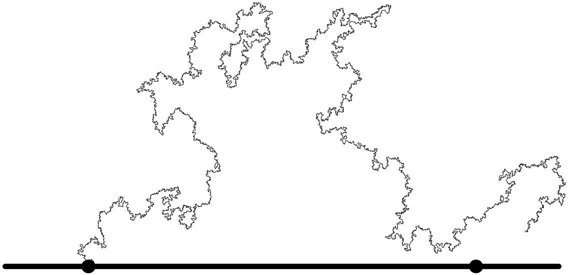
What is logarithmic in Schramm-Loewner Evolutions?

Kalle Kytölä

Advanced Conformal Field Theory and Applications
Workshop 1: Logarithmic Conformal Field Theory, October 3-7, 2011

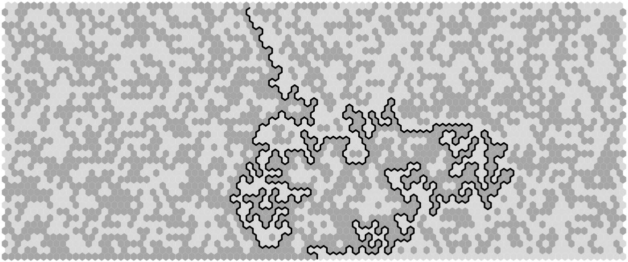
Random geometry of critical phenomena and LCFT

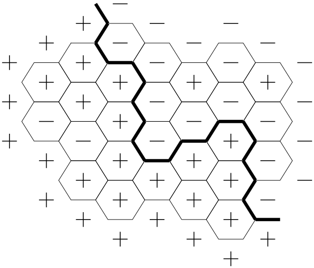


- Geometric questions have lead to logarithmic CFT
  - polymers, percolation ([Saleur '87], [Cardy '99], . . .)
  - non-diagonalizable transfer matrices ([Saleur], [Saint-Aubin & Pearce & Rasmussen '09], . . .)
- Schramm-Loewner evolutions (SLE)
  - random conformally invariant fractal curves ([Schramm '00])
  - some rigorous results about scaling limits of interfaces in critical models of statistical mechanics
  - breakthrough: Fields medals Werner (2006), Smirnov (2010)

Percolation

$p = p_c$



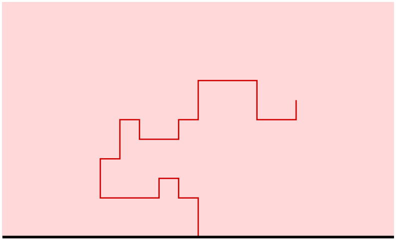


Ising model

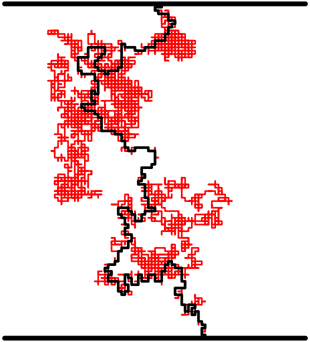
$T = T_c$

Random curves in 2D critical phenomena

Self-avoiding walk



Loop erased random walk



Chordal SLE$_\kappa$ in the half-plane

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In the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \Im\mathrm{m}\ z > 0\}$, from 0 to ∞

Loewner's equation for g_t

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t(z) = \frac{2}{g_t(z) - X_t}$$

Driving process Brownian motion

$$X_t = \sqrt{\kappa}\ B_t$$

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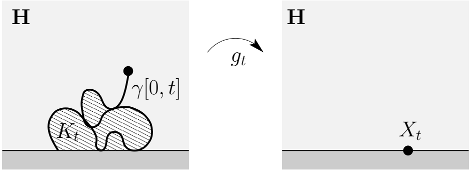

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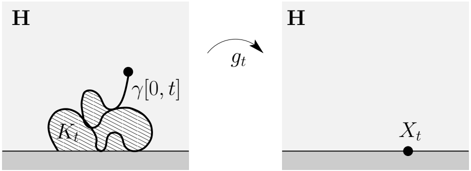

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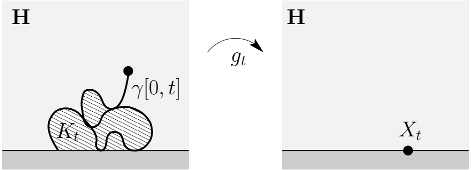

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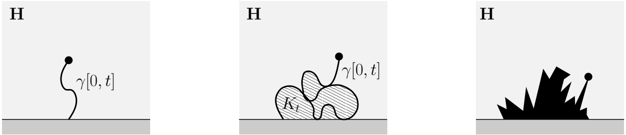
- Laurent expansion at infinity (defines a_m, $m = 2, 3, \ldots$)

$$g_t(z) = z + \sum_{m=2}^{\infty} a_m(t)\ z^{1-m}$$

Some properties of chordal SLE$_\kappa$

Phases: [Rohde & Schramm]

- $0 \leq \kappa \leq 4$: The curve γ is simple (no self intersections)
- $4 < \kappa < 8$: The curve γ is self-touching (but not self crossing)
- $\kappa \geq 8$: The curve γ is space filling (Peano curve)



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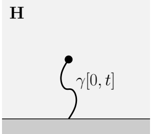

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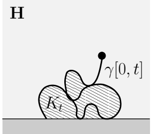

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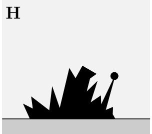

$8 \leq \kappa$

Fractal dimensions? [Rohde & Schramm, Beffara, ...]

