Anomalous dimensions for deformed supergroup WZW models

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IHP Program on Advanced Conformal Field Theory

Workshop on Logarithmic CFT and Representation Theory, 5.10.2011

Based on work with T. Creutzig, G. Götz, A. Konechny, V. Mitev and V. Schomerus (in various combinations).

For a recent review see "Conformal superspace σ-models", J.Geom.Phys. 61 (2011) 1703-1716 (not on the arXiv).

Geometrical QFTs: σ-models and their Applications

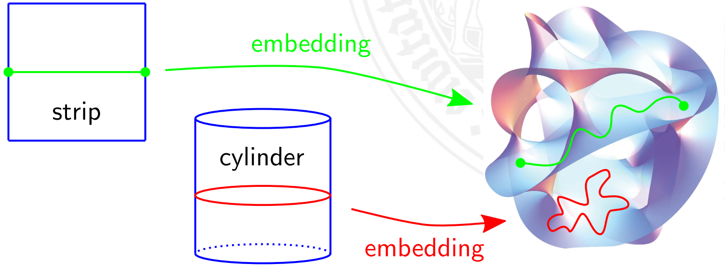
σ-models in a nutshell

World-sheet

2D surface
(w/wo boundaries or handles)

Target space

(Pseudo-)Riemannian manifold
(extra structure: gauge fields, ...)



$\boxed{\sigma\text{-models} = \text{(quantum) field theories}}$

Adding supersymmetry...

Appearances of superspace σ-models

- String theory
  - Quantization of strings in flux backgrounds [Berkovits]
  - String theory / gauge theory correspondence [Maldacena]
  - Moduli stabilization in string phenomenology [KKLT] [...]
- Disordered systems
  - Quantum Hall systems
  - Self avoiding random walks, polymer physics, ...
  - Efetov's supersymmetry trick

Conformal invariance

- String theory: Diffeomorphism + Weyl invariance
- Statistical physics: Critical points / 2$^{\text{nd}}$ order phase transitions

Supergroups and supercosets

Two important classes

- Supergroups
- Supercosets

Outline of the talk

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1. Supergroup WZW models and their deformations
   - Harmonic analysis on supergroups G
   - Free fermion resolution
   - Deformations preserving G and $G \times G$
2. The compactified free boson and its dualities
3. Supercoset σ-models
   - Overview & Applications
   - Conformal invariance
4. A particular example
   - Supersphere σ-models
   - Duality with deformed supergroup WZW models

Supergroup WZW models

Ingredients

- Lie supergroup G, based on simple Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\underline{0}} \oplus \mathfrak{g}_{\underline{1}}$
- Non-degenerate invariant form $\langle\cdot,\cdot\rangle$ (normalized)
- The level $k \in \mathbb{Z}$, corresponding to

$$k\omega_{\text{top}}(g) \; = \; k\big\langle g^{-1}dg, [g^{-1}dg, g^{-1}dg]\big\rangle \in H_3(G) \cong \mathbb{Z}$$

Action functional

$$\mathcal{S}^{\text{WZW}}[g] \; = \; k\mathcal{S}_{\text{metric}}[g] + k\mathcal{S}_{\text{top}}[g]$$

Supergroup WZW models: Solution strategy

Symmetry

$$\text{Isometry } G \times G \quad \Rightarrow \quad \begin{cases} \text{Affine Lie superalgebra } \mathrm{J}(z),\ \bar{\mathrm{J}}(\bar{z}) \\ \mathrm{J}^\mu(z)\,\mathrm{J}^\nu(w) \;=\; \frac{k\kappa^{\mu\nu}}{(z-w)^2} + \frac{if^{\mu\nu}{}_\lambda\,\mathrm{J}^\lambda(w)}{z-w} \end{cases}$$

Solving the supergroup WZW model

$$\underbrace{\begin{matrix} \text{Supergeometry} & \qquad & \infty\text{-dim symmetry} \\ \text{(Harmonic analysis)} & \qquad & \text{(Affine Lie superalgebra)} \end{matrix}}_{\text{Full spectrum}}$$

Supergeometry & Harmonic analysis

$$\left.\begin{matrix} \text{Laplacian non-diagonalizable} \\ \text{Non-chiral state space} \end{matrix}\right\} \quad \Rightarrow \quad \text{Logarithmic CFT}$$

[Schomerus,Saleur'05] [Götz,TQ,Schomerus'06] [Saleur,Schomerus'06] [TQ,Schomerus'07]

Preparations

Step by step

- Some relevant modules of $\mathfrak{g}$
- The algebra of functions on G
- Harmonic analysis
- Lift to the affine Lie superalgebra

Modules of Lie superalgebras

List of relevant modules

- Simple modules $\mathcal{L}_\mu$
- Kac modules $\mathcal{K}_\mu$
- Projective covers $\mathcal{P}_\mu$
- Projective modules $\mathcal{B}_\mu$

Realization: Projective modules

- As an induced module from $\mathfrak{g}_{\underline{0}}$ using *all* fermionic generators

$$\mathcal{B}_\mu \;=\; \mathrm{Ind}_{\mathfrak{g}_{\underline{0}}}^{\mathfrak{g}}(L_\mu) \;\to\; L_\mu \otimes \bigwedge(\mathfrak{g}_{\underline{1}})$$

