Coset Graphs and Modular Invariants in Logarithmic Minimal Models

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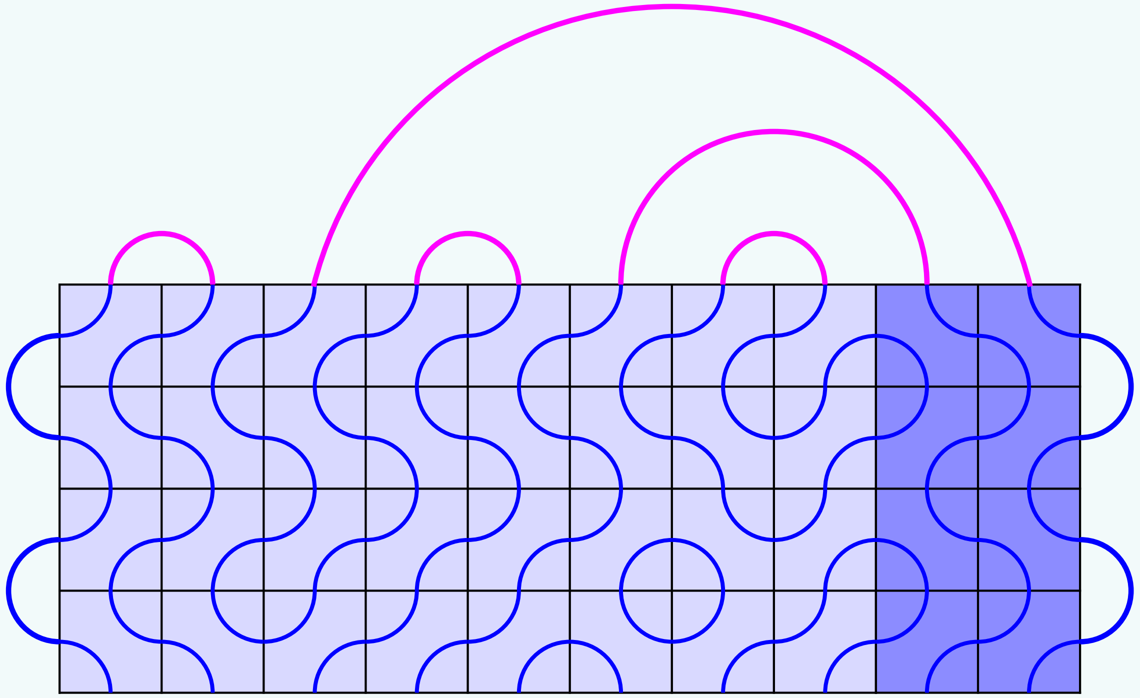

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- P.A. Pearce, JR, P. Ruelle, *Integrable boundary conditions and $\mathcal{W}$-extended fusion in the logarithmic minimal models $\mathcal{LM}(1,p)$*, J. Phys. A: Math. Theor. 41 (2008).
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Lattice Approaches to (Logarithmic) CFT

Rational minimal models (BPZ & ABF 1984)

$$\mathcal{M}(p,p'); \qquad (p,p')=1, \quad 1<p<p'$$

Conventional lattice approach to CFT

local degrees of freedom	$\Rightarrow$	symmetric transfer matrices	$\Rightarrow$	diagonalizable transfer matrices	$\Rightarrow$	L_0 is diagonalizable	$\Rightarrow$	rational CFT

Logarithmic CFT (Knizhnik 1987, Rozansky-Saleur 1992, Gurarie 1993, essentially everyone present here today!)

- The Virasoro mode L_0 is non-diagonalizable and exhibits **non-trivial Jordan blocks**.

Paradigm shift in lattice approach

logarithmic CFT	$\Rightarrow$	non-local degrees of freedom

- Statistical systems with **non-local** degrees of freedom are associated with **Logarithmic** CFTs. Examples are critical dense polymers and critical percolation.

Logarithmic minimal models (Pearce-Rasmussen-Zuber 2006)

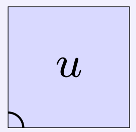
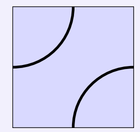
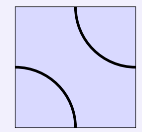
$$\mathcal{LM}(p,p'); \qquad (p,p')=1, \quad 1\le p<p'$$

Other lattice approaches to logarithmic CFT (Mahieu-Ruelle 2001, Read-Saleur 2007)

- Abelian sandpile model, quantum spin chains, ...