- $\dim(\gamma[0,\infty]) = 1 + \kappa/8$ for $0 \leq \kappa \leq 8$
- $\dim(\partial K_t) = 1 + 2/\kappa$ for $\kappa \geq 4$

Models of statistical physics and chordal SLEs

Model	SLE_κ	CFT?
loop-erased random walk[1]	$\kappa = 2$	$c = -2$ CFT
Ising spin cluster boundary[2]	$\kappa = 3$	fermionic CFT, $c = 1/2$
free field level lines[3]	$\kappa = 4$	free boson, $c = 1$ CFT
Ising FK clusters[2]	$\kappa = \frac{16}{3}$	fermionic CFT, $c = 1/2$
percolation cluster boudary[4]	$\kappa = 6$	$c = 0$, Cardy's formula
uniform spanning tree[1]	$\kappa = 8$	$c = -2$ CFT
self-avoiding walk	$\kappa = \frac{8}{3}$?	CFT of $c = 0$
q-Potts and FK model	$\kappa = \kappa(q)$?	
$O(n)$ model	$\kappa = \kappa(n)$?	
$\vdots$	$\vdots$	

[1] Lawler & Schramm & Werner : Ann. Probab. **32** no. 1B (2004).
[2] Smirnov & Chelkak 2011(?)
[3] Schramm & Sheffield : (2006)
[4] Smirnov: C.R.Acad.Sci Paris **333** (2001), Camia & Newman: (2006)

Variants of SLE, $\mathrm{SLE}_\kappa(\rho)$

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t(z) = \frac{2}{g_t(z) - X_t}, \qquad g_0(z) = z$$

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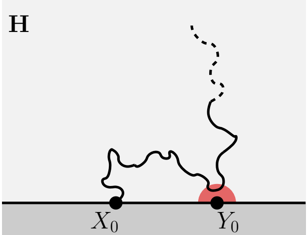
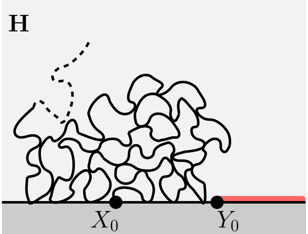


Example variant 1:

chordal SLE_κ from X_0 to ∞,
conditioned to visit $Y_0 \in \partial\mathbb{H}$

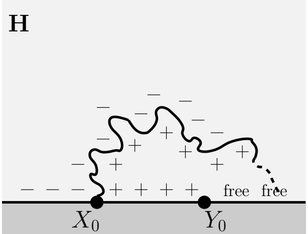
$= \mathrm{SLE}_\kappa(\rho = \kappa - 8)$



Example variant 2:

chordal SLE_κ from X_0 to ∞, conditioned to avoid the arc $[Y_0, \infty) \subset \partial\mathbb{H}$

$= \mathrm{SLE}_\kappa(\rho = \kappa - 4)$



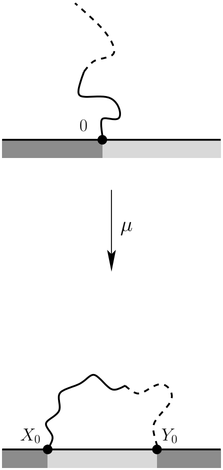
Example variant 3:

dipolar SLE_κ [Bauer & Bernard & Houdayer]
(role of Y_0 and ∞ symmetric)

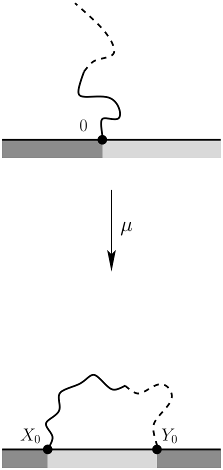
$= \mathrm{SLE}_\kappa(\rho = \frac{\kappa-6}{2})$

e.g. Ising interface with *plus-minus-free* boundary conditions [Hongler & Kytölä]

Coordinate change: [. . . , Schramm & Wilson, . . .]



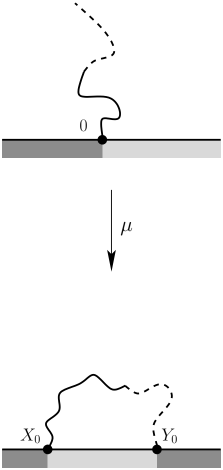
More example variants by coordinate changes



Coordinate change: [. . . , Schramm & Wilson, . . .]

