# Rényi-Shannon entropy of Luttinger liquids

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"CFT, topology, and information" IHP, November 2-4, 2011

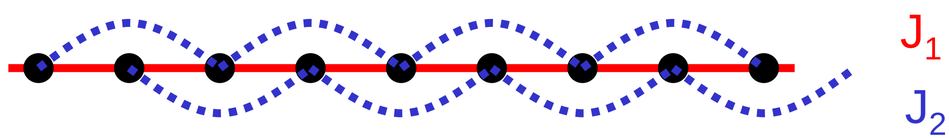
# Some critical spin chains

Critical phase
c=1, Luttinger Liquids

- XXZ chain with anisotropy parameter Δ

$$\mathcal{H} = \sum_j \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z\right) - h \sum_j \sigma_j^z$$

- $J_1$-$J_2$ Heisenberg spin chain (“zig-zag”)




Critical point
c=1/2

- Ising chain in transverse field

$$\mathcal{H} = -\mu \sum_{j=0}^{L-1} \sigma_j^x \sigma_{j+1}^x - \sum_{j=0}^{L-1} \sigma_j^z$$

# Shannon-Rényi entropies

$$|g\rangle = \sum_{i=1}^{2^L} \psi_i |i\rangle \quad \text{ground-state}$$

$$|i\rangle = |\downarrow\uparrow\uparrow\downarrow\cdots\rangle, \cdots \quad \text{"Ising configurations"}$$

$$p_i = |\langle i|g\rangle|^2 \qquad \sum_i p_i = 1 \qquad \text{probabilities (normalized)}$$

$$S_{n=1} = -\sum_i p_i \log(p_i)$$

Shannon entropy

$$S_n = \frac{1}{1-n} \log \underbrace{\left( \sum_i (p_i)^n \right)}_{Z_n \text{ : Partition function}}$$

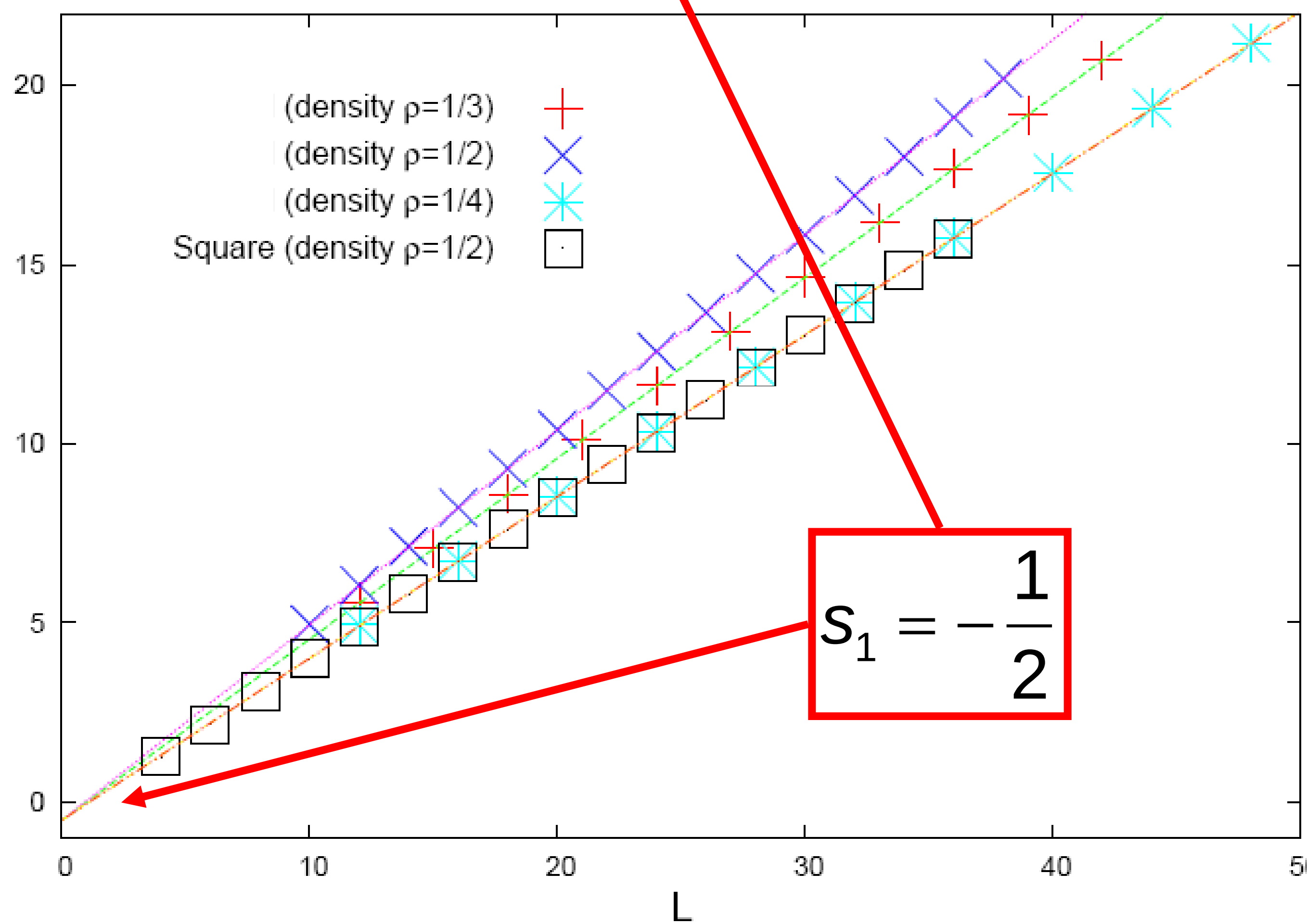
Shannon-Rényi entropy

$(p_i)^n \sim \exp(-nE_i)$
$n \sim$ inverse temperature

NB: These entropies are *entanglement entropies* of some completely different (2d) wave-functions.
Stéphan, Furukawa, GM and Pasquier, Phys Rev. B 80, 184421 (2009)

# Numerics: XXZ free fermions Δ=0 – Rényi index n=1

$$S_{n=1}(L) = -\sum_i p_i \ln p_i = aL + s_1$$




$$s_1 = -\frac{1}{2}$$

Stéphan, Furukawa, GM and Pasquier, Phys Rev. B 80, 184421 (2009)

# Subleading constants are (almost always) interesting !

- Local, microscopic and non-universal contributions
… are expected to contribute to the **extensive term** :

$$S_n(L) = a_n L + \cdots$$

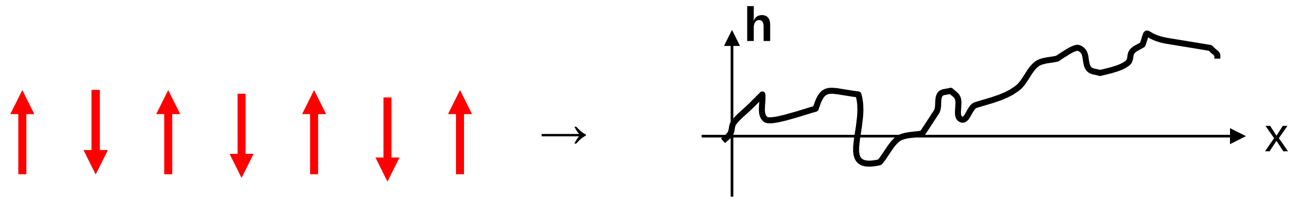
- The (possibly universal) effects of long-distance correlations
may however appear in **subleading** terms :

$$S_n(L) = a_n L + s_n + \cdots$$

How can we compute/understand these subleading constants ?