Logarithmic Minimal Models $\mathcal{LM}(p,p')$

Face operators defined in planar Temperley-Lieb algebra (Jones 1999)

$X(u) =$  $= \dfrac{\sin(\lambda-u)}{\sin\lambda}$  $+ \dfrac{\sin u}{\sin\lambda}$  ;

$$X_j(u) = \frac{\sin(\lambda-u)}{\sin\lambda}\, I + \frac{\sin u}{\sin\lambda}\, e_j$$

$1 \le p < p'$ coprime integers, $\lambda = \dfrac{(p'-p)\pi}{p'} =$ crossing parameter

$u =$ spectral parameter, $\beta = 2\cos\lambda =$ fugacity of loops

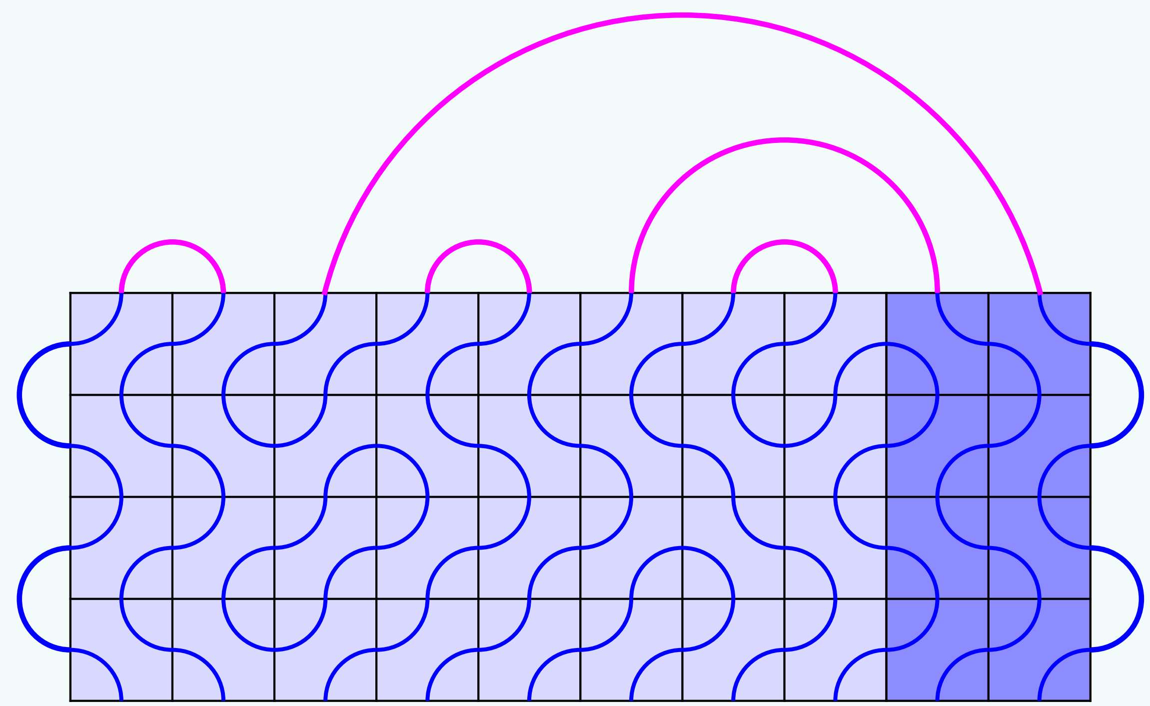
Planar Algebra
(Temperley-Lieb Algebra)

↓ YBE

Nonlocal Statistical Mechanics
(Yang-Baxter Integrable Link Models)

↓ continuum limit ↑ lattice realization

Logarithmic CFTs
(Logarithmic Minimal Models)



- Non-local degrees of freedom (connectivities)
- Inf. families of integrable boundary conditions
- Transfer matrices act on spaces of link states