- γ = trace of SLE$_\kappa(\rho)$
- $\mu : \mathbb{H} \to \mathbb{H}$ Möbius with
  $\mu(X_0) = \widetilde{X}_0$, $\mu(Y_0) = \infty$, $\mu(\infty) = \widetilde{Y}_0$

The curve $\mu \circ \gamma$ is the trace of an SLE$_\kappa(\rho')$ with $\rho' = \kappa - 6 - \rho$



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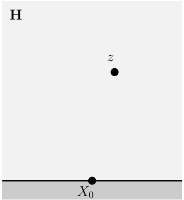
More example variants:

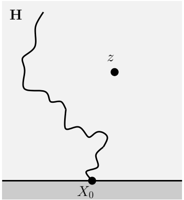
* Chordal SLE_κ from X_0 to Y_0 is $\mathrm{SLE}_\kappa(\rho = \kappa - 6)$
* Chordal SLE_κ from X_0 to Y_0 conditioned to "visit infinity" is $\mathrm{SLE}_\kappa(\rho = 2)$
* Chordal SLE_κ from X_0 to Y_0 conditioned to avoid $[Y_0, \infty)$ is $\mathrm{SLE}_\kappa(\rho = -2)$
* etc.

A typical SLE calculation: "Schramm's formula"



chordal SLE_κ from $X_0 = x$

Does the curve leave $z \in \mathbb{H}$ on its left or right?



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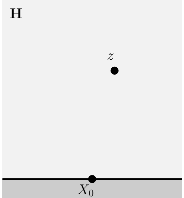

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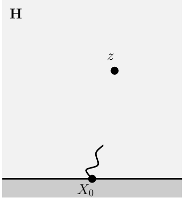

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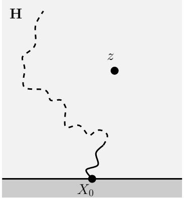

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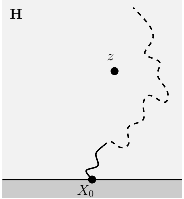

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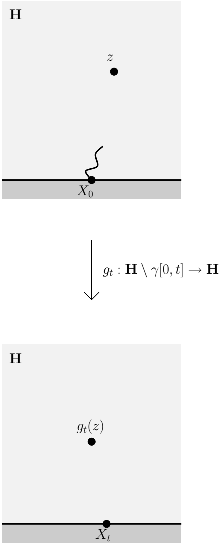

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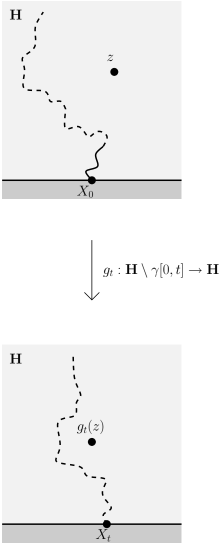

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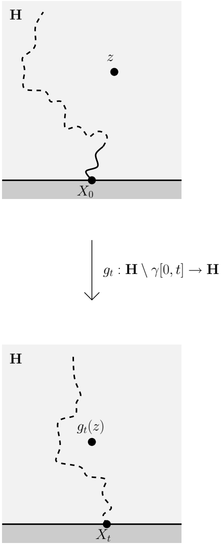

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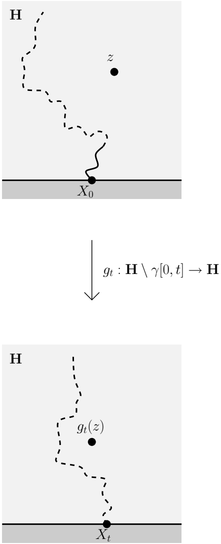

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$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \mathsf{E}\big[P(X_t, g_t(z), \overline{g_t(z)})\big] = \mathsf{E}\big[(\mathcal{A}P)\big]$$

where $\mathcal{A} = \frac{2}{z-x}\frac{\partial}{\partial z} + \frac{2}{\bar{z}-x}\frac{\partial}{\partial \bar{z}} + \frac{\kappa}{2}\frac{\partial^2}{\partial x^2}$

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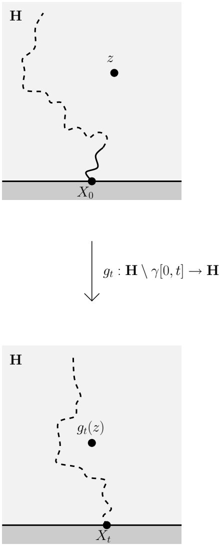

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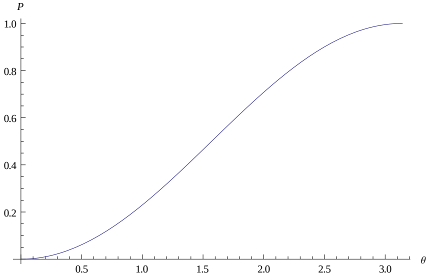
* To find P, solve the diff. eq. $\mathcal{A}P = 0$ with appropriate boundary conditions!

Schramm's formula

$$\begin{aligned} P(x,z,\overline{z}) &= \langle \psi_{h_{1,2}}(x)\ \phi_{h=\overline{h}=0}(z,\overline{z})\ \psi_{h_{1,2}}(\infty)\rangle \\ &= \frac{1}{2} - \frac{\mathrm{i}\,\Gamma(\frac{4}{\kappa})}{\sqrt{\pi}\Gamma(\frac{8-4\kappa}{2\kappa})} \frac{z+\overline{z}-2x}{z-\overline{z}}\ {}_2F_1\left(\frac{1}{2},\frac{4}{\kappa};\frac{3}{2};(\frac{z+\overline{z}-2x}{z-\overline{z}})^2\right) \end{aligned}$$