# Continuum limit

- Microscopic configurations → coarse grained **height** field (bosonization)




- Effective action in the critical phase (c=1): Gaussian

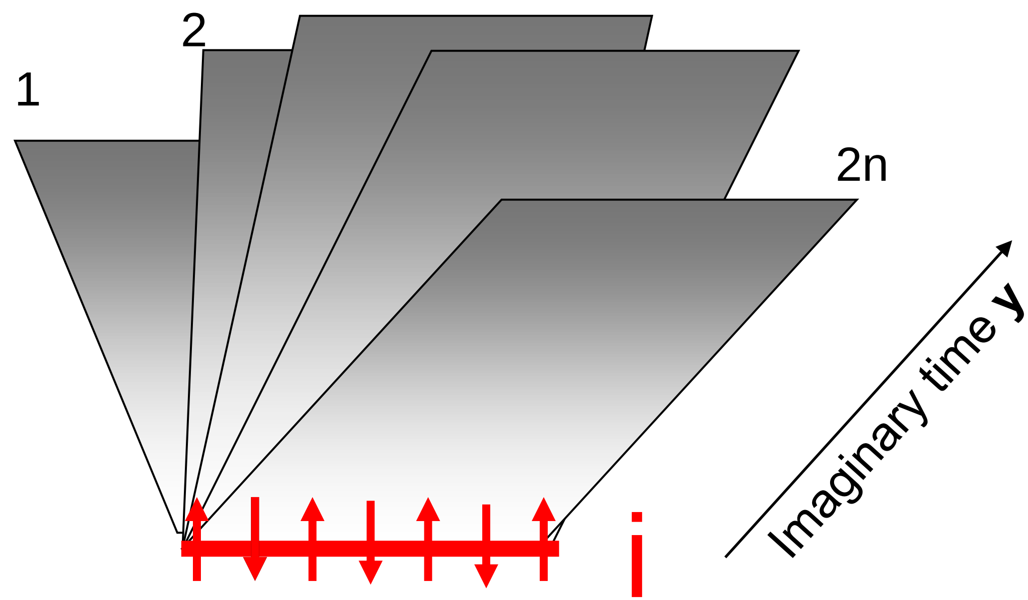
$$h(x,y) \equiv h(x,y) + 2\pi R \quad \text{compactified free field}$$

$$S_g[h] = \frac{g}{4\pi} \int (\vec{\nabla} h)^2 \underbrace{dx}_{\text{space}} \underbrace{dy}_{\text{Imaginary time}} \qquad Z_g = \int D(h) \exp(-S_g[h])$$

Luttinger parameter $\tilde{R} = \sqrt{2g}R$

# Usual approach: replicas and Rényi “book”

$(\rho_i)^n \sim$

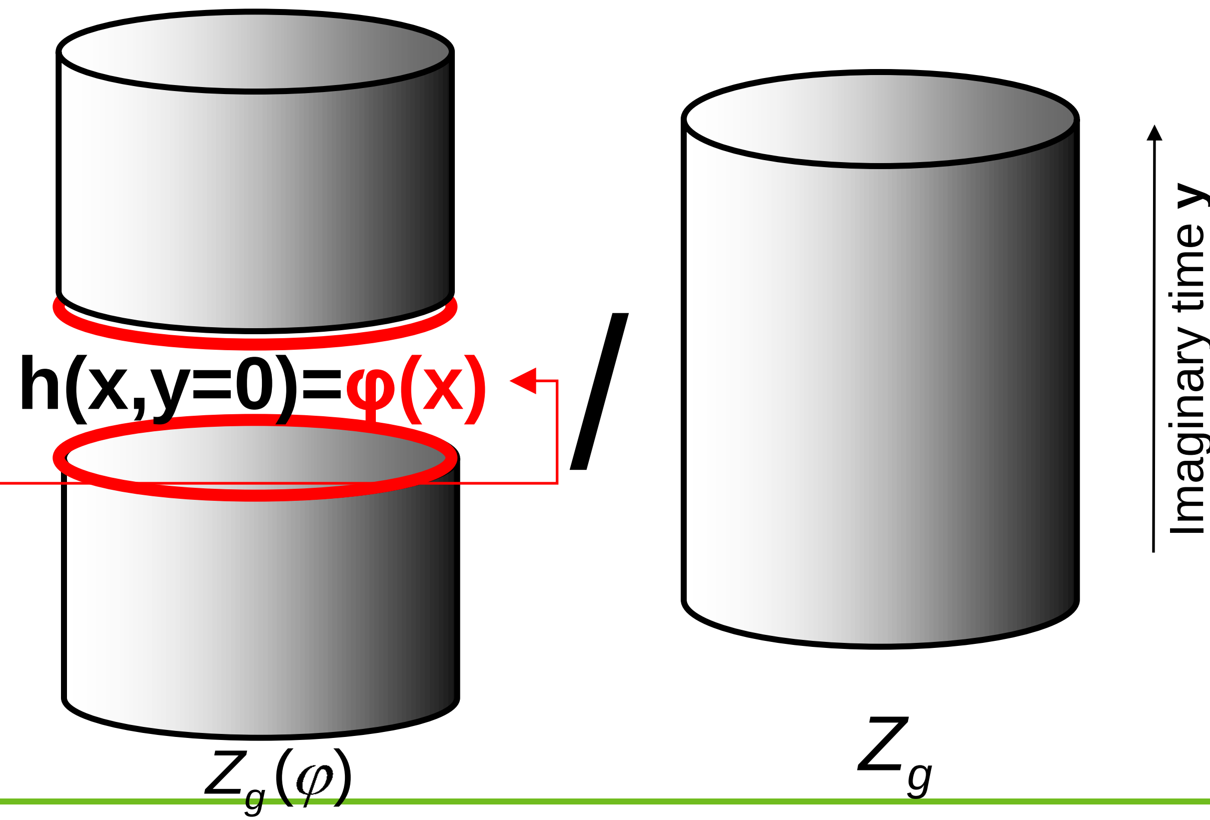


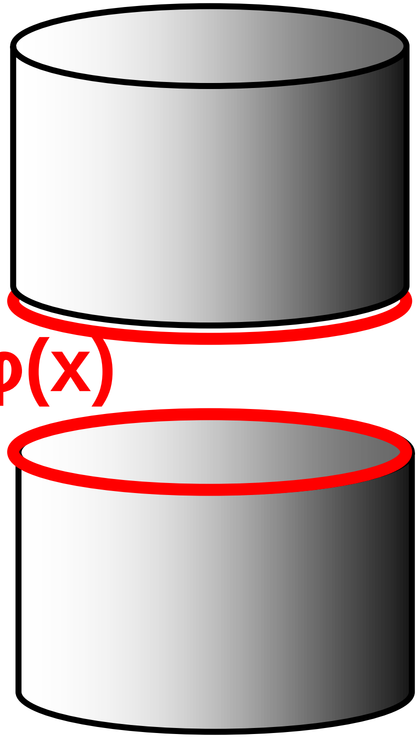
# Free field calculation – without replicas (1/4)

$$h(x,y) \equiv h(x,y) + 2\pi R \quad \text{compactified free field}$$

$$S_g[h] = \frac{g}{4\pi} \int \left(\vec{\nabla} h\right)^2 dx\,dy \qquad Z_g = \int D(h) \exp(-S_g[h])$$