Trilogy and Central Questions

Theory	$\mathcal{M}(p,p')$	$\mathcal{LM}(p,p')$	$\mathcal{WLM}(p,p')$
Degrees of freedom	local heights	non-local loops	non-local loops with infinitely thick boundary conditions
$p=1$	$\mathcal{M}(1,p')=\emptyset$	$\mathcal{LM}(1,2)$: crit. dense polymers	$\mathcal{WLM}(1,2)$: triplet model
$p>1$	Baxter-Forrester RSOS	$\mathcal{LM}(p,p')$	$\mathcal{LM}(p,p')$ with $\mathcal{W}$-boundaries
CFT content	$c=1-\frac{6(p-p')^2}{pp'}$ finite # irred reps	$c=1-\frac{6(p-p')^2}{pp'}$ infinite # indec reps	$c=1-\frac{6(p-p')^2}{pp'}$ **finite** sets of $\mathcal{W}$-indec reps

Central Questions:

- To what extent do $\mathcal{W}$-extended logarithmic minimal models $\mathcal{WLM}(p,p')$ resemble rational CFTs?
- For example, is there a Verlinde-like formula?

- Are the $\mathcal{W}$-extended logarithmic minimal models $\mathcal{WLM}(p,p')$ classified by graphs?
- If so, is it the same graphs describing the bulk and boundary theories?

- Considering that $c^{\text{eff}}=1$ for the logarithmic minimal models, does the $c=1$ compactified boson play a role in their description?

Projective Representations in $\mathcal{WLM}(p,p')$

Representations associated with boundary conditions

There are $2pp'$ $\mathcal{W}$-projective representations associated with boundary conditions

$$\hat{\mathcal{R}}^{r,s}_{\kappa p,\kappa' p'}, \qquad \kappa,\kappa'=1,2; \qquad 0\le r\le p-1, \quad 0\le s\le p'-1$$

This notation assumes that

$$\hat{\mathcal{R}}^{r,s}_{2p,p'}=\hat{\mathcal{R}}^{r,s}_{p,2p'}, \quad \hat{\mathcal{R}}^{r,s}_{2p,2p'}=\hat{\mathcal{R}}^{r,s}_{p,p'}, \quad \hat{\mathcal{R}}^{0,0}_{p,p'}=\mathcal{W}(\Delta_{p,p'}), \quad \hat{\mathcal{R}}^{0,0}_{2p,p'}=\mathcal{W}(\Delta_{2p,p'})$$

and for later convenience, we extend it by

$$\hat{\mathcal{R}}^{p,s}_{p,p'}\equiv\hat{\mathcal{R}}^{0,s}_{p,2p'}, \qquad \hat{\mathcal{R}}^{r,p'}_{p,p'}\equiv\hat{\mathcal{R}}^{r,0}_{2p,p'}, \qquad \hat{\mathcal{R}}^{p,p'}_{p,p'}\equiv\hat{\mathcal{R}}^{0,0}_{2p,2p'}\equiv\hat{\mathcal{R}}^{0,0}_{p,p'}$$

Rank of a $\mathcal{W}$-projective representation

$$\text{rank}(\hat{\mathcal{R}}^{r,s}_{\kappa p,\kappa' p'})=d_{r,s}-\lfloor\tfrac{d_{r,s}}{4}\rfloor$$

where the **degree** $d_{r,s}$ is defined by

$$d_{r,s}=d^{(p)}_r d^{(p')}_s=(2-\delta^{(p)}_{r,0})(2-\delta^{(p')}_{s,0}), \qquad \delta^{(N)}_{m,n}=\begin{cases}1, & m=n \text{ mod } N\\ 0, & \text{otherwise}\end{cases}$$

- In summary

$$\#[\text{Proj}]=2pp', \qquad \begin{cases}\#[\text{rank 1}] &= 2\\ \#[\text{rank 2}] &= 2(p+p'-2)\\ \#[\text{rank 3}] &= 2(p-1)(p'-1)\end{cases}$$

- $\mathcal{W}$-extension believed to be w.r.t. $\mathcal{W}_{p,p'}$ of Feigin-Gainutdinov-Semikhatov-Tipunin (2006).