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Realization: Projective covers

- As indecomposable submodule of $\mathcal{B}_\mu$

$$\mathcal{B}_\mu \;=\; \bigoplus m_{\mu\nu}\mathcal{P}_\nu$$

- By Frobenius reciprocity, the multiplicity satisfies

$$m_{\mu\nu} \;=\; \dim \mathrm{Hom}_{\mathfrak{g}}(\mathcal{B}_\mu, \mathcal{L}_\nu) \;=\; \dim \mathrm{Hom}_{\mathfrak{g}_0}(L_\mu, \mathcal{L}_\nu|_{\mathfrak{g}_0})$$

Modules of Lie superalgebras

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- Projective modules $\mathcal{B}_\mu$

Realization: Kac modules

- Rather complicated in general
- In case of $\mathbb{Z}_2$-compatible $\mathbb{Z}$-grading localized in *three* degrees:

$$\mathfrak{g} \;=\; \mathfrak{g}_{\underline{0}} \oplus \mathfrak{g}_{\underline{1}} \;=\; \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \qquad \text{(“type I”)}$$

one simply induces using *half* of the fermionic generators

$$\mathcal{K}_\mu \;=\; \mathrm{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{g}}(L_\mu) \;\to\; L_\mu \otimes \bigwedge(\mathfrak{g}_{-1})$$

Modules of Lie superalgebras

List of relevant modules

- Simple modules $\mathcal{L}_\mu$
- Kac modules $\mathcal{K}_\mu$
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- Projective modules $\mathcal{B}_\mu$

Realization: Simple modules

- As (possibly trivial) quotients of Kac modules $\mathcal{K}_\mu$ or projective covers $\mathcal{P}_\mu$ by their unique maximal submodule
- Typical: $\mathcal{L}_\mu \cong \mathcal{P}_\mu$
- Atypical: otherwise

The algebra of functions on supergroups

The algebra of functions on G

- Definition: $\mathcal{F}(G) = \mathcal{F}(G_0) \otimes \bigwedge(\mathfrak{g}_{\underline{1}}^*)$
- Action of $\mathfrak{g} \oplus \mathfrak{g}$

Decomposition

- Peter-Weyl Theorem for G_0

$$\mathcal{F}(G_0)\Big|_{\mathfrak{g}_0 \oplus \mathfrak{g}_0} = \bigoplus L_\mu \otimes L_\mu^*$$

- This induces

$$\mathcal{F}(G)\Big|_{\mathfrak{g}_0 \oplus \mathfrak{g}} = \bigoplus L_\mu \otimes \Big(L_\mu \otimes \bigwedge(\mathfrak{g}_{\underline{1}})\Big)^* = \bigoplus L_\mu \otimes \mathcal{B}_\mu^*$$

$$= \bigoplus m_{\mu\nu}\, L_\mu \otimes \mathcal{P}_\nu^* = \bigoplus \mathcal{L}_\nu|_{\mathfrak{g}_0} \otimes \mathcal{P}_\nu^*$$

Peter-Weyl Theorem for supergroups

Peter-Weyl Theorem for supergroups

- Left-right regular action

$$\mathcal{F}(G)\big|_{\mathfrak{g}\oplus\mathfrak{g}} = \bigoplus_{\mu \text{ typ}} \mathcal{L}_\mu \otimes \mathcal{L}_\mu^* \oplus \bigoplus_{[\sigma] \text{ atyp}} \mathcal{I}_{[\sigma]}$$

- Restriction to either left or right action yields

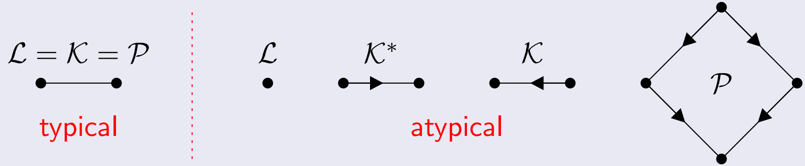
$$\mathcal{F}(G)\big|_{\mathfrak{g}} = \bigoplus_{\mu \text{ typ}} \dim(\mathcal{L}_\mu^*)\,\mathcal{L}_\mu \oplus \bigoplus_{\sigma \text{ atyp}} \dim(\mathcal{L}_\sigma^*)\,\mathcal{P}_\sigma$$

- Laplacian Δ is not diagonalizable on the spaces $\mathcal{I}_{[\sigma]}$

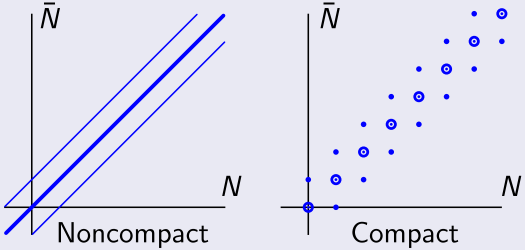
["Common lore"] [TQ,Schomerus'07] [Mitev,TQ,Schomerus'11]

Example: Harmonic analysis on GL(1|1)

