An integrable and critical modification of the Chalker-Coddington network model

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Based on

Y.I., Paul Fendley and John Cardy, `1103.3368`

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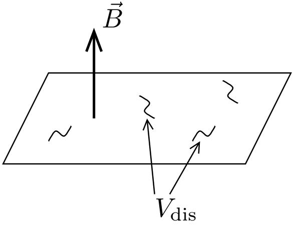
Outline

1. The Chalker-Coddington model
2. Exactly solvable truncated model
3. Results

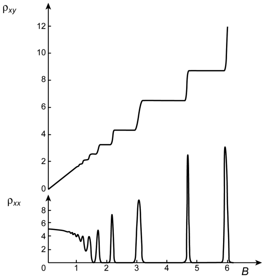
1. The Chalker-Coddington model

1.1. Integer Quantum Hall Effect

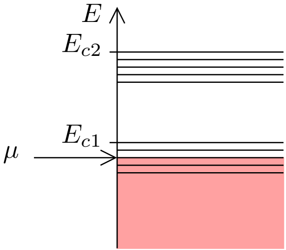


- 2D electron gas in strong transverse $\vec{B}$.
- Disordered electric potential $V_{\rm dis}$.
- Neglect interactions $\Rightarrow$ single-particle.

Conductivity plateaux :

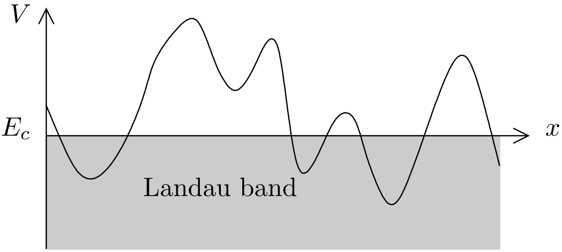


1.2. Landau-level picture



- Single e^- in disordered potential V_{dis} and transverse $\vec{B}$
- Chemical potential $\mu \propto \|\vec{B}\|$
- Disorder $\Rightarrow$ broadened Landau levels

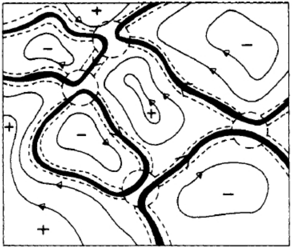
1.3. Real-space picture, plateau transition



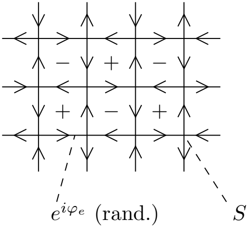
- Inside Landau band :
  - e^- "trapped" in potential wells
  - Wavefunction ψ is localised
  - Exponential decay of correlations $\overline{\psi(0)^\alpha \psi(r)^\alpha} \sim \exp(-r/\xi)$
- At critical energy E_c :
  - Typical e^- trajectories become very long
  - Meet many saddle points of V_{dis}
  - Wavefunction ψ is delocalised
  - Algebraic decay of correlations $\overline{\psi(0)^\alpha \psi(r)^\alpha} \sim r^{-2X_\alpha}$

1.4. The Chalker-Coddington model

Potential landscape



Lattice model



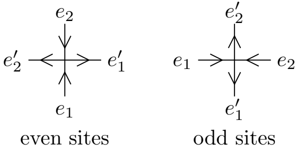
Definition of network model :

- One-parameter ($E = \mu$) scaling
- Discrete wavefunction $|\psi\rangle = \sum_e \psi_e |e\rangle$
- Edges ($\sim$ equipotential lines) $\rightarrow$ random phases $\{\exp(i\varphi_e)\}$
- Vertices ($\sim$ saddle points) $\rightarrow$ 2×2 scattering matrix S

- Time-evolution operator :

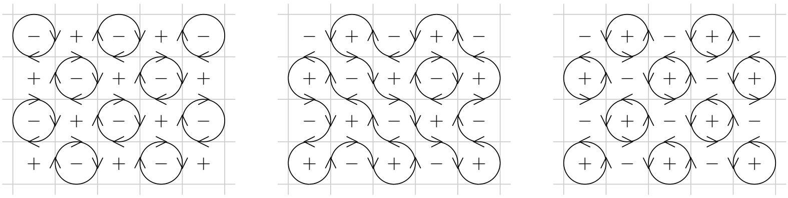
$$\mathcal{U} = \prod_{\text{vertex } v} \left[\sum_{e_\ell \xrightarrow{v} e'_j} \exp(i\varphi_{e'_j})\; S_{j\ell}\; |e'_j\rangle\langle e_\ell| \right]$$