$\mathcal{W}$-Projective Fusion Algebra

$$\hat{\mathcal{R}}_{\kappa p,p'}^{r,s} \otimes \hat{\mathcal{R}}_{p,\kappa' p'}^{r',s'} = \frac{d_{r,s}d_{r',s'}}{4}\Bigg(\Big\{\bigoplus_{r''}^{p-|r-r'|-1} \oplus \bigoplus_{r''}^{|p-r-r'|-1}\Big\}\Big\{\bigoplus_{s''}^{p'-|s-s'|-1} \oplus \bigoplus_{s''}^{|p'-s-s'|-1}\Big\}\hat{\mathcal{R}}_{\kappa p,\kappa' p'}^{r'',s''}$$

$$\oplus \Big\{\bigoplus_{r''}^{p-|p-r-r'|-1} \oplus \bigoplus_{r''}^{|r-r'|-1}\Big\}\Big\{\bigoplus_{s''}^{p'-|p'-s-s'|-1} \oplus \bigoplus_{s''}^{|s-s'|-1}\Big\}\hat{\mathcal{R}}_{\kappa p,\kappa' p'}^{r'',s''}$$

$$\oplus \Big\{\bigoplus_{r''}^{p-|r-r'|-1} \oplus \bigoplus_{r''}^{|p-r-r'|-1}\Big\}\Big\{\bigoplus_{s''}^{p'-|p'-s-s'|-1} \oplus \bigoplus_{s''}^{|s-s'|-1}\Big\}\hat{\mathcal{R}}_{\kappa p,(2\cdot\kappa')p'}^{r'',s''}$$

$$\oplus \Big\{\bigoplus_{r''}^{p-|p-r-r'|-1} \oplus \bigoplus_{r''}^{|r-r'|-1}\Big\}\Big\{\bigoplus_{s''}^{p'-|s-s'|-1} \oplus \bigoplus_{s''}^{|p'-s-s'|-1}\Big\}\hat{\mathcal{R}}_{\kappa p,(2\cdot\kappa')p'}^{r'',s''}\Bigg)$$

where $1 \cdot 1 = 2 \cdot 2 = 1$, $2 \cdot 1 = 1 \cdot 2 = 2$ and

$$\bigoplus_{n}^{N} R_n = \bigoplus_{n=\epsilon(N),\,\text{by}\,2}^{N} R_n, \qquad \epsilon(N) = \tfrac{1}{2}(1-(-1)^N) = N \pmod 2$$

- This fusion algebra does not contain an identity but is both associative and commutative and, despite appearances, the multiplicities are all non-negative integers.

$\mathcal{W}$-Projective Grothendieck Group

Projective characters $[\ell = 0, 1]$

$$\begin{aligned}
\chi\big[\hat{\mathcal{R}}^{0,0}_{(\ell+1)p,p'}\big](q) &= \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}} q^{(2k+\ell)^2pp'/4} \\
\chi\big[\hat{\mathcal{R}}^{a,0}_{(\ell+1)p,p'}\big](q) &= \frac{2}{\eta(q)}\sum_{k\in\mathbb{Z}} q^{(a+(2k+\ell)p)^2p'/4p} \\
\chi\big[\hat{\mathcal{R}}^{0,b}_{p,(\ell+1)p'}\big](q) &= \frac{2}{\eta(q)}\sum_{k\in\mathbb{Z}} q^{(b+(2k+\ell)p')^2p/4p'} \\
\chi[\hat{\mathcal{R}}^{a,b}_{(\ell+1)p,p'}](q) &= \frac{2}{\eta(q)}\sum_{k\in\mathbb{Z}} \Big[q^{(ap'-bp+(2k+\ell)pp')^2/4pp'} + q^{(ap'+bp+(2k+\ell)pp')^2/4pp'}\Big]
\end{aligned}$$

- The number of linearly independent $\mathcal{W}$-projective characters is given by

$$\tfrac{1}{2}(p+1)(p'+1)$$

Projective Grothendieck generators

$$\mathcal{G}_{r,s} = [\hat{\mathcal{R}}^{r,s}_{p,p'}], \qquad \mathcal{G}_{r,s} = \mathcal{G}_{p-r,p'-s}, \qquad 0 \le r \le p\ , \quad 0 \le s \le p'$$

- Such an equivalence class is uniquely characterized by the conformal weight

$$\Delta_{r,s} = \Delta_{p-r,p'-s} = \frac{(p'r-ps)^2-(p-p')^2}{4pp'}, \qquad 0 \le r \le p\ , \quad 0 \le s \le p'$$

- The projective Grothendieck generators can be organized into a Kac table with a $\mathbb{Z}_2$ Kac-table symmetry.