$(x=0,\ z=e^{\mathrm{i}\theta})$



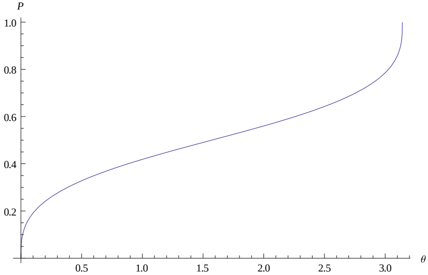
$\kappa = \frac{8}{3}$

$P = \frac{1}{2}(1-\cos(\theta))$

Schramm's formula

$$P(x,z,\bar z) = \langle \psi_{h_{1,2}}(x)\ \phi_{h=\bar h=0}(z,\bar z)\ \psi_{h_{1,2}}(\infty)\rangle$$
$$= \frac{1}{2} - \frac{\mathrm{i}\,\Gamma(\frac{4}{\kappa})}{\sqrt{\pi}\,\Gamma(\frac{8-4\kappa}{2\kappa})} \frac{z+\bar z-2x}{z-\bar z}\ {}_2F_1\left(\frac{1}{2},\frac{4}{\kappa};\frac{3}{2};\left(\frac{z+\bar z-2x}{z-\bar z}\right)^2\right)$$

$(x=0,\ z=e^{\mathrm{i}\theta})$



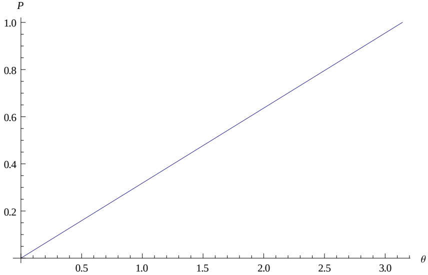
$\kappa = 6$

$P = \text{(hypergeom.)}$

Schramm's formula

$$
\begin{aligned}
P(x, z, \bar{z}) &= \langle \psi_{h_{1,2}}(x)\ \phi_{h=\bar{h}=0}(z, \bar{z})\ \psi_{h_{1,2}}(\infty) \rangle \\
&= \frac{1}{2} - \frac{\mathfrak{i}\ \Gamma(\frac{4}{\kappa})}{\sqrt{\pi}\Gamma(\frac{8-4\kappa}{2\kappa})} \frac{z + \bar{z} - 2x}{z - \bar{z}}\ {}_2F_1\left(\frac{1}{2}, \frac{4}{\kappa}; \frac{3}{2}; (\frac{z + \bar{z} - 2x}{z - \bar{z}})^2\right)
\end{aligned}
$$

$(x = 0,\ z = e^{\mathfrak{i}\theta})$



$\kappa = 4$

$P = \frac{1}{\pi}\,\theta$

Schramm's formula

$$P(x,z,\overline{z}) = \langle \psi_{h_{1,2}}(x)\ \phi_{h=\overline{h}=0}(z,\overline{z})\ \psi_{h_{1,2}}(\infty)\rangle$$
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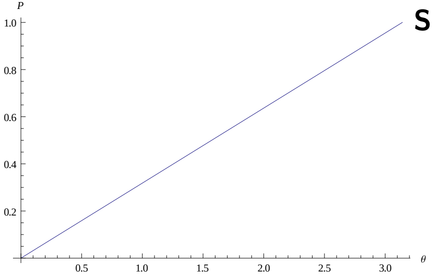

Summary: "typical SLE calculation"

$\kappa = 4$

$P = \frac{1}{\pi}\theta$

Schramm's formula

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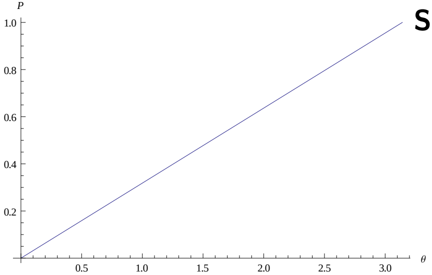

$\kappa = 4$

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Summary: "typical SLE calculation"

- P is a CFT correlation function
  - field $\phi(z,\overline{z})$ has conformal weight $h=\overline{h}=0$

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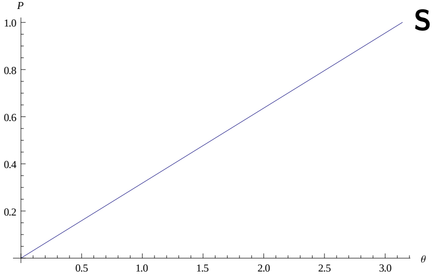

$\kappa = 4$

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Summary: "typical SLE calculation"

- P is a CFT correlation function
  - * field $\phi(z,\bar{z})$ has conformal weight $h=\bar{h}=0$
- Importance of martingales
  - * generator: $\mathcal{A} = \frac{2}{z-x}\frac{\partial}{\partial z} + \frac{2}{\bar{z}-x}\frac{\partial}{\partial \bar{z}} + \frac{\kappa}{2}\frac{\partial^2}{\partial x^2}$
  - * local martingales: $\mathrm{Ker}\ \mathcal{A}$

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&= \frac{1}{2} - \frac{\mathrm{i}\,\Gamma(\frac{4}{\kappa})}{\sqrt{\pi}\,\Gamma(\frac{8-4\kappa}{2\kappa})}\,\frac{z+\bar{z}-2x}{z-\bar{z}}\ {}_2F_1\left(\frac{1}{2},\frac{4}{\kappa};\frac{3}{2};\left(\frac{z+\bar{z}-2x}{z-\bar{z}}\right)^2\right)
\end{aligned}
$$