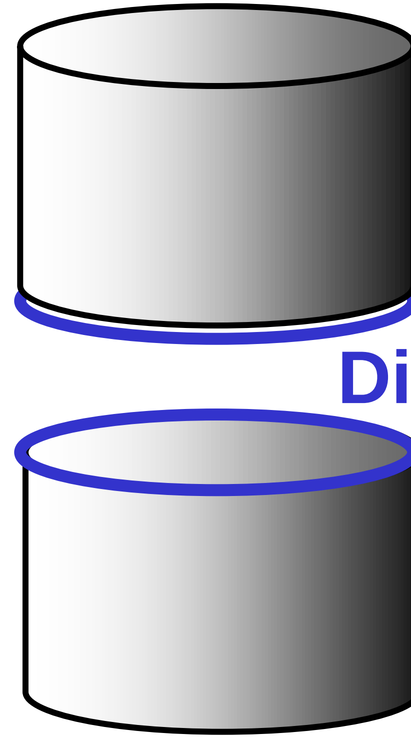
$$p_g(\varphi) = Z_g(\varphi) / Z_g =$$






$Z_g(\varphi)$

$$= \exp(-S_g[h_\varphi]) \times$$




$Z_g(0)$

$h = h_\varphi + \tilde{h}$

$S[h] = S[h_\varphi] + S[\tilde{h}]$

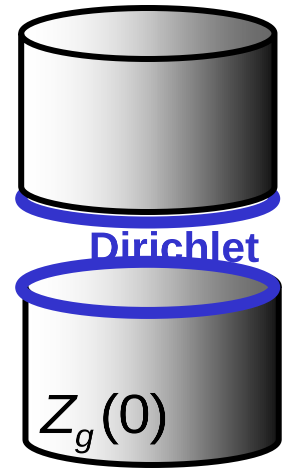
$\Delta h_\varphi = 0 \;;\; h_\varphi(x, y=0) = \varphi(x)$

$\tilde{h}$ : Dirichlet

$$\rightarrow Z_g(\varphi) = \exp(-S_g[h_\varphi]) \cdot Z_g(0)$$

# Free field calculation – without replicas (3/4)

$$Z_g(\varphi) = \exp(-S_g[h_\varphi]) \times$$




$$Z_g(\varphi)^n = \underbrace{\exp(-n \cdot S_g[h_\varphi])}_{S_{ng}[h_\varphi]} \times \left(Z_g(0)\right)^n$$

$$= \frac{Z_{ng}(\varphi)}{Z_{ng}(0)}$$

$$p_g(\varphi)^n = \left(\frac{Z_g(\varphi)}{Z_g}\right)^n = \frac{Z_{ng}(\varphi)}{Z_{ng}(0)} \times \left(\frac{Z_g(0)}{Z_g}\right)^n$$

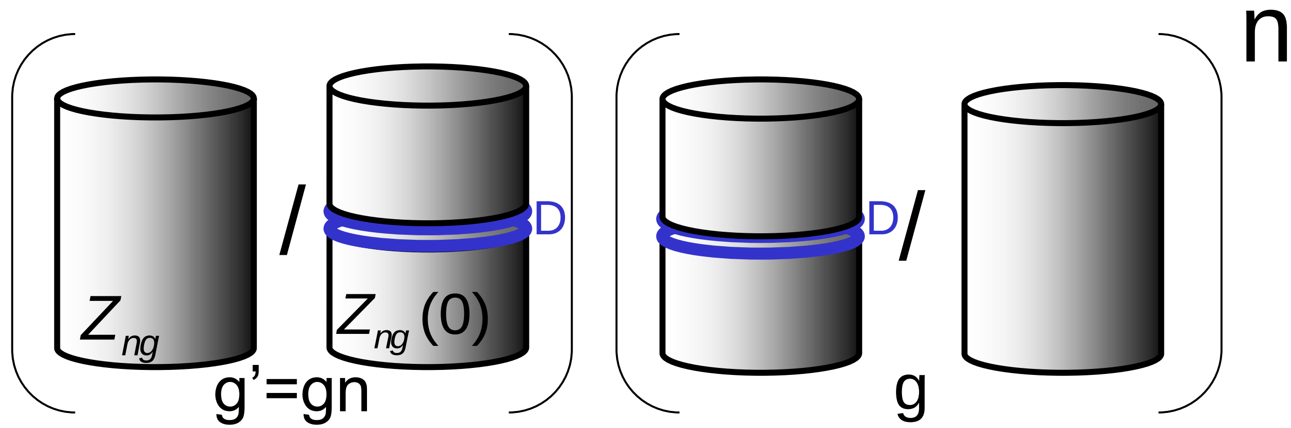
$$\sum_\varphi p_g(\varphi)^n = \frac{Z_{ng}}{Z_{ng}(0)} \times \left(\frac{Z_g(0)}{Z_g}\right)^n$$

=Standard ratio of cylinder partition functions.

n-dependence in the stiffness g'=n.g

# Free field calculation – without replicas (4/4)

$$\sum_\varphi \left(p_\varphi\right)^n =$$




$$\text{Standard result}: Z_g / Z_g(0) = \left(Z_{\text{free}} / Z_{\text{Dirichlet}}\right)^2 = \sqrt{2gR} \quad (g_D{}^2 \text{ of Affleck-Ludwig})$$

$$\text{we finally obtain}: \sum_\varphi \left(p_\varphi\right)^n = \left(\sqrt{2gR}\right)^{1-n} \sqrt{n}$$

Fendley, Saleur and Warner, Nucl. Phys. B **430**, 577 (1994)

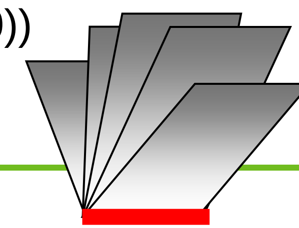
$$\text{And finally}:$$

$$S_n = \frac{1}{1-n} \ln\left(\sum_\varphi p_\varphi{}^n\right)$$

$$S_n = \ln(\tilde{R}) + \frac{1}{2(1-n)} \ln(n) \xrightarrow[n\to 1]{} \ln(\tilde{R}) - \frac{1}{2}$$

