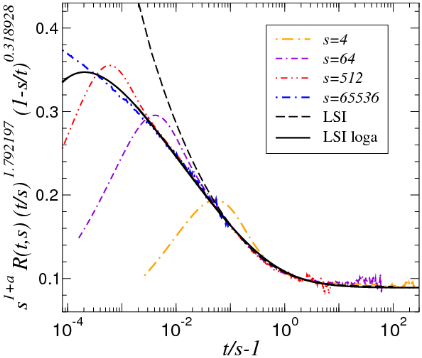
BUT : logarithmic factor for $y \gg 1$ **?**

logar. LSI works at least down to $y \simeq 1.002$, with $a' - a \simeq -0.002$.

An alternative interpretation : $R(t,s) = \left\langle \psi(t)\widetilde{\psi}(s)\right\rangle$

good collapse $\Rightarrow$ **no** logarithmic corrections $\Rightarrow$ $\boxed{x' = \widetilde{x}' = 0}$

$$h_R(y) = \left(1-\frac{1}{y}\right)^{a-a'}\left[h_0 - g_{12,0}\widetilde{\xi}'\ln(1-1/y) - \frac{1}{2}f_0\widetilde{\xi}'^2\ln^2(1-1/y) - g_{21,0}\xi'\ln(y-1) + \frac{1}{2}f_0\xi'^2\ln^2(y-1)\right]$$



no logarithmic growth for $y \to \infty$

$\Rightarrow$ $\boxed{\xi' = 0}$

only $\widetilde{\xi}' \neq 0$ remains !

logar. LSI works at least down to $y \simeq 1.005$, with $a' - a \simeq 0.17$.

(B) **assumption :** $R(t,s) = \left\langle \psi(t)\widetilde{\psi}(s) \right\rangle$ $1D$ **KPZ equation/RSOS model**

good collapse $\Rightarrow$ **no** logarithmic corrections $\Rightarrow$ $\boxed{x' = \widetilde{x}' = 0}$

no logarithmic factors for $y \gg 1 \Rightarrow$ $\boxed{\xi' = 0}$

$\Rightarrow$ only $\widetilde{\xi}' = 1$ remains

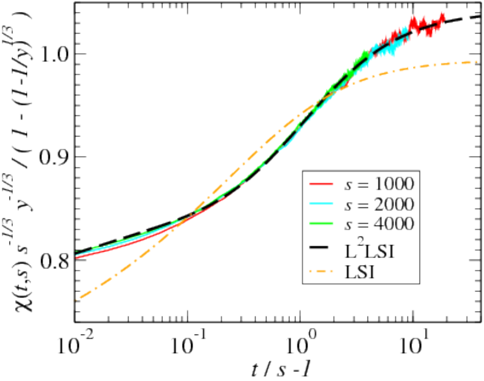
$$f_R(y) = y^{-\lambda_R/z} \left(1 - \frac{1}{y}\right)^{-1-a'} \left[h_0 - g_0 \ln\left(1 - \frac{1}{y}\right) - \frac{1}{2} f_0 \ln^2\left(1 - \frac{1}{y}\right) \right]$$

use specific values of $1D$ KPZ class $\frac{\lambda_R}{z} - a = 1$

find integrated autoresponse $\chi(t,s) = \int_0^s \mathrm{d}u\, R(t,u) = s^{1/3} f_\chi(t/s)$

$$\begin{aligned} f_\chi(y) &= y^{1/3} \left\{ A_0 \left[1 - \left(1 - \frac{1}{y}\right)^{-a'} \right] \right. \\ &\quad \left. + \left(1 - \frac{1}{y}\right)^{-a'} \left[A_1 \ln\left(1 - \frac{1}{y}\right) + A_2 \ln^2\left(1 - \frac{1}{y}\right) \right] \right\} \end{aligned}$$

with free parameters A_0, A_1, A_2 and a'



non-log LSI with $a = a'$:
deviations $\approx 20\%$

non-log LSI with $a \neq a'$:
works up to $\approx 5\%$

log LSI : works **better than** $\approx 0.1\%$

R	a'	A_0	A_1	A_2
$\langle \phi\widetilde{\phi} \rangle$ – LSI	-0.500	0.662	0	0
$\langle \phi\widetilde{\psi} \rangle$ – L^{1}LSI	-0.500	0.663	$-6 \cdot 10^{-4}$	0
$\langle \psi\widetilde{\psi} \rangle$ – L^{2}LSI	-0.8206	0.7187	0.2424	-0.09087

logarithmic LSI works at least down to $y \simeq 1.01$, with
$a' - a \approx -0.4873$ (can we make a conjecture ?)

9. Conclusions

- physical ageing occurs naturally in many **irreversible** systems relaxing towards **non**-equilibrium stationary states
  considered here : absorbing phase transitions & surface growth
- scaling phenomenology essentially the same as in simple magnetic systems
- **but** finer differences in relationships between non-equilibrium exponents
- a major difference w/ equilibrium : intrinsic **absence** of time-translation-invariance $\Rightarrow$ **2** scaling dimensions
- shape of scaling functions :
  logarithmic local scale-invariance ?
  performed numerical experiments on auto-response function :
  (i) $1D$ critical directed percolation (ii) $1D$ KPZ equation

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