- Scattering matrix :



$$S = \begin{pmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{pmatrix}$$

- Parameter λ controls plateau transition (analogous to μ)



$\lambda = 0$ $\lambda = \frac{\pi}{4}$ $\lambda = \frac{\pi}{2}$

1.5. SUSY lattice path-integral

- Action [$b(e)$= complex Gaussian var.]

$$\mathcal{A}[b] = \sum_{\text{vertex } v} \sum_{e_\ell \xrightarrow{v} e'_j} \exp(i\varphi_{e'_j})\ S_{j\ell}\ b(e'_j)^* b(e_\ell)$$

- Green's function

$$G(e', e) := \langle e' |\, (1 - \mathcal{U})^{-1}\, | e \rangle = \frac{\int [\mathrm{D}b]\ b^*(e')b(e)\ \exp \mathcal{A}[b]}{\int [\mathrm{D}b] \exp \mathcal{A}[b]}$$

- Use Grassmann $\{f(e)\}$:

$$\int [\mathrm{D}f] \exp \mathcal{A}[f] = \left(\int [\mathrm{D}b] \exp \mathcal{A}[b] \right)^{-1}$$

$$\Rightarrow G(e', e) = \int [\mathrm{D}b][\mathrm{D}f]\ b^*(e')b(e)\ \exp(\mathcal{A}[b] + \mathcal{A}[f])$$

1.6. Correlation functions

- Mean squared Green's function :

$$\left|G(e', e)\right|^2 = \int [\mathrm{D}b_{1,2}][\mathrm{D}f_{1,2}]\ (b_1^* b_2)(e')\ (b_1 b_2^*)(e) \times \exp(\mathcal{A}[b_1] + \mathcal{A}[f_1] + \mathcal{A}^*[b_2] + \mathcal{A}^*[f_2])$$

- $2N$-point correlation function :

$$\mathcal{G}_{2N}(e_1', \ldots e_N' | e_1, \ldots e_N) := \int [\mathrm{D}b_{1,2}][\mathrm{D}f_{1,2}] \prod_{j=1}^{N} (b_1^* b_2)(e_j') \prod_{\ell=1}^{N} (b_1 b_2^*)(e_\ell) \times \exp(\mathcal{A}[b_1] + \mathcal{A}[f_1] + \mathcal{A}^*[b_2] + \mathcal{A}^*[f_2])$$

- Question : CFT for the $\overline{\mathcal{G}_{2N}}$?

1.7. Relation to Logarithmic CFT

- CC model$= \begin{cases} \text{path integral with loop weight } n = 0 \\ n \to 0 \text{ replicas of the pure model} \end{cases}$

- J. Cardy's argument [`cond-mat/9911024`]
  These $n \to 0$ limits include correlators of the form

  $$\langle D(r)D(0)\rangle \propto r^{-2x}(\log r + \dots)$$

  together with a 'partner'

  $$\langle C(r)D(0)\rangle \propto r^{-2x}\,, \qquad \langle C(r)C(0)\rangle = 0$$

  (for replicas, $C := \sum_a E_a$ and $D := E_a$)

- Expect the CFT for $\{\overline{\mathcal{G}_{2N}}\}$ to be a LCFT !

2. Exactly solvable truncated model

2.1. Truncation scheme

- $\{b(e)\}$=bosons $\Rightarrow$ lattice paths for $\mathcal{G}_{2N}$ are in infinite number !
- Existing schemes [Kondev, Marston, Tsai, Zirnbauer, 90's]
  - Equivalent spin chain

$$\mathcal{H} = V \otimes V^* \otimes \cdots \otimes V \otimes V^*$$

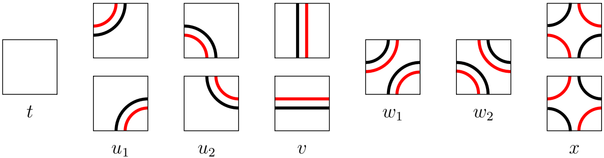
    V, V^* infinite-dim. (Fock-space) repr. of $gl(2|2)$
  - Truncate V, V^* in sectors of the SUSY charges Q_{ab}
  - $\Rightarrow$ series of non-critical, SUSY spin chains
  - Observe numerically convergence to IQHE
- Our scheme
  - Truncate path integral $\Rightarrow$ well-defined loop model
  - Fine-tune Boltzmann weights $\Rightarrow$ series of critical, integrable loop models
  - Identify corresponding CFTs