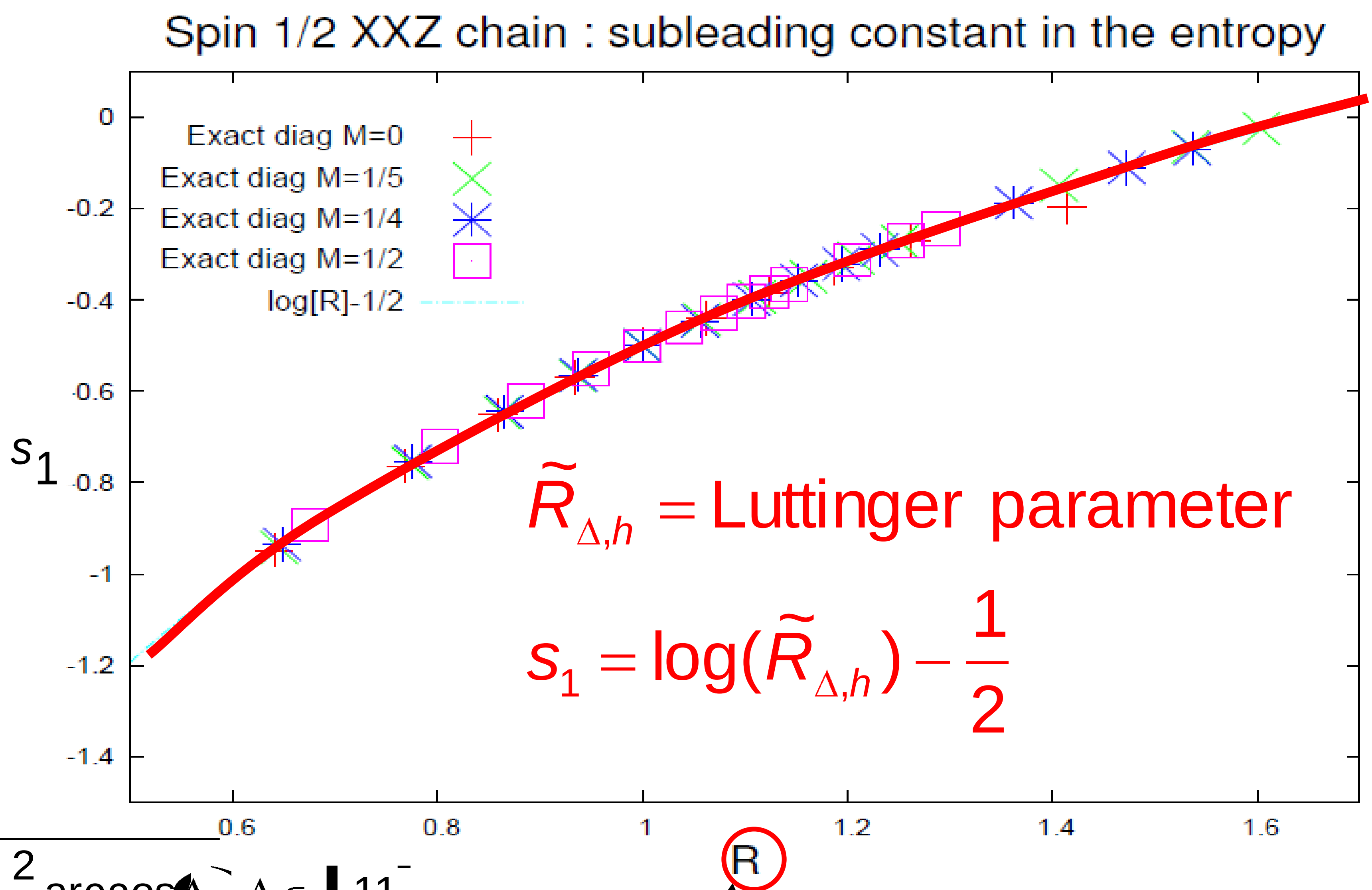
$$\text{with } \tilde{R} = \sqrt{2gR}$$

In agreement with the boundary state construction for the n-sheeted geometry: **Oshikawa arXiv:1007.3739** ; (+Hsu & Fradkin, J. Stat. Mech , P09004 (2010))



# Numerics – XXZ chain at Δ≠0 – Rényi index n=1

Stéphan, Furukawa, GM, Pasquier . Phys. Rev. B 80, 184421 (2009)




$$\tilde{R}_{\Delta,h=0} = \sqrt{2 - \frac{2}{\pi}\arccos(\Delta)}, \quad \Delta \in [-1,1]$$

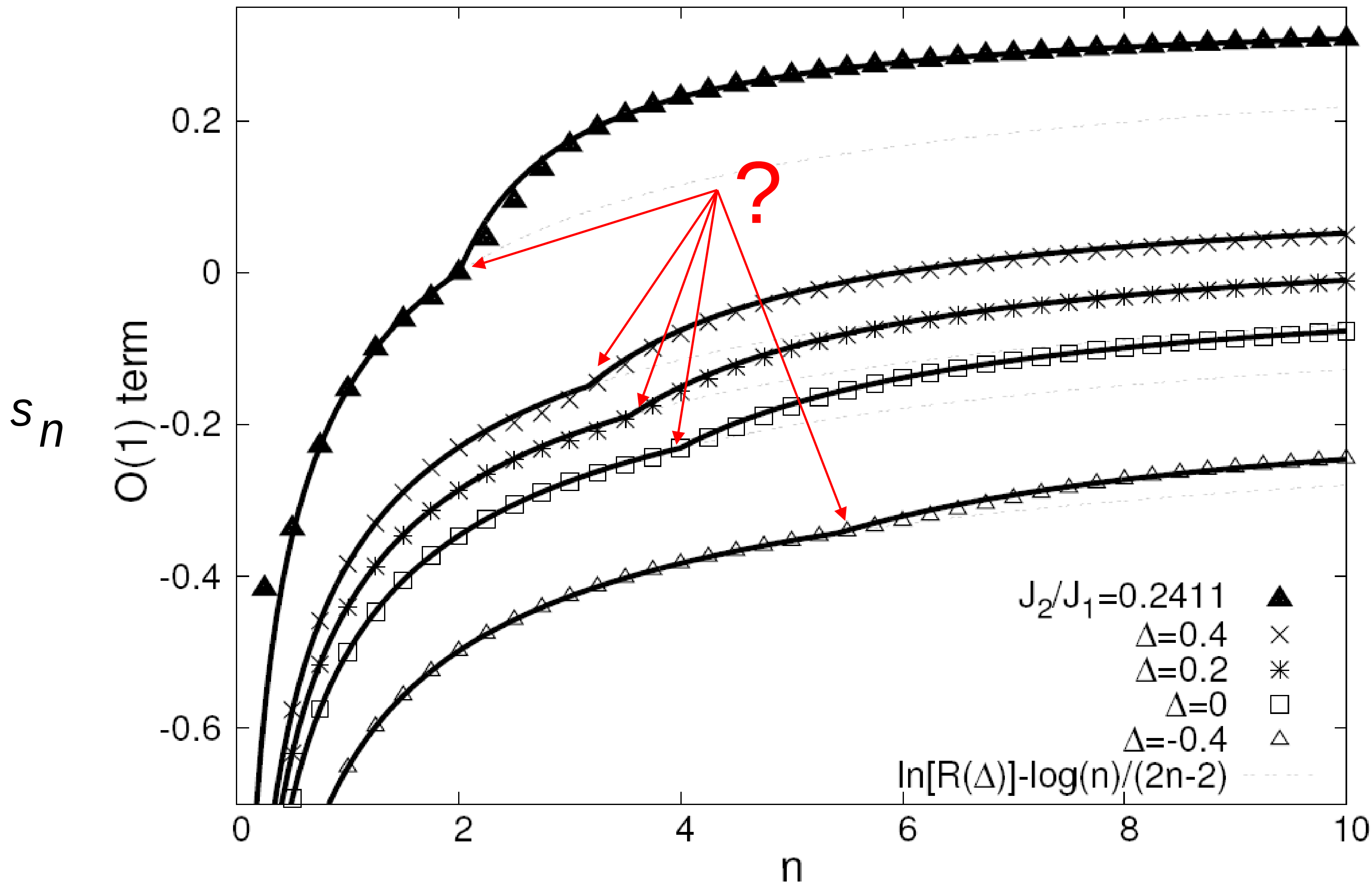
$$S_{n=1} = \ln(\widetilde{R}) - \frac{1}{2}$$