Relevant representations



The space of functions $\mathcal{F}(\mathrm{GL}(1|1))$



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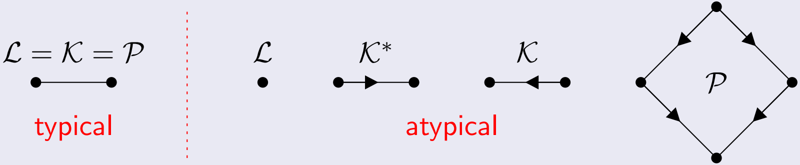
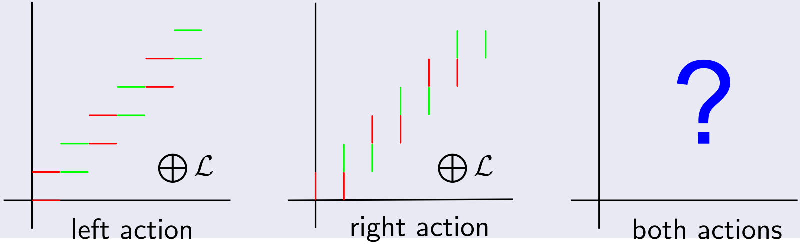


The space of functions $\mathcal{F}(\mathrm{GL}(1|1))$

In the typical sectors one finds the spectrum



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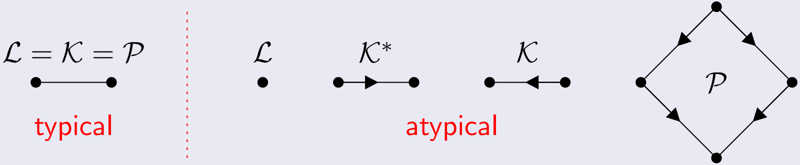

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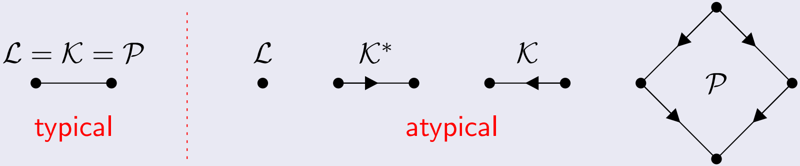
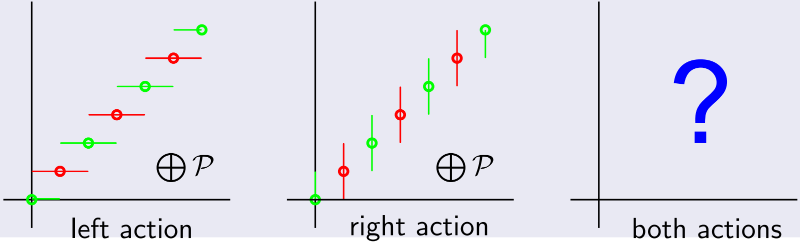


The space of functions $\mathcal{F}(\mathrm{GL}(1|1))$

In the atypical sectors of the WZW model, however, one obtains



Example: Harmonic analysis on GL(1|1)

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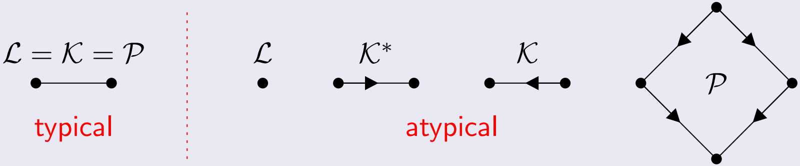

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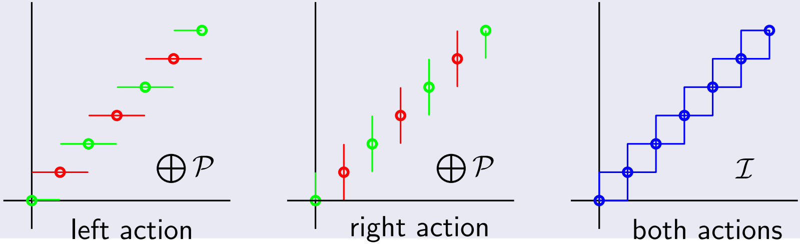


Lift to the full WZW model

Free fermion resolution (type I only)

$$\text{Supergroup WZW model on } G \quad = \quad \begin{cases} \text{Bosonic WZW model on } G_0 \\ \text{Symplectic fermions } (\to \mathfrak{g}_{\pm 1}) \\ \text{Interactions} \end{cases}$$

[Schomerus,Saleur'05] [Götz,TQ,Schomerus'06] [Saleur,Schomerus'06] [TQ,Schomerus'07]

Comparison with standard free field constructions

	Standard	Here
Grading	$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus \mathfrak{g}_\alpha$	$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$
Gauss decomposition	$\exp(X^+)\, t\, \exp(X^-)$	$\exp(F^+)\, g_0\, \exp(F^-)$

Deformations of supergroup WZW models

Deformation operator

Observation

Supergroup WZW models with vanishing Killing form possess non-standard marginal deformations [Bershadsky,Vaintrob,Zhukov]

Marginal deformations of supergroup WZW models

$$\text{Full supersymmetry } G\times G: \qquad \mathcal{S}_{\text{def}} = \int\langle \mathrm{J}, \mathrm{Ad}_g(\bar{\mathrm{J}})\rangle$$

$$\text{Diagonal supersymmetry } G: \qquad \mathcal{S}_{\text{def}} = \int\langle \mathrm{J}, \bar{\mathrm{J}}\rangle$$