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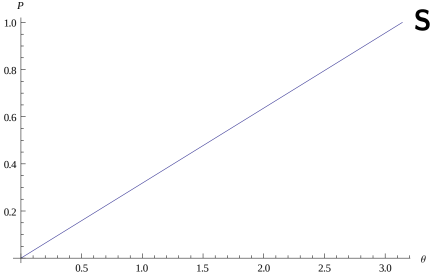

$\kappa = 4$

$P = \frac{1}{\pi}\,\theta = \frac{1}{2\pi\mathrm{i}}\log(z/\bar{z})$

Summary: "typical SLE calculation"

- P is a CFT correlation function
  - * field $\phi(z,\bar{z})$ has conformal weight $h=\bar{h}=0$
- Importance of martingales
  - * generator: $\mathcal{A} = \frac{2}{z-x}\frac{\partial}{\partial z} + \frac{2}{\bar{z}-x}\frac{\partial}{\partial \bar{z}} + \frac{\kappa}{2}\frac{\partial^2}{\partial x^2}$
  - * local martingales: $\mathrm{Ker}\ \mathcal{A}$
- Case $\kappa \to 4$ is logarithmic

SLE local martingales

Say $\mathrm{SLE}_\kappa(\rho)$ data at time t consists of

- positions of driving process X_t and passive point Y_t
- coefficients $a_m(t)$ of the map $g_t(z) = \sum_{m \geq 2} a_m(t) z^{1-m}$

[Kytölä: Rev. Math. Phys. **19** (2007)]

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Consider a function $\varphi(x, y; a_2, a_3, \ldots)$.

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Consider a function $\varphi(x, y; a_2, a_3, \ldots)$. The drift is

$$\frac{\mathrm{d}}{\mathrm{d}s} \mathsf{E}\Big[\frac{\varphi(X_{t+s}, Y_{t+s}; a_2(t+s), \ldots)}{Z(X_{t+s}, Y_{t+s})} \;\Big|\; \text{data at time } t \Big] = \Big(\frac{A_{\kappa;\rho}\, \varphi}{Z} \Big)$$

where the "generator" $A_{\kappa;\rho}$ is a second order differential operator

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If $\varphi \in \mathrm{Ker}\, A_{\kappa;\rho}$, the process $\frac{\varphi(X_t, Y_t; a_2(t), \ldots)}{Z(X_t, Y_t)}$ is a local martingale

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where the "generator" $A_{\kappa;\rho}$ is a second order differential operator

$$\begin{aligned} A_{\kappa;\rho} &= \frac{\kappa}{2} \frac{\partial^2}{\partial x^2} + \frac{2}{y-x} \frac{\partial}{\partial y} - \frac{\rho(\rho+4-\kappa)/(2\kappa)}{(y-x)^2} \\ &\quad + 2 \sum_{l\geq 2} \Big(\sum_{\substack{0\leq m \leq l-2 \\ 0 \leq r \leq \lfloor \frac{l-2-m}{2} \rfloor}} x^m (-1)^r \frac{(m+r)!}{m!\, r!} \sum_{\substack{k_1,\ldots k_r \geq 2 \\ \sum k_j = l-m-2}} a_{k_1} \cdots a_{k_r} \Big) \frac{\partial}{\partial a_l} \end{aligned}$$

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$\mathrm{Ker}\, A_{\kappa;\rho} =$ "the space of $\mathrm{SLE}_\kappa(\rho)$ local martingales"

[Kytölä: Rev. Math. Phys. **19** (2007)]

A representation of Virasoro algebra

Let $c = \frac{(6-\kappa)(3\kappa-8)}{2\kappa}$, $h_x = \frac{6-\kappa}{2\kappa}$, $h_y = \frac{\rho(\rho+4-\kappa)}{4\kappa}$

A representation of Virasoro algebra

Let $c = \frac{(6-\kappa)(3\kappa-8)}{2\kappa}$, $h_x = \frac{6-\kappa}{2\kappa}$, $h_y = \frac{\rho(\rho+4-\kappa)}{4\kappa}$ and

$$L_0 = \sum_{l\geq 2} l a_l \frac{\partial}{\partial a_l} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + h_x + h_y\,, \qquad L_1 = \sum_{l\geq 3} ((l-2)a_{l-1})\frac{\partial}{\partial a_l} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\,, \qquad L_2 = -\frac{\partial}{\partial a_2} + \sum_{l\geq 4} ((l-3)a_{l-n})\frac{\partial}{\partial a_l}$$

$$L_{-1} = \sum_{l\geq 2} \Big((l+2)a_{l+1} + \sum_{\substack{m_1,m_2\geq 2 \\ m_1+m_2=l+1}} a_{m_1}a_{m_2} \Big) \frac{\partial}{\partial a_l} + (x^2 - 3a_2)\frac{\partial}{\partial x} + (y^2 - 3a_2)\frac{\partial}{\partial y} + 2xh_x + 2yh_y$$