Kac Tables of Critical Percolation $\mathcal{LM}(2,3)/\mathcal{WLM}(2,3)$

s							
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	12	$\frac{65}{8}$	5	$\frac{21}{8}$	1	$\frac{1}{8}$	$\cdots$
9	$\frac{28}{3}$	$\frac{143}{24}$	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\cdots$
8	7	$\frac{33}{8}$	2	$\frac{5}{8}$	0	$\frac{1}{8}$	$\cdots$
7	5	$\frac{21}{8}$	1	$\frac{1}{8}$	0	$\frac{5}{8}$	$\cdots$
6	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\cdots$
5	2	$\frac{5}{8}$	0	$\frac{1}{8}$	1	$\frac{21}{8}$	$\cdots$
4	1	$\frac{1}{8}$	0	$\frac{5}{8}$	2	$\frac{33}{8}$	$\cdots$
3	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\frac{10}{3}$	$\frac{143}{24}$	$\cdots$
2	0	$\frac{1}{8}$	1	$\frac{21}{8}$	5	$\frac{65}{8}$	$\cdots$
1	0	$\frac{5}{8}$	2	$\frac{33}{8}$	7	$\frac{85}{8}$	$\cdots$
	1	2	3	4	5	6	r

$\hat{s}$				
2	$\frac{5}{8}$	2	$\frac{33}{8}$	7
1	$\frac{1}{8}$	1	$\frac{21}{8}$	5
0	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\frac{10}{3}$
	0	1	2	3 $\hat{r}$

$\mathcal{W}$-**Irreducible** (Non-Minimal)

$\hat{s}$				
2	$\frac{5}{8}, \frac{5}{8}$	0,2 0,2	$\frac{1}{8}, \frac{33}{8}$	0,1 2,7
1	$\frac{1}{8}, \frac{1}{8}$	0,1 0,1	$\frac{5}{8}, \frac{21}{8}$	0,1 2,5
0	$-\frac{1}{24}$	$\frac{1}{3}, \frac{1}{3}$	$\frac{35}{24}$	$\frac{1}{3}, \frac{10}{3}$
	0	1	2	3 $\hat{r}$

Projective Covers

s			
3			
2		0	
1		0	
0			
	0	1	2 r

Minimal/Projective Covers

s			
3	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$
2	$\frac{5}{8}$	0	$\frac{1}{8}$
1	$\frac{1}{8}$	0	$\frac{5}{8}$
0	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$
	0	1	2 r

Proj Grothendieck
(Bordered Minimal Kac)

$$d_{r,s} = \begin{cases} \text{degree} \\ 2^{\text{rank}-1} \end{cases} = (2-\delta^{(p)}_{r,0})(2-\delta^{(p')}_{s,0}) = \begin{cases} 1, & \text{corner (mid blue)} \\ 2, & \text{edge (light blue)} \\ 4, & \text{interior (white)} \end{cases} \qquad \delta^{(N)}_{m,n} = \begin{cases} 1, & m = n \bmod N \\ 0, & \text{otherwise} \end{cases}$$

Projective Grothendieck Ring of $\mathcal{WLM}(p,p')$

Multiplication rules

$$\mathcal{G}_{r,s} * \mathcal{G}_{r',s'} = d_{r,s} d_{r',s'} \sum_{r''=\epsilon(p+r+r'+1),\,\text{by }2}^{p-\epsilon(r+r'+1)} \sum_{s''=\epsilon(p'+s+s'+1),\,\text{by }2}^{p'-\epsilon(s+s'+1)} \mathcal{G}_{r'',s''}$$

- It follows that, up to the multiplicities $d_{r,s}d_{r',s'} \in \{1,2,4,8,16\}$, there are only **two** possible linear combinations of generators arising as the result of a simple multiplication in the projective Grothendieck ring.