Data

Field	$\mathrm{J}^\mu(z)$	$\bar{\mathrm{J}}^\nu(\bar{z})$	$\mathrm{Ad}_g \to : \phi_{\mu\nu}(g(z,\bar{z}))$:	$\mathrm{Ad}_g(\bar{J})$
Representation	(ad, 0)	(0, ad)	(ad, ad)	(ad, 0)
Dimension $(h,\bar{h})$	(1, 0)	(0, 1)	(0, 0)	(0,1)

Deformation operator

Observation

Supergroup WZW models with vanishing Killing form possess non-standard marginal deformations [Bershadsky,Vaintrob,Zhukov]

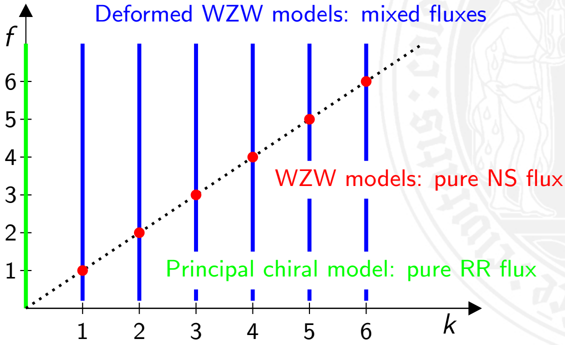
Marginal deformations of supergroup WZW models

$$
\begin{aligned}
\text{Full supersymmetry } G\times G: &\qquad \mathcal{S}_{\text{def}} = \int\langle \mathrm{J},\mathrm{Ad}_g(\bar{\mathrm{J}})\rangle \\
\text{Diagonal supersymmetry } G: &\qquad \mathcal{S}_{\text{def}} = \int\langle \mathrm{J},\bar{\mathrm{J}}\rangle
\end{aligned}
$$

Data

Field	$\mathrm{J}^\mu\bar{\mathrm{J}}_\mu$	$:\mathrm{J}^\mu\phi_{\mu\nu}\bar{\mathrm{J}}^\nu:$
Representation	$(\mathrm{ad},\mathrm{ad})$	$(0,0)$
Dimension $(h,\bar{h})$	$(1,1)$	$(1,1)$

The moduli space of supergroup σ-models



String theory ($AdS_3 \times S^3$)
[Berkovits,Vafa,Witten]
[Bershadsky,Vaintrob,Zhukov]
[Götz,TQ,Schomerus]
[TQ,Schomerus,Creutzig]
[Ashok,Benichou,Troost]
[Benichou,Troost]

Condensed matter theory (IHQE)
[Zirnbauer]
[Bhaseen,Kog.,Sol.,Tan.,Tsvelik]
[Tsvelik]
[Obuse,Sub.,Fur.,Gruz.,Ludwig]

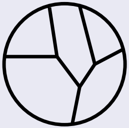
Action functional

$$\mathcal{S}^{\mathrm{WZW}}[g] \;=\; f\,\mathcal{S}_{\mathrm{metric}}[g] + k\,\mathcal{S}_{\mathrm{top}}$$

Sketch of conformal invariance

The β-function vanishes identically...

$$\beta = \sum_{\substack{\text{certain} \\ G\text{-invariants}}}$$



$$= 0$$

Ingredients: Invariant metric: $\kappa^{\mu\nu}$

Structure constants: $f^{\mu\nu\lambda}$

Sketch of conformal invariance

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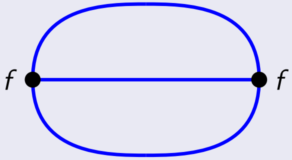

$= 0$

Ingredients: Invariant metric: $\kappa^{\mu\nu}$

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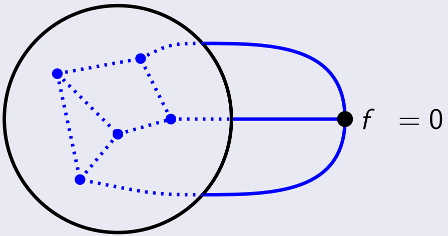
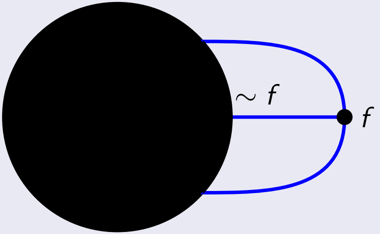


There is a unique invariant rank 3 tensor!

[Bershadsky,Zhukov,Vaintrob'99] [Babichenko'06]

Sketch of conformal invariance

The β-function vanishes identically...



$= 0$

There is a unique invariant rank 3 tensor!

[Bershadsky,Zhukov,Vaintrob'99] [Babichenko'06]

Solution strategy

How to deal with deformed supergroup WZW models?