$$\begin{aligned} L_{-2} = &\sum_{l\geq 2} \Big((l+4)a_{l+2} - 4a_2 a_l + \sum_{\substack{m_1,m_2\geq 2 \\ m_1+m_2=l+2}} 3a_{m_1}a_{m_2} + \sum_{\substack{m_1,m_2,m_3\geq 2 \\ m_1+m_2+m_3=l+2}} a_{m_1}a_{m_2}a_{m_3} \Big) \frac{\partial}{\partial a_l} \\ &+ (x^3 - 4xa_2 - 5a_3)\frac{\partial}{\partial x} + (y^3 - 4ya_2 - 5a_3)\frac{\partial}{\partial y} + (3x^2 - 4a_2)h_x + (3y^2 - 4a_2)h_y - \frac{c}{2}a_2 \end{aligned}$$

$$L_n = \frac{1}{(n-2)!}\Big[\underbrace{[[\cdots[L_2, L_1], \ldots], L_1], L_1}_{n-2 \text{ times}} \Big]\,, \qquad L_{-n} = \frac{1}{(n-2)!}\Big[\underbrace{L_{-1}, [L_{-1}, [\ldots, [L_{-1}}_{n-2 \text{ times}}, L_{-2}] \cdots]] \Big] \quad \text{for } n > 2.$$

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SLE$_\kappa(\rho)$ local martingales form a Virasoro module

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Let $c = \frac{(6-\kappa)(3\kappa-8)}{2\kappa}$, $h_x = \frac{6-\kappa}{2\kappa}$, $h_y = \frac{\rho(\rho+4-\kappa)}{4\kappa}$ and

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$\text{SLE}_\kappa(\rho)$ local martingales form a Virasoro module

$$[L_n, L_m] = (n-m)\,L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}$$

A representation of Virasoro algebra

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SLE$_\kappa(\rho)$ local martingales form a Virasoro module

$$[L_n, L_m] = (n-m)\, L_{n+m} + \tfrac{c}{12}(n^3-n)\delta_{n+m,0}$$
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L_n preserve Ker $A_{\kappa;\rho}$: if $A_{\kappa;\rho}\,\varphi = 0$ then also $A_{\kappa;\rho}\,(L_n\varphi) = 0$

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* Virasoro algebra $\mathfrak{Vir}$: basis $\ldots, L_{-2}, L_{-1}, L_0, L_1, L_2, \ldots, C$
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Highest weight modules: $\mathcal{H}$ a h.w. module of highest weight h, $w \in \mathcal{H}$ (*highest weight vector*)

(i) $L_0 \, w = h \, w$ (and $C \, w = c \, w'$)

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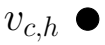
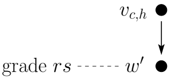
- Any h.w.m. $\mathcal{H}$ is isomorphic to a quotient of $V_{c,h}$ by a submodule.

Submodules of the Verma module

IRREDUCIBLE VERMA MODULES



REDUCIBLE VERMA MODULES $\kappa \notin \mathbf{Q}$



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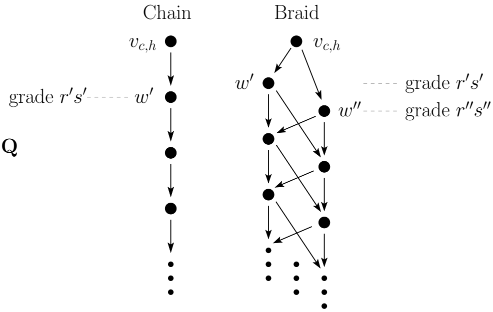

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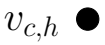

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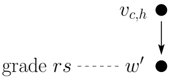

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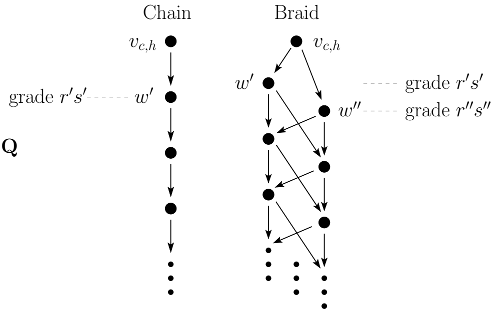

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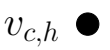

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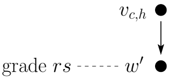

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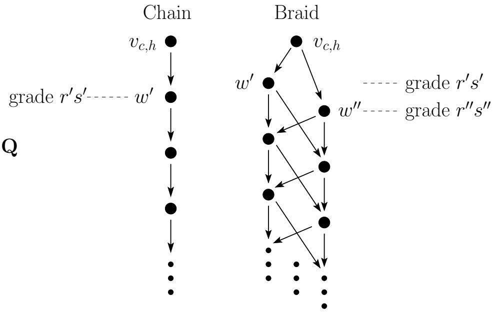

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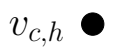

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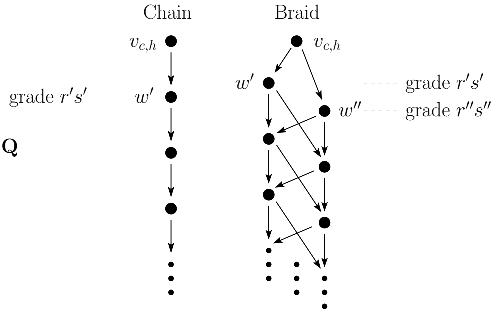