**OK with XXZ chain**

$$S_n = \ln(\widetilde{R}) - \frac{1}{2}\frac{\ln(n)}{(n-1)}$$

**?**

# Numerics for Rényi index n ≠ 1

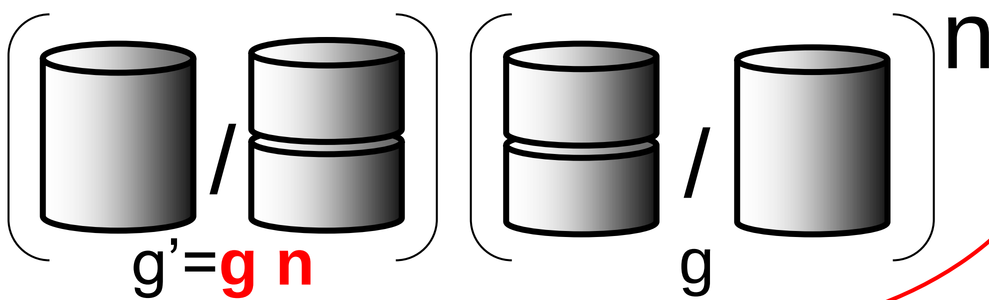


# Boundary phase transition

- Compactified field $h \equiv h + 2\pi R$
- Allowed vertex operators $V_d \cos(d \cdot \frac{h}{R})$
- XXZ chain: **d=2** (Umklapp), …
- $V_d$ renormalizes to zero at the boundary if $d^2 > 2gR^2$

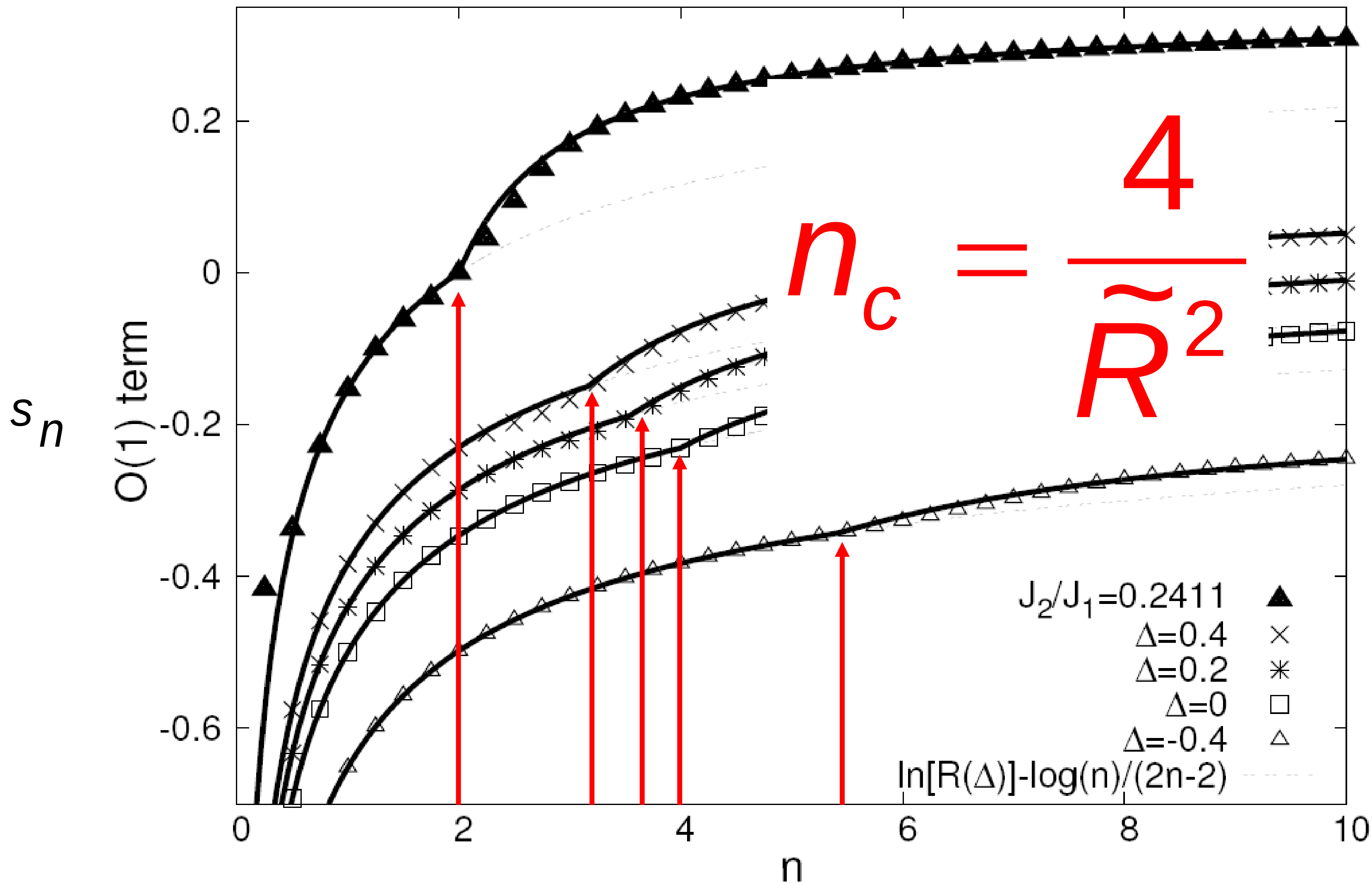
(otherwise locks the field in a one of the *d* minima of the cosine)

- What about :




- **Critical value of n :** $d^2 = 2gn_cR^2 \quad \Rightarrow n_c = \frac{4}{\tilde{R}^2}$

# Critical Rényi index $n_c$

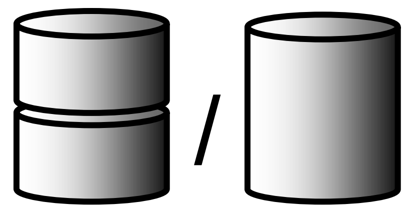


## Locked phase n> $n_c$

$$S_n = \frac{1}{1-n}\log\left(\sum_i (p_i)^n\right) \xrightarrow{n>n_c} \frac{1}{1-n}\log\left(2\,(p_{\max})^n\right)$$

$$|i_{\max}\rangle = |\uparrow\downarrow\uparrow\downarrow\cdots\uparrow\downarrow\rangle \quad \text{or} \quad |i_{\max}\rangle = |\downarrow\uparrow\downarrow\uparrow\cdots\downarrow\uparrow\rangle$$ **d=2**