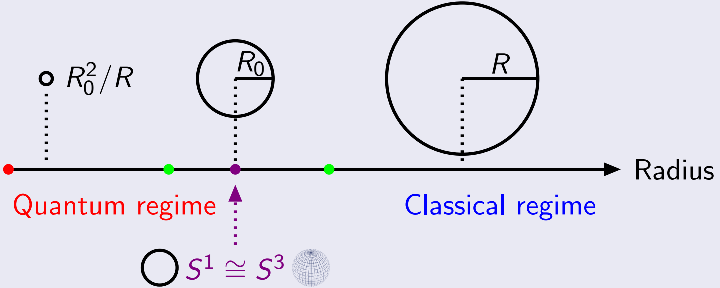
- Quasi-abelian perturbation theory
  $\rightarrow$ This talk [Bershadsky,Zhukov,Vaintrob] [TQ,Schomerus,Creutzig] [Mitev,TQ,Schomerus] [Konechny,TQ]

- Non-chiral current algebras
  $\rightarrow$ Troost's talk [Ashok,Benichou,Troost] [Benichou,Troost] [Konechny,TQ]

$U(1)$ WZW models

The compactified free boson

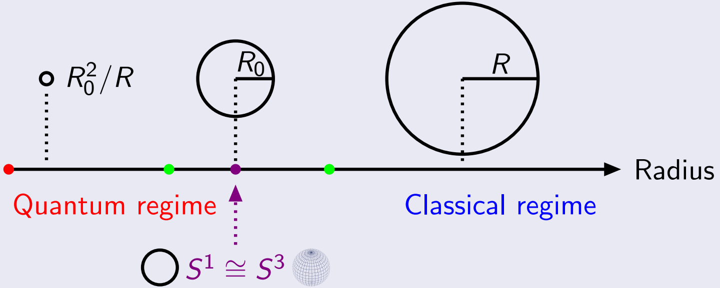
Free boson theories



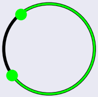
Two lessons

- There is an equivalence: $R \leftrightarrow R_0^2/R$ ("T-duality")
- In the quantum regime geometry starts to loose its meaning

Free boson theories



An open string partition function



$$Z_{\mathrm{op}}(q,z|R) \;=\; \mathrm{tr}\Big[z^{\mathrm{P}}\, q^{\mathrm{Energy}(R)}\Big] \;=\; \frac{1}{\eta(q)} \sum_{w\in\mathbb{Z}} z^{w}\, q^{\frac{w^2}{2R^2}}$$

Goals of the remaining talk

Next steps

- Tackle supercoset σ-models
- Employ deformations of supergroup WZW models
- Use quasi-abelian perturbation theory, i.e. reduce calculations to the free boson case

Supercosets

Appearance of supercosets

String backgrounds as supercosets...

Minkowski	$\mathrm{AdS}_5 \times S^5$	$\mathrm{AdS}_4 \times \mathbb{CP}^3$	$\mathrm{AdS}_2 \times S^2$
$\frac{\text{super-Poincaré}}{\text{Lorentz}}$	$\frac{\mathrm{PSU}(2,2\|4)}{\mathrm{SO}(1,4)\times\mathrm{SO}(5)}$	$\frac{\mathrm{OSP}(6\|2,2)}{\mathrm{U}(3)\times\mathrm{SO}(1,3)}$	$\frac{\mathrm{PSU}(1,1\|2)}{\mathrm{U}(1)\times\mathrm{U}(1)}$

[Metsaev,Tseytlin] [Berkovits,Bershadsky,Hauer,Zhukov,Zwiebach] [Arutyunov,Frolov]

Supercosets in statistical physics...

IQHE	Dense polymers	Dense polymers
(non-conformal)	$S^{2S+1\|2S}$	$\mathbb{CP}^{S-1\|S}$
$\frac{\mathrm{U}(1,1\|2)}{\mathrm{U}(1\|1)\times\mathrm{U}(1\|1)}$	$\frac{\mathrm{OSP}(2S+2\|2S)}{\mathrm{OSP}(2S+1\|2S)}$	$\frac{\mathrm{U}(S\|S)}{\mathrm{U}(1)\times\mathrm{U}(S-1\|S)}$

[Weidenmüller] [Zirnbauer]

[Read,Saleur] [Candu,Jacobsen,Read,Saleur]

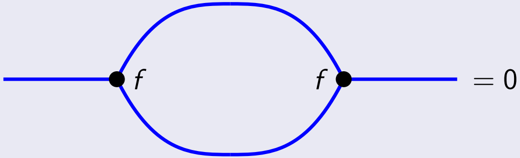
A unifying construction

Definition of the cosets

$$G/H : \quad gh \sim g$$

Some additional requirements for conformal invariance

- $H \subset G$ is invariant subgroup under an automorphism
- Ricci flatness ("super Calabi-Yau") $\Leftrightarrow$ vanishing Killing form

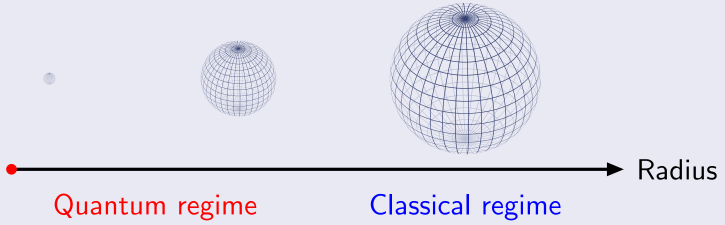


Examples: Cosets of $\mathrm{PSU}(N|N)$, $\mathrm{OSP}(2S+2|2S)$, $\mathrm{D}(2,1;\alpha)$.