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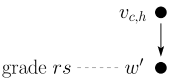

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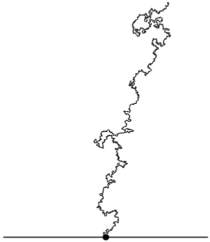
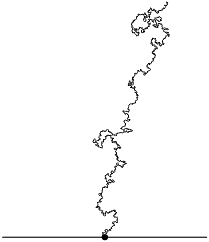


Chordal SLE in $\mathbb{H}$ from 0 to ∞:

[Bauer & Bernard: Phys. Lett. B **557** (2003) and Ann. Henri Poincaré **5** (2004)]

Example of the space of local martingales



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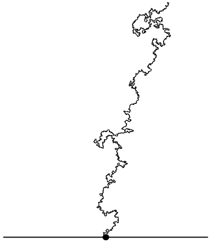

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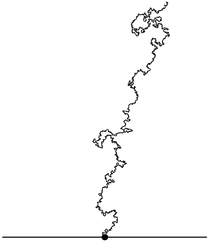

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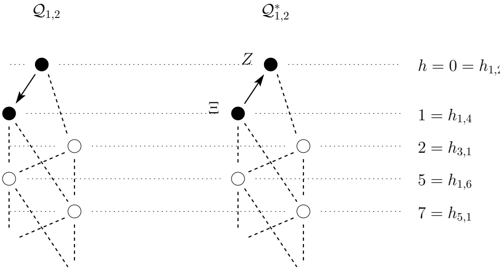
Example: $\kappa = 6$

- $Z = 1 \in \mathrm{Ker}\ A_\kappa$ (constant martingale) generates irreducible h.w.m. $c = 0, h = 0$
- $\Xi = x \in \mathrm{Ker}\ A_\kappa$ (Brownian motion) generates the entire $\mathcal{Q}^*_{\kappa;1,2}$



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- According to the singular vector structure of $\mathcal{H}^{\mathrm{L}}$,$\mathcal{H}^{\mathrm{R}}$ and corresp. Vermas, such extensions are parametrized by an affine subspace of a vector space of dim. 0, 1 or 2 [KK & Ridout 2009]

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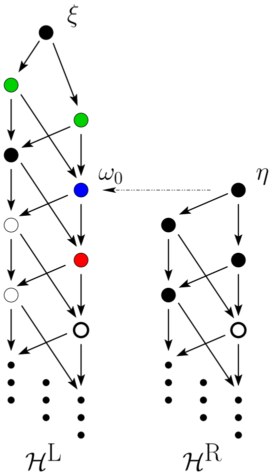

Staggered modules of length two

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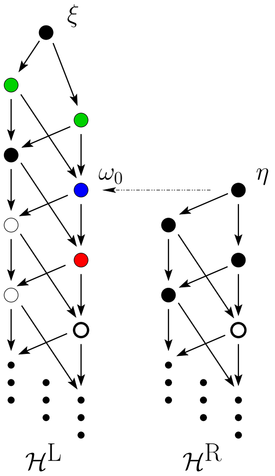

Necessary conditions:

* $\ell = h^{\mathrm{R}} - h^{\mathrm{L}} \in \mathbb{N}$ a grade of singular $0 \neq \omega_0 = S\xi \in \mathcal{H}^{\mathrm{L}}$, denote rank r
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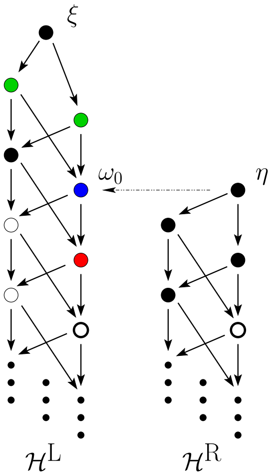

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Notation:

* $b = \#$ singulars of rank $\mathrm{r} - 1$ in $\mathcal{H}^{\mathrm{L}}$
* $a = \#$ non-zero sing. of rank $\mathrm{r} + \bar{\mathrm{r}} - 1$ in $\mathcal{H}^{\mathrm{L}}$
* $d = \#$ indep. null vectors of rank $\bar{\mathrm{r}}$ in $\mathcal{H}^{\mathrm{R}}$

invariants $\beta \in \mathbb{C}^b$ and affine func. $\alpha(\beta) \in \mathbb{C}^{a \times d}$

* β uniquely identifies $\mathcal{S}$ if it exists
* $\mathcal{S}$ exists iff $\alpha(\beta) = 0$

$\mathrm{SLE}_\kappa(\rho)$ local martingales, $\rho = (\kappa - 4)/2$

* partition function $Z = (x-y)^{(\kappa-4)/2\kappa} \in \mathrm{Ker}\, A_{\kappa;\rho}$
  - constant martingale
  - $L_0 Z = h_Z\, Z$ with $h_Z = \frac{8-\kappa}{16} = h_{0,1}(\kappa)$
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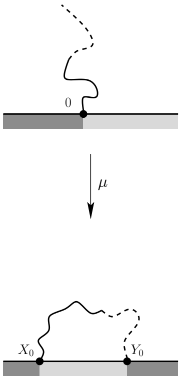
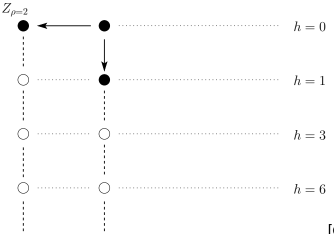
Structure of the staggered module $\mathcal{U}\Lambda$?