$$p_{\max} = \frac{Z_{\text{Dirichlet}}}{Z_{\text{free}}} =$$

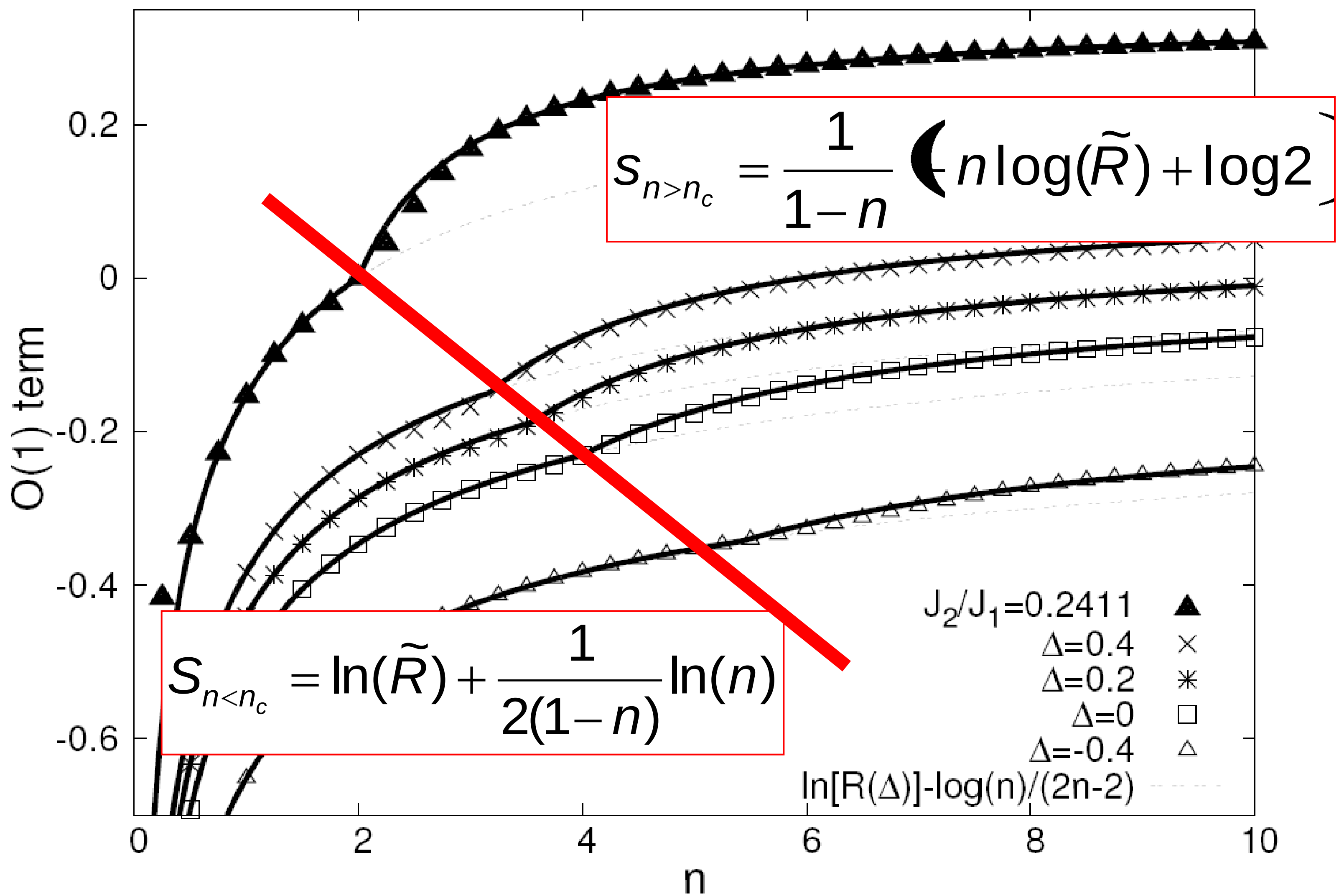


$$= \tilde{R}^{-1}$$

$$s_{n>n_c} = \frac{1}{1-n}\left(-n\log(\tilde{R}) + \log 2\right)$$

… in agreement with numerics

# Subleading entropy constant: summary




Stéphan, GM, Pasquier, arXiv:1104.2544

# Open chains

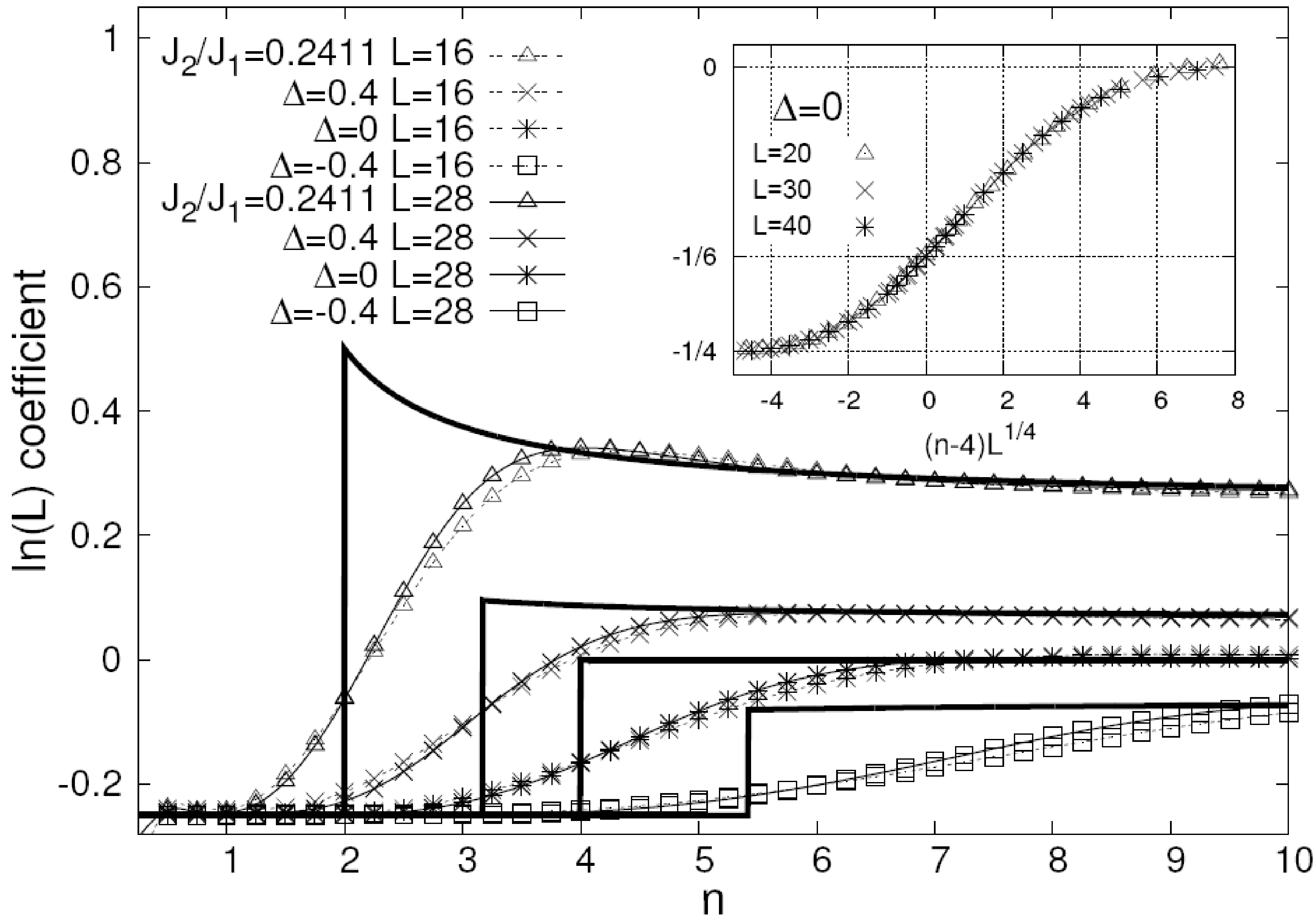
# Luttinger liquid with open boundary conditions

$$S_n(L) = a_n L + b_n \log(L) + \cdots$$

- n < $n_c$ $\quad -\frac{1}{4}$ Zaletel, Bardarson and Moore, PRL 2011
- n > $n_c$ $\quad \frac{n}{n-1}\left(\frac{R^2}{4} - \frac{1}{4}\right)$ Stéphan, GM, Pasquier arXiv:1104.2544
- n = $n_c$ $\quad$ ?