For complete list, see [Candu,Creutzig,Mitev,Schomerus]

Properties of conformal supercoset models

The moduli space of generic supercoset theories



Properties at a glance

- Supersymmetry G: $g \mapsto kg$ (realized geometrically)
- Conformal invariance [Kagan,Young] [Babichenko] [Candu,Creutzig,Mitev,Schomerus]
- Integrability [Pohlmeyer] [Lüscher] ... [Bena,Polchinski,Roiban] [Young]

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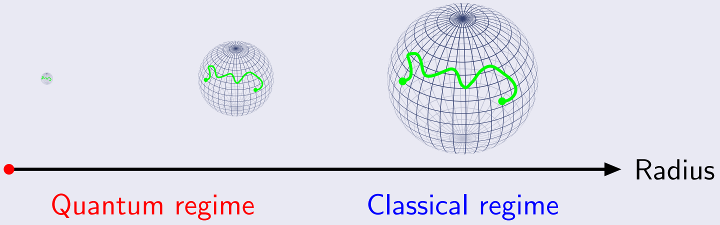


The general open string partition function

$$Z(q,z|R) = \mathrm{tr}\left[z^{\text{Cartan}}\, q^{\text{Energy}(R)}\right] = \sum_{\Lambda} \underbrace{\psi_\Lambda(q,R)}_{\text{Dynamics}}\ \underbrace{\chi_\Lambda(z)}_{\text{Symmetry}}$$

Supersphere σ-Models

The supersphere $S^{3|2}$

Realization of $S^{3|2}$ as a submanifold of flat superspace $\mathbb{R}^{4|2}$

$$\vec{X} = \begin{pmatrix} \vec{x} \\ \eta_1 \\ \eta_2 \end{pmatrix} \qquad \text{with} \qquad \vec{X}^2 = \vec{x}^2 + 2\eta_1\eta_2 = R^2$$

Symmetry

$$\mathrm{O}(4) \times \mathrm{SP}(2) \xrightarrow[\text{symmetrization}]{\text{super-}} \mathrm{OSP}(4|2)$$

Realization as a supercoset

$$S^{3|2} = \frac{\mathrm{OSP}(4|2)}{\mathrm{OSP}(3|2)}$$

The supersphere σ-model

Action functional

$$\mathcal{S}_\sigma = \int \partial_\mu \vec{X} \cdot \partial^\mu \vec{X} \qquad \text{with} \qquad \vec{X}^2 = R^2$$

The space of states for freely moving open strings

$$\prod X^{a_i} \prod \partial_t X^{b_j} \prod \partial_t^2 X^{c_k} \cdots \qquad \text{and} \qquad \vec{X}^2 = R^2$$

$\Rightarrow$ Products of coordinate fields and their derivatives

Large volume partition function

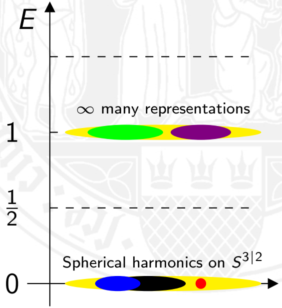
- "Single particle energies" add up $\rightarrow \#$ derivatives
- Partition function is pure combinatorics [Candu,Saleur] [Mitev,TQ,Schomerus]

Sketch of the large volume partition function

Basis not appropriate for $R \ll \infty$

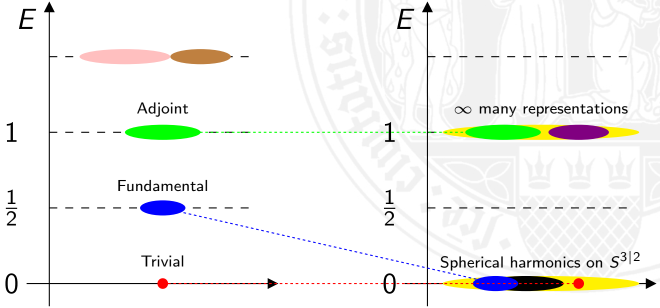
$\partial_t X^i \prod X^{a_j} \longrightarrow$ 1

$\prod X^{a_j} \longrightarrow$ 0



Weak curvature ($R = \infty$)

Sketch of the large volume partition function

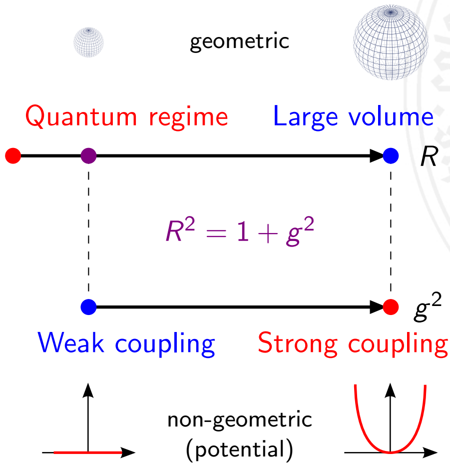


Quantum regime ($R = 1$)

Weak curvature ($R = \infty$)

A Duality for Supersphere σ-Models

A world-sheet duality for superspheres?