2.2. Truncation to a loop model

- Path integral
  - Generic factor in $\mathcal{A}[b]$:

$$
\begin{aligned}
e^{Kb(e_j)^* b(e_\ell)} &= 1 + Kb(e_j)^* b(e_\ell) + \frac{K^2}{2}[b(e_j)^* b(e_\ell)]^2 + \dots \\
&= \bullet \cdots \bullet \quad + \quad \bullet \rightarrow \bullet \quad + \quad \bullet \rightrightarrows \bullet \quad + \dots
\end{aligned}
$$

  - Keep only two first terms $\left[e^{Kb(e_j)^* b(e_\ell)}\right]_{\mathrm{tr}} := 1 + Kb(e_j)^* b(e_\ell)$
  - Average on the $\varphi(e)$
    $\Rightarrow$ { edges of **forward** path } = { edges of **backward** path }
- Resulting $n = 0$ loop model



$$
Z_{\mathrm{loop}} = \sum_{\mathrm{config.}\ C} t^{\#t(C)} u_1^{\#u_1(C)} \dots x^{\#x(C)} \times n^{\#\mathrm{loops}(C)}
$$

2.3. Two-colour algebras and the BWM algebra

- Single Temperley-Lieb algebra $\{e_j\}$

$$e_j^2 = n\ e_j\,, \qquad e_j e_{j\pm 1} e_j = e_j$$

- Baxterisation : $n = -2\cos 2\theta, \quad \check{R}_j(u) := \frac{\sin(2\theta-u)\ \mathbf{1} - \sin u\ e_j}{\sin(2\theta-u)}$

$$\begin{cases} \check{R}_j(u)\check{R}_{j+1}(u+v)\check{R}_j(v) = \check{R}_{j+1}(v)\check{R}_j(u+v)\check{R}_{j+1}(u) \\ \check{R}_j(u)\check{R}_j(-u) = \mathbf{1} \end{cases}$$

- Braiding limit

$$b_j^{\pm} := \check{R}_j(\pm i\infty) \Rightarrow \begin{cases} b_j b_{j+1} b_j = b_{j+1} b_j b_{j+1} \\ b_j^+ b_j^- = b_j^- b_j^+ = \mathbf{1} \end{cases}$$

- Double-TL algebra $\mathrm{TL} \otimes \mathrm{TL}$ contains the Birman-Wenzl-Murakami braid-monoid algebra :

$$E_j := e_j \otimes e_j\,, \qquad B_j^{\pm} := b_j^{\pm} \otimes b_j^{\pm}$$

2.4. Integrable two-colour loop model

- Our two-colour loop model is based on two copies of the *dilute* TL algebra : $\mathrm{dTL} \otimes \mathrm{dTL}$
- For dTL generators $g_j \in \{e_j, b_j^+, b_j^-, I_j, \ldots\}$ apply the same trick : $G_j := g_j \otimes g_j$
- The G_j generate the *dilute* BWM algebra
- dBMW was Baxterised in [Grimm-Warnaar, '95] :

$$
\begin{aligned}
n &= -2\cos 2\theta \\
t &= -\cos(2\varphi - 3\theta) - \cos 5\theta + \cos 3\theta + \cos\theta \\
u_1 &= -2\sin 2\theta \sin(\varphi - 3\theta) \\
u_2 &= 2\sin 2\theta \sin\varphi \\
v &= x = 2\sin\varphi \sin(\varphi - 3\theta) \\
w_1 &= 2\sin(\varphi - 2\theta)\sin(\varphi - 3\theta) \\
w_2 &= 2\sin\varphi \sin(\varphi - \theta)
\end{aligned}
$$

3. Results on the loop model

3.1. Conformal Field Theory at roots of unity

- Two types of critical regimes :
  - regime I : loop flavours are decoupled $\rightarrow$ (Coulomb Gas)^2
  - regime II : loop flavours are coupled

- Roots of unity :

$$n = -(q + q^{-1}), \qquad q = e^{\frac{i\pi}{k+2}}, \qquad k = 1, 2, 3 \ldots$$

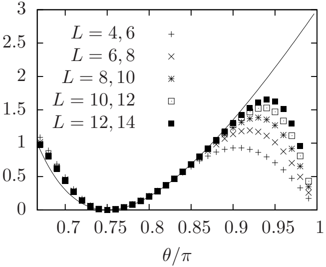
- In the coupled regime :
  - CFT : $\frac{\mathrm{SU}(2)_k \times \mathrm{SU}(2)_k}{\mathrm{SU}(2)_{2k}}$ coset WZW model
  - Central charge : $c = \frac{3k^2}{(k+1)(k+2)}$
  - Energy dimension : $X_T = \frac{2k^2}{k+1}$

  - Question : Analytic continuation for $k \rightarrow 0$?