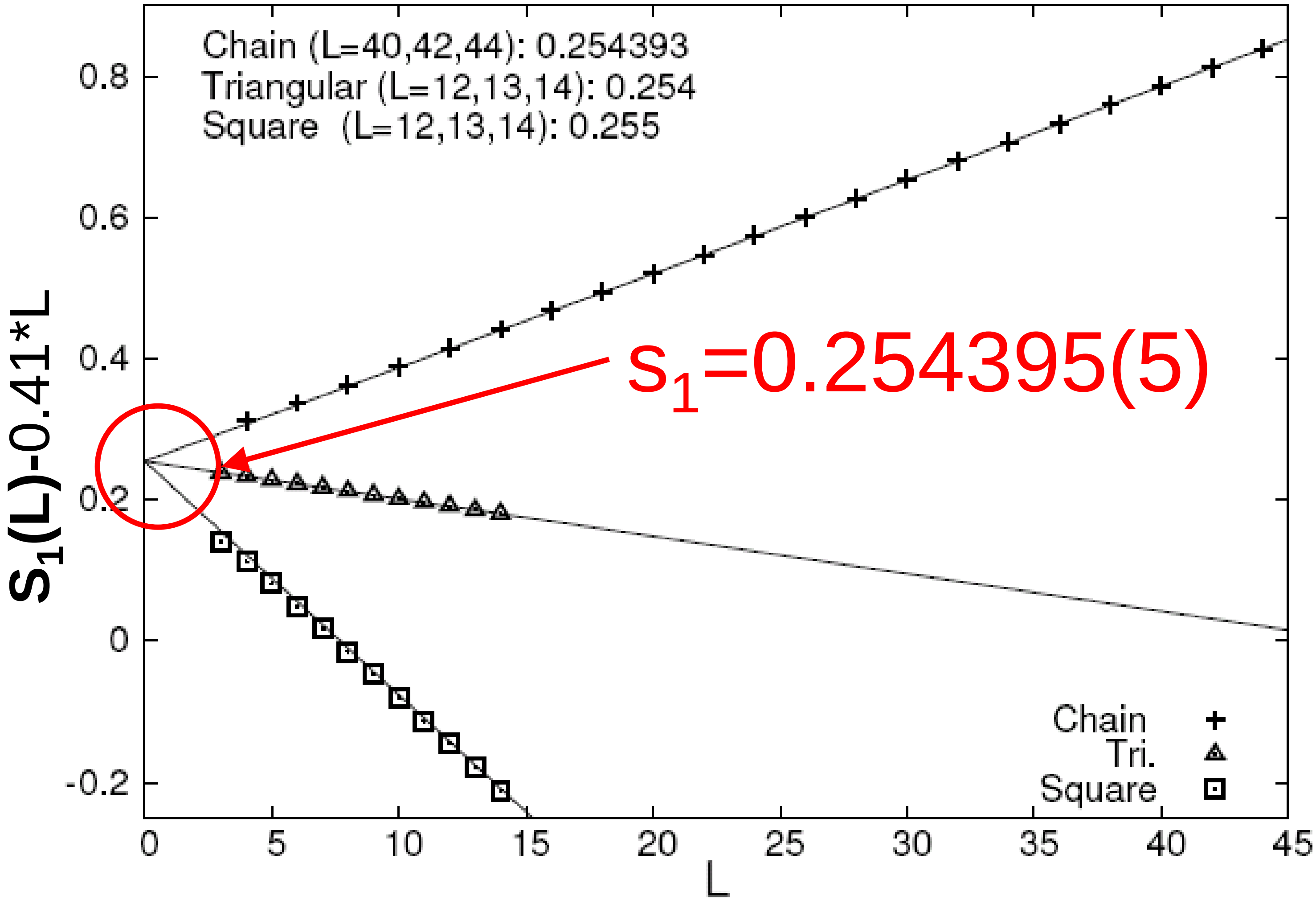
# Open chains and logarithms in the Shannon-Rényi entropy