Supersphere σ-model

$$Z_\sigma(q, z|R)$$

$$\|$$

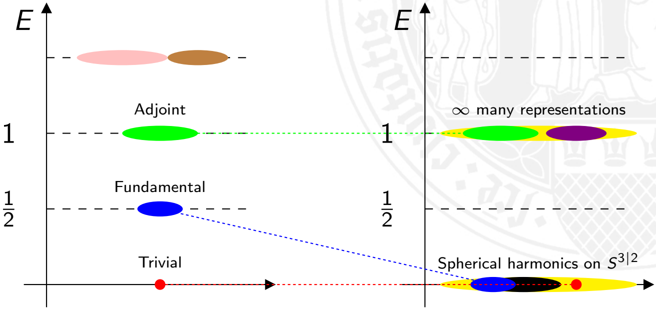
$$Z_{\mathrm{GN}}(q, z|g^2)$$

OSP$(4|2)$ **Gross-Neveu model**

[Candu,Saleur][2] [Mitev,TQ,Schomerus]

Interpolation of an open string spectrum

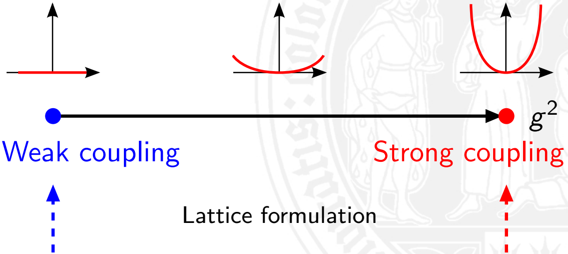
In the two extreme limits the spectrum has the form...



Quantum regime ($R = 1$)

Weak curvature ($R = \infty$)

Evidence for the duality

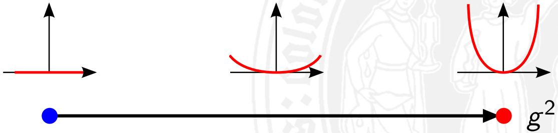


Free fields / WZW model
Kac-Moody symmetry

Free theory
Combinatorics

[Candu,Saleur]^2 [Mitev,TQ,Schomerus]

Evidence for the duality



Weak coupling

Strong coupling

Lattice formulation

Free fields / WZW model
Kac-Moody symmetry

Free theory
Combinatorics

$$\boxed{\text{Goal:} \quad Z_{\mathrm{GN}}(q,z|g^2) \;=\; \sum_\Lambda \psi_\Lambda^\sigma(q,g^2)\,\chi_\Lambda(z)}$$

[Candu,Saleur] [Mitev,TQ,Schomerus]

OSP(4|2) Gross-Neveu Model

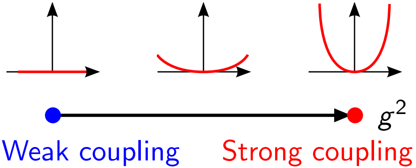
The OSP(4|2) Gross-Neveu model

Field content

- Fundamental OSP(4|2)-multiplet $(\psi_1, \psi_2, \psi_3, \psi_4, \beta, \gamma)$
- All these fields have scaling dimension $1/2$

Formulation as a Gross-Neveu model

$$\mathcal{S}_{\text{GN}} = \mathcal{S}_{\text{free}} + g^2\,\mathcal{S}_{\text{int}} \qquad \begin{cases} \mathcal{S}_{\text{free}} = \int \left[\psi\bar{\partial}\psi + \beta\bar{\partial}\gamma + h.c.\right] \\ \mathcal{S}_{\text{int}} = \int \left[\psi\bar{\psi} + \beta\bar{\gamma} - \gamma\bar{\beta}\right]^2 \end{cases}$$



Goal: $Z_{\text{GN}}(q, z|g^2)$

An open string spectrum

Formulation as a deformed OSP(4|2) WZW model

$$\mathcal{S}_{\text{GN}} = \mathcal{S}_{\text{WZW}} + g^2\,\mathcal{S}_{\text{def}} \qquad \text{with} \qquad \mathcal{S}_{\text{def}} = \int \langle \text{J}, \bar{\text{J}} \rangle$$

Solution at $g = 0$

- At $g = 0$ there is an OSP(4|2) Kac-Moody algebra symmetry

$$\text{J}^\mu(z)\,\text{J}^\nu(w) = \frac{k\,\kappa^{\mu\nu}}{(z-w)^2} + \frac{i f^{\mu\nu}{}_\lambda \text{J}^\lambda(w)}{z-w}$$

- Partition functions can be constructed using combinatorics

An open string partition function for $g = 0$

$$Z_{\text{GN}}(g^2 = 0) = \sum_\Lambda \underbrace{\psi_\Lambda^{\text{WZW}}(q)}_{\text{energy levels}} \quad \underbrace{\chi_\Lambda(z)}_{\text{OSP}(4|2)\text{ content}}$$

An open string spectrum

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- Partition functions can be constructed using combinatorics

An open string partition function for all g

$$Z_{\mathrm{GN}}(g^2) = \sum_\Lambda \underbrace{q^{-\frac{1}{2}\frac{g^2}{1+g^2}C_\Lambda}}_{\text{anomalous dimension}} \; \underbrace{\psi_\Lambda^{\mathrm{WZW}}(q)}_{\text{energy levels}} \; \underbrace{\chi_\Lambda(z)}_{\text{OSP(4|2) content}}$$

An annulus partition function

Specific boundary conditions in the OSP(4|2) WZW model...