Conformal partition functions associated with projective boundary conditions

$$Z_{(r,s)|(r',s')}(q) = \chi[\mathcal{G}_{r,s} * \mathcal{G}_{r',s'}](q) = \sum_{r''=\epsilon(p+r+r'+1),\,\text{by }2}^{p-\epsilon(r+r'+1)} \sum_{s''=\epsilon(p'+s+s'+1),\,\text{by }2}^{p'-\epsilon(s+s'+1)} d_{r,s} d_{r',s'} \chi[\mathcal{G}_{r'',s''}](q)$$

- Here we have assigned $\mathcal{G}_{r,s}$ the common character of the representatives within its equivalence class

$$\chi[\mathcal{G}_{r,s}](q) = \chi[\hat{\mathcal{R}}^{r,s}_{p,p'}](q)$$

$c = 1$ Compactified Boson Revisited

- The $c = 1$ boson on the circle of compactification radius $R = \sqrt{2p'/p}$, where p, p' are coprime integers, exhibits an extended symmetry with $2n = 2pp'$ primary operators.
- The conformal weights and $u(1)$ characters are

$$\Delta_j = \min\Big[\frac{j^2}{4n}, \frac{(2n-j)^2}{4n}\Big], \quad \varkappa_j^n(q) = \varkappa_{2n-j}^n(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} q^{(j+2kn)^2/4n}, \quad j = 0, 1, \ldots, 2n$$

- The **$\mathcal{W}$-projective characters** are expressible in terms of the $u(1)$ characters

$$\chi[\mathcal{G}_{r,s}](q) = d_{r,s}\varkappa_{r,s}^n(q), \qquad \varkappa_{r,s}^n(q) = \tfrac{1}{2}\Big[\varkappa_{rp'-sp}^n(q) + \varkappa_{rp'+sp}^n(q)\Big], \qquad 0 \le r \le p; \quad 0 \le s \le p'$$

- The modular transformations and modular matrix S^{Circ} are

$$\varkappa_j^n(e^{-2\pi i/\tau}) = \sum_{k=0}^{2n-1} S_{jk}^{\text{Circ}}\, \varkappa_k^n(e^{2\pi i\tau}), \qquad S_{jk}^{\text{Circ}} = \frac{1}{\sqrt{2n}} e^{-\pi ijk/n}$$

- For each pair p, p', there is a modular invariant partition function

$$Z_{p,p'}^{\text{Circ}}(q) = \sum_{j=0}^{2n-1} \varkappa_j^n(q)\varkappa_{\omega_0 j}^n(\bar{q}), \qquad \omega_0 = r_0p' + s_0p \pmod{2n}$$

- The Bezout pair (r_0, s_0) and Bezout number ω_0 are uniquely determined by

$$r_0p' - s_0p = 1, \qquad 1 \le r_0 \le p-1, \qquad 1 \le s_0 \le p'-1, \qquad ps_0 < p'r_0$$

- The Bezout number ω_0 acts as a conjugation on characters

$$\varkappa_{\omega_0(rp'\pm sp)}^n(q) = \varkappa_{rp'\mp sp}^n(q), \qquad r = 0, 1, \ldots, p; \quad s = 0, 1, \ldots, p'$$

$A_n^{(2)}$ Graph Fusion Algebra, I

- The $c\!=\!1$ boson fusion algebra is associated with the cyclic directed graph $\mathbb{Z}_{2n}$ with $2n$ nodes

$$\phi_i \times \phi_j = \sum_{k=0}^{2n-1} N_{ij}{}^k \phi_k, \qquad N_{ij}{}^k = \delta^{(2n)}_{i+j,k}$$

  where i, j, k and their sums are interpreted as integers mod $2n$.

- This algebra is realized by powers of the cyclic shift matrix Ω^ω where ω is coprime to $2n = 2pp'$ and $\Omega^{2n} = I$.

- Consider the composites $\phi_r + \phi_{-r} = 2\cos(r\varphi/\sqrt{n})$, $r \neq 0, n$. Setting $X = N_1 = \Omega^\omega + \Omega^{-\omega}$ etc, the corresponding algebra

$$\langle N_r = \tfrac{1}{2} d_r^{(n)} (\Omega^{r\omega} + \Omega^{-r\omega}),\ r = 0, 1, \ldots, n \rangle$$

  is realized by

$$N_r = d_r^{(n)} T_r(\tfrac{X}{2}), \qquad 0 \le r \le n; \qquad T_{n+1}(\tfrac{X}{2}) - T_{n-1}(\tfrac{X}{2}) = 0$$

  where $T_r(X)$ is the r'th Chebyshev polynomial of the first kind.