Stéphan, GM, Pasquier, arXiv:1104.2544



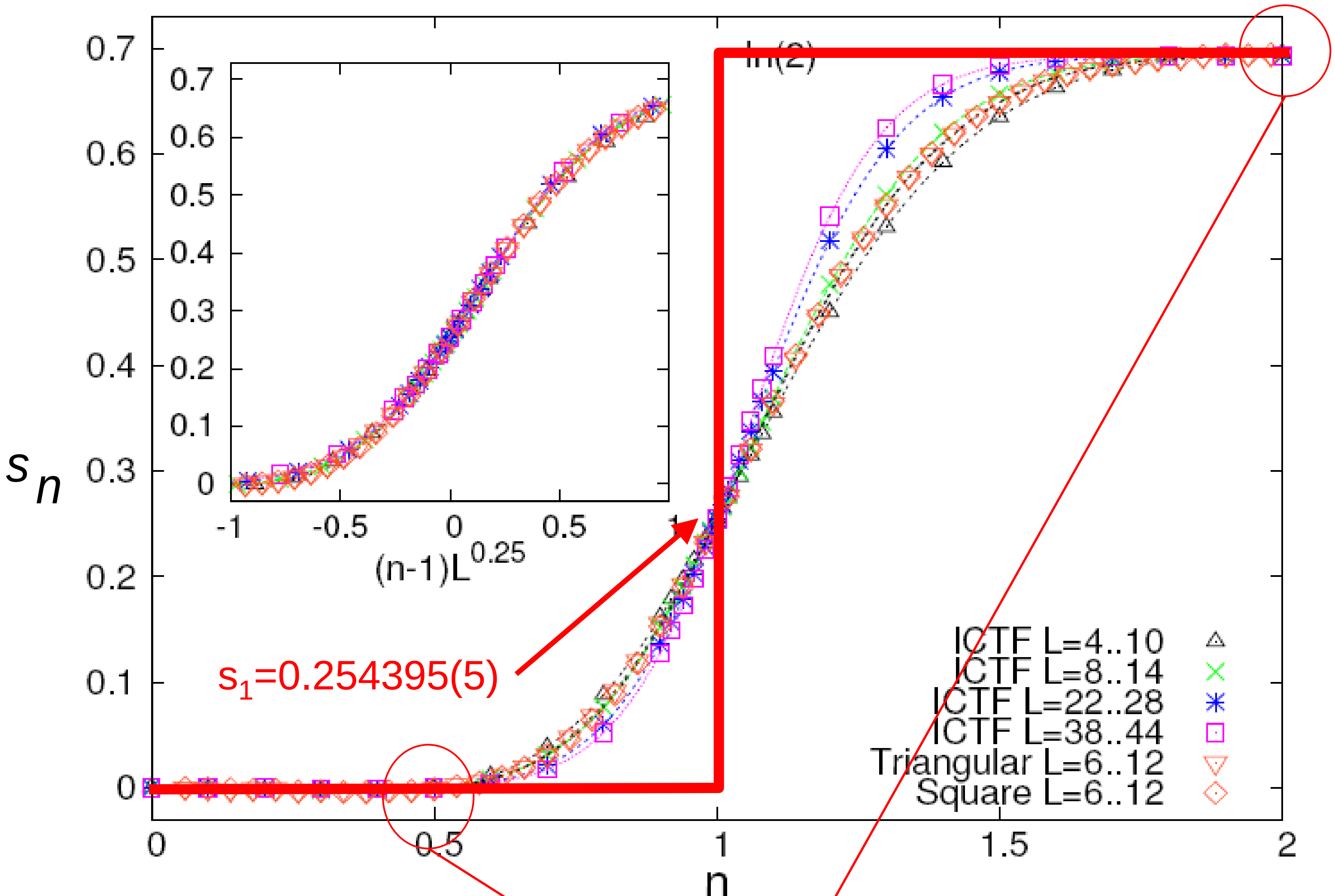
# Ising model

# Critical Ising model: subleading constant in the n=1 entropy



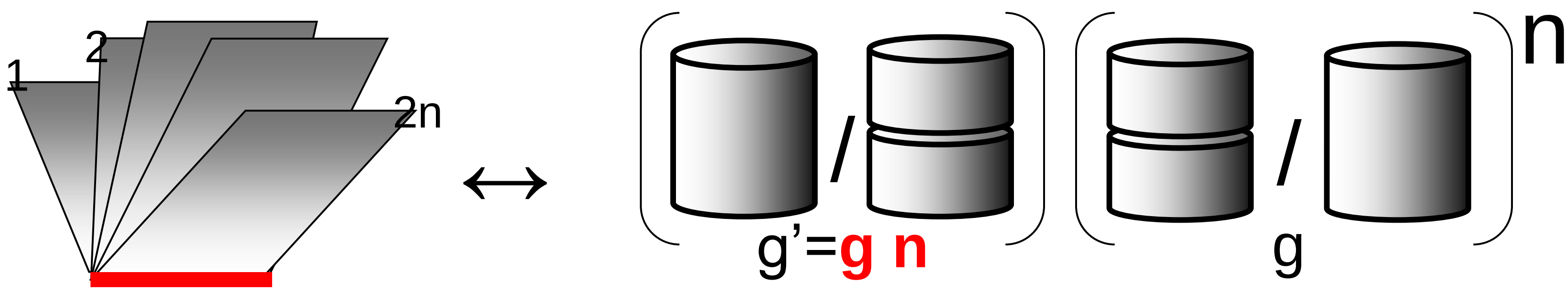

Calculations at critical point, Rényi parameter n=1

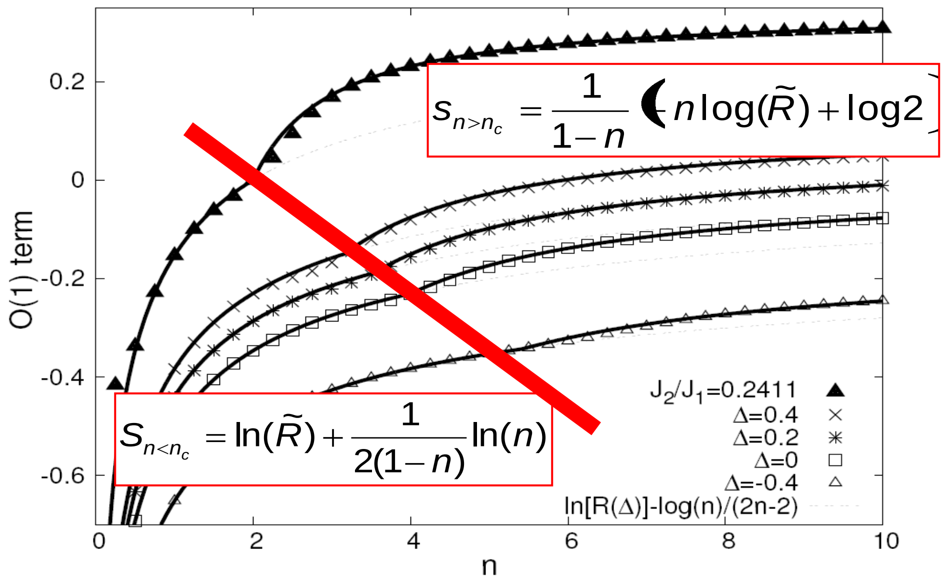
# Critical Ising chain: subleading constant in the entropy, n≠1

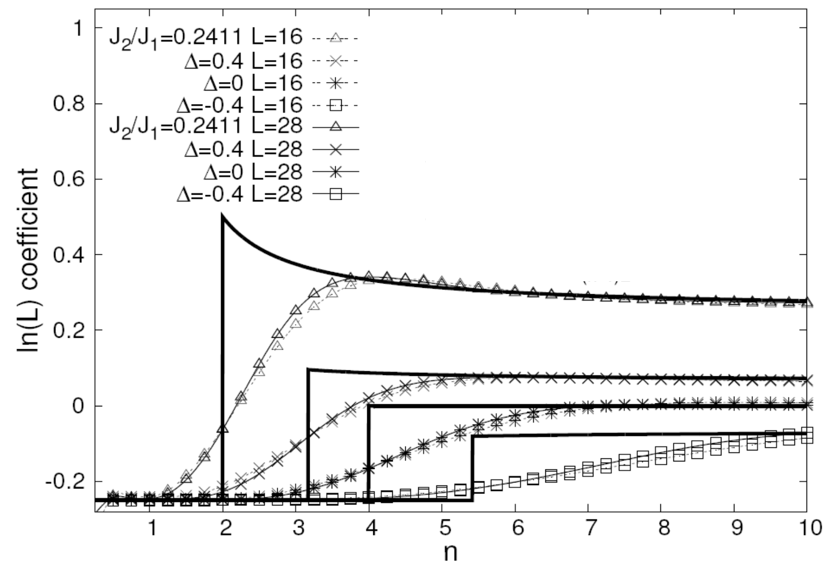



**n=1/2** and **n=∞** are **exactly solvable**
(Affleck-Ludwig boundary entropy
with respectively free and ferro. boundary conditions)

# Summary




J.-M. Stéphan, S. Furukawa, GM and V. Pasquier, Phys Rev. B 80, 184421 (2009)
J.-M. Stéphan, GM, V. Pasquier arXiv:1104.2544
J.-M. Stéphan, GM, and V. Pasquier, Phys. Rev. B 82, 125455 (2010)
J.-M. Stéphan, GM and F. Alet, Phys. Rev. B 82, 180406(R) (2010)

Ising:
**0.254395**(5) = ?

# Entropy for the Ising model & basis choice

$$\mathcal{H} = -\mu \sum_{j=0}^{L-1} \sigma_j^x \sigma_{j+1}^x - \sum_{j=0}^{L-1} \sigma_j^z$$

$$|g\rangle = \text{groundstate}$$

$$|i\rangle = |\uparrow\downarrow\uparrow\downarrow\downarrow \cdots\rangle \text{ z - basis}$$

$$p_i^z = |\langle i|g\rangle|^2$$

$$S^z(L,\mu) = -\sum_i p_i^z \log(p_i^z)$$

$$|j\rangle = |\rightarrow\leftarrow\leftarrow\leftarrow \rightarrow \cdots\rangle \text{ x - basis}$$

$$p_i^x = |\langle j|g\rangle|^2$$

$$S^x(L,\mu) = -\sum_i p_i^x \log(p_i^x)$$

Duality: $S^z(L,\mu) = S^x(L,1/\mu) - \ln(2)$

Critical point:

$S_0^z = -0{,}438754 = 0{,}254393 - \ln(2) = S_0^x - \ln(2)$