- $h^{\mathrm{L}} = h^{\mathrm{R}}$: no β-invariants to be determined
- $h^{\mathrm{L}} = h_{0,1}(\kappa)$: generically the Verma modules irreducible!

at $\kappa = 8$ (uniform spanning tree, $c = -2$)

$$\mathrm{SLE}_\kappa\left(\tfrac{\kappa-4}{2}\right) \;=\; \mathrm{SLE}_\kappa(\kappa-6) \qquad \text{and} \qquad h_{0,1}(\kappa) = h_{1,1}(\kappa) = h_{1,3}(\kappa)$$





[Gurarie '93] [Gaberdiel & Kausch '96]

* $\chi_{\mathcal{U}\,Z}(q) =$
  $1 + q^2 + q^3 + 2q^4 + 2q^5 + 4q^6 + \cdots = \chi_{\mathcal{Q}_{1,1}}(q)$
* $\chi_{\mathcal{U}\,\Lambda}(q) =$
  $2 + 1q + 3q^2 + 3q^3 + 6q^4 + 7q^5 + 12q^6 + \cdots$

$$0 \longrightarrow \mathcal{Q}_{1,1} \longrightarrow \mathcal{U}\,\Lambda \longrightarrow \mathcal{Q}_{1,3} \longrightarrow 0$$

Example staggered module: local martingales of $\mathrm{SLE}_\kappa(-2)$

$\mathrm{SLE}_\kappa(\rho)$ local martingales, $\rho = -2$

* partition function $Z = (x-y)^{-2/\kappa} \in \mathrm{Ker}\, A_{\kappa;\rho}$
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  - $L_0 Z = h_Z\, Z$ with $h_Z = 0 = h_{1,1}(\kappa)$
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* $\Lambda = \frac{4-\kappa}{\kappa}(x-y)^{1-\frac{2}{\kappa}} \log(x-y) + \frac{4-\kappa}{8}\,(x-y)^{-\frac{2}{\kappa}}\,(x+y)$ in kernel of $A_{\kappa;\rho}$
  - $(L_0 - 1)\Lambda = L_{-1}Z = \frac{4-\kappa}{\kappa}(x-y)^{1-\frac{2}{\kappa}}$

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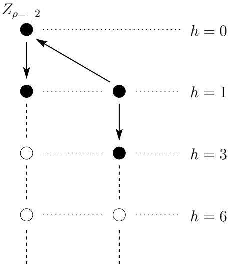
Structure of the staggered module $\mathcal{U}\Lambda$?

- $\omega_0 = (L_0 - 1)\Lambda = L_{-1}Z$ is a rank one singular vector
  - (at most) one β-invariant: $L_1\Lambda = \beta Z$ with $\beta = 1 - \frac{\kappa}{4}$

at $\kappa = 8$ (uniform spanning tree, $c = -2$):
$h^{\mathrm{L}} = h_{1,1}(\kappa) = h_{1,3}(\kappa)$, $h^{\mathrm{R}} = h_{2,1}(\kappa) = h_{1,5}(\kappa)$.



[Gaberdiel & Kausch '96]

* $\chi_{\mathcal{U}\, Z_{\rho=-2}}(q) = 1 + q + 2q^2 + 2q^3 + 4q^4 + 5q^5 + 8q^6 + \cdots = \chi_{\mathcal{Q}_{1,3}}(q)$
* $\chi_{\mathcal{U}\,\Lambda}(q) = 1 + 2q + 3q^2 + 4q^3 + 7q^4 + 10q^5 + 14q^6 + \cdots$
* $0 \longrightarrow \mathcal{Q}_{1,3} \longrightarrow \mathcal{U}\,\Lambda \longrightarrow \mathcal{Q}_{1,5} \longrightarrow 0$
* invariant β of the module: $L_1\,\Lambda = \beta\, Z_{\rho=-2}$, here $\beta = -1$.

SLE$_\kappa(\rho)$ local martingales, $\rho = \frac{-\kappa-4}{2}$

Partition function $Z = (x-y)^{-\frac{1}{2}-\frac{2}{\kappa}} \in \mathrm{Ker}\, A_{\kappa;\rho}$

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Structure of the staggered module $\mathcal{U}\Lambda$?

- (generically) one β-invariant: $(L_1^2 - \frac{\kappa}{4} L_2)\Lambda = \beta Z$ with $\beta = -\frac{1}{16}(\kappa-4)(\kappa-2)(\kappa+4)$.

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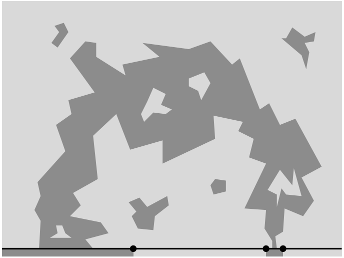
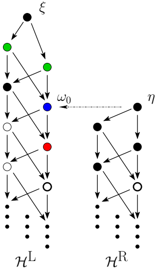
- still no examples of staggered modules with two β-invariants





Thank you!