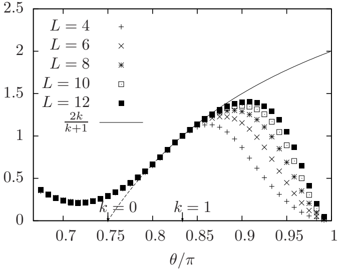
3.2. Numerical study of truncated model

- Method = exact transfer-matrix diagonalisation

central charge



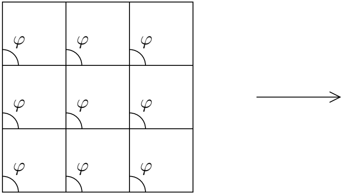
X_T



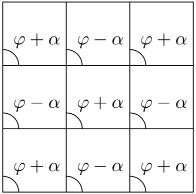
- For $k \geq 1$: c and X_T match coset-CFT prediction
- At $k = 0$:
  - $c = 0$, as expected in disordered critical model
  - Energy dimension $X_T \simeq 0.29 \leftrightarrow$ fractal dim. of path $d_f \simeq 1.71$
  - "One-leg" exponent $X_{\mathcal{G}_2} = 0$ (proba. conservation in original CC model)

3.3. Correlation-length exponent

- Off-critical CC model



CC at $\lambda = \lambda_c$
homogeneous loop model



CC at $\lambda \neq \lambda_c$
staggered loop model

- Correlation length $\xi \sim |\alpha|^{-\nu}$
- Numerical result estimate : $\nu \simeq 1.1$

3.4. Relation between X_T and ν

- Effective massive Hamiltonian

$$H_{\rm eff}(n) = H_{\rm crit} + \alpha^r \; H_{\rm pert}$$

- Scaling law

$$\xi \propto |\alpha^r|^{1/y_T} \quad \Rightarrow \quad d - X_T = \frac{r}{\nu}$$

r is *constant* on the critical line (when n is varied)

- Exactly solvable point : $n = 1$ (free fermions)

$$H_{\rm eff}(n=1) = \int {\rm d}x \left(i\Psi^\dagger \partial_x \Psi + {\rm const} \times \alpha^2 \; \Psi^\dagger \Psi \right)$$

- Throughout the coupled critical regime :

$$\boxed{2 - X_T = \frac{2}{\nu}}$$

3.5. Summary

- Defined integrable critical loop model
- At roots of unity ($n = -2\cos\frac{\pi}{k+2}$), identifed CFT : $\frac{\mathrm{SU}(2)_k \times \mathrm{SU}(2)_k}{\mathrm{SU}(2)_{2k}}$ WZW coset model
- At $n = 0$, loop model $\equiv$ truncation of the CC model
- Fractal dimension of loops $d_f \simeq 1.71$
- Correlation-length exponent $\nu \simeq 1.1$, scaling relation $2 - X_T = \frac{2}{\nu}$
- A bit far from $\nu_{\mathrm{CC}} \simeq 2.34...$

3.6. Discussion : higher-order truncations

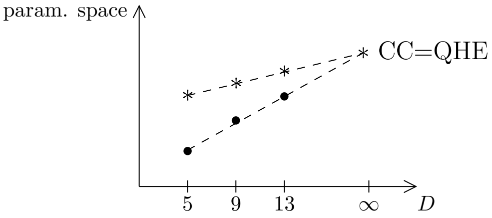
- Average on random phases $\overline{(\dots)}$ imposes on every edge :

$$N_{b_1} + N_{f_1} = N_{b_2^*} + N_{f_2^*}$$

- Possible terms on a given edge $(m = 1, 2, 3, \dots)$:

$$1, \quad b_1^{m-1} f_1 (b_2^{m-1} f_2)^*, \quad b_1^m (b_2^{m-1} f_2)^*, \quad b_1^{m-1} f_1 b_2^{m*}, \quad b_1^m b_2^{m*}$$

- Truncated models : $D = 4p + 1$ terms per edge



$*$ = our critical truncation

$\bullet$ = non-critical truncation (Marston,Kondev,Tsai $\sim$ '96)

Perspectives

- Full CFT spectrum form Bethe-Ansatz solution
- $k \to 0$ limit : log-CFT features ?
- Relation to a super-spin chain ?
- Higher-order truncations : fusion of dBMW integrable model

Thank you for your attention !