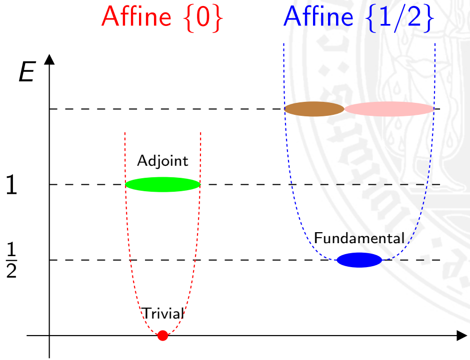
For a certain class of open strings one obtains

$$Z_{\mathrm{GN}}(g^2=0) \;=\; \underbrace{\chi_{\{0\}}(q,z)}_{\text{vacuum}} + \underbrace{\chi_{\{1/2\}}(q,z)}_{\text{fundamental}}$$

The problem (yet again...)

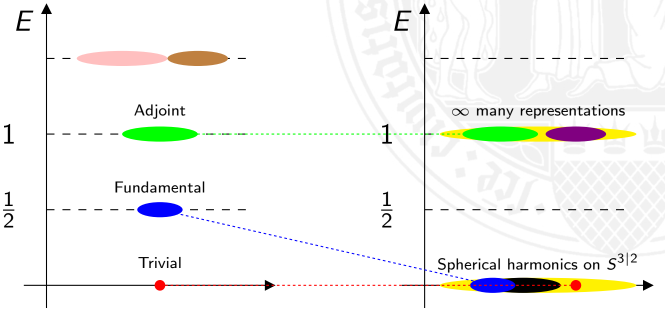
$\boxed{\text{Organize this into representations of OSP}(4|2)\text{!}}$

What did we achieve now?



$\boxed{\textbf{WZW model } (g = 0)}$

$$Z_{\mathrm{GN}}(g^2) \;=\; \sum_{\Lambda} \underbrace{q^{-\frac{1}{2}\frac{g^2}{1+g^2}C_\Lambda}}_{\text{anomalous dimension}} \; \underbrace{\psi_\Lambda^{\mathrm{WZW}}(q)}_{\text{energy levels}} \; \underbrace{\chi_\Lambda(z)}_{\mathrm{OSP}(4|2)\text{ content}}$$



WZW model ($g = 0$)

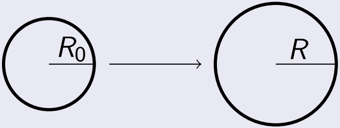
Strong deformation ($g = \infty$)

Supersphere σ-model at $R \to \infty$

Quasi-abelian Deformations

Radius deformation of the free boson revisited

Consider a deformation...



$$R = R_0\sqrt{1+\gamma}$$

Freely moving open strings on a circle of radius R...

$$Z(q,z|R) \;=\; \frac{1}{\eta(q)}\sum_{w\in\mathbb{Z}} z^w\, q^{\frac{w^2}{2R^2}} \;=\; \frac{1}{\eta(q)}\sum_{w\in\mathbb{Z}} q^{\frac{w^2}{2R_0^2(1+\gamma)}}\,\chi_w(z)$$

Anomalous dimensions

$$\delta_\gamma E_w \;=\; \frac{w^2}{2R_0^2}\left[\frac{1}{1+\gamma}-1\right] \;=\; -\frac{\gamma}{1+\gamma}\,\frac{w^2}{2R_0^2} \;=\; -\frac{\gamma}{1+\gamma}\,C_2(w)$$

Quasi-abelianness of supergroup WZW theories

The effective deformation for conformal dimensions

- The combinatorics of the perturbation series is determined by the current algebra

$$J^{\mu}(z)\,J^{\nu}(w) = \frac{k\,\kappa^{\mu\nu}}{(z-w)^2} + \frac{i f^{\mu\nu}{}_{\lambda} J^{\lambda}(w)}{z-w} \sim \frac{k\,\kappa^{\mu\nu}}{(z-w)^2}$$

- Vanishing Killing form $\Rightarrow$ the perturbation is quasi-abelian (for the purposes of calculating anomalous dimensions)

[Bershadsky,Zhukov,Vaintrob] [TQ,Schomerus,Creutzig]

- In the OSP(4|2) WZW model a representation Λ shifts by

$$\delta E_{\Lambda}(g^2) = -\frac{1}{2}\frac{g^2 C_{\Lambda}}{1+g^2}$$

Conclusions

Conclusions and Outlook

Conclusions

- Supergroup WZW models and its deformations provide interesting geometric examples of logarithmic CFTs
- Using supersymmetry we determined the full spectrum of anomalous scaling dimensions for certain annulus partition functions in Gross-Neveu models as a function of the moduli
- Our results provided strong evidence for a duality between supersphere σ-models and Gross-Neveu models

Outlook

- Conformal invariance $\leftrightarrow$ Integrability
- Correlation functions?
- Application to other geometries and phenomena...