- The multiplication rules are given explicitly by

$$N_r N_{r'} = \sum_{r''=0}^{n} N_{rr'}{}^{r''} N_{r''} = \frac{d_r^{(n)} d_{r'}^{(n)}}{2} \Big(\frac{1}{d^{(n)}_{|r-r'|}} N_{|r-r'|} + \frac{1}{d^{(n)}_{n-|n-r-r'|}} N_{n-|n-r-r'|} \Big), \qquad 0 \le r, r' \le n$$

$A_n^{(2)}$ Graph Fusion Algebra, II

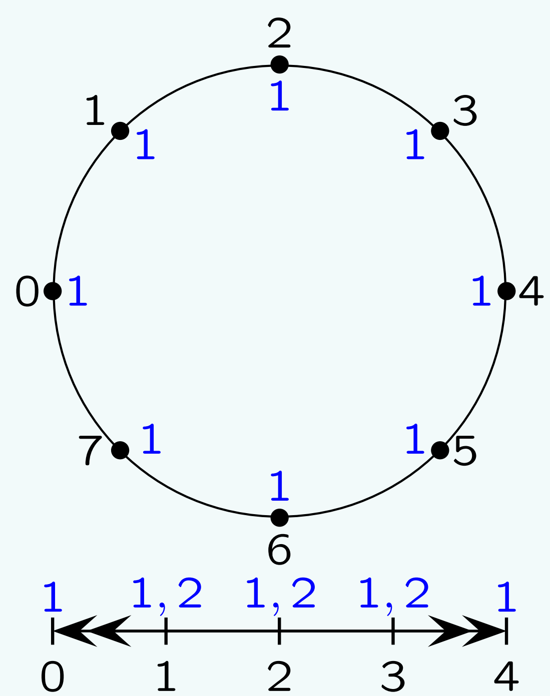
Regular representation

$$N_1 = \begin{pmatrix} 0 & 1 & & & & & \\ 2 & 0 & 1 & & & & \\ & 1 & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & 1 & \\ & & & & 1 & 0 & 2 \\ & & & & & 1 & 0 \end{pmatrix} = \text{asymmetric}, \qquad \lambda_r^{(n)} = 2\cos\frac{r\pi}{n} = \text{eigvals}, \qquad r = 0, 1, \ldots, n$$

Twisted $A_n^{(2)}$ graph

- The fundamental generator X can be viewed as the adjacency matrix for the **twisted** affine Dynkin diagram $A_n^{(2)}$. The latter can be obtained by folding the affine Dynkin diagram $A_{2n-1}^{(1)}$ with $2n$ nodes

$A_{2n-1}^{(1)}$



$A_n^{(2)}$

$$C = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad A_{2n-1}^{(1)} C = C A_n^{(2)}$$

[Left/Right Perron-Frobenius eigenvectors indicated]

- $2I - N_1^T$ is a generalized, **symmetrizable** Cartan matrix. So $D^{1/2} N_1^T D^{-1/2}$ is symmetric and $A = N_1^T$ is similar to a real symmetric matrix with real eigenvalues and eigenvectors

$$D = \text{diag}(d_0, d_1, \ldots, d_n) = \text{diag}(1, 2, 2, \ldots, 2, 1), \qquad d_r = a_r = \text{Coxeter labels}$$

Coset Graph $A^{(2)}_{p,p'}$

Definition [The $\mathbb{Z}_2$ quotient is taken with respect to the $\mathbb{Z}_2$ Kac-table symmetry, cf. p.0-14]

$$A^{(2)}_{p,p'} = A^{(2)}_p \otimes A^{(2)}_{p'}/\mathbb{Z}_2, \qquad \tfrac{1}{2}(p+1)(p'+1) \text{ nodes}$$

Eigenvalues of adjacency matrix

$$\lambda^{p,p'}_{r,s} = \lambda^{p,p'}_{p-r,p'-s} = 4\cos\frac{r\pi}{p}\cos\frac{s\pi}{p'}, \qquad r = 0, 1, \ldots, p; \quad s = 0, 1, \ldots, p'$$

$A^{(2)}_{p,p'}$ **graph fusion algebra** [generated by $\frac{1}{2}(p+1)(p'+1)$ matrices, where $X = N_{1,0}$, $Y = N_{0,1}$]

$$N_{r,s}N_{r',s'} = \sum_{r'',s''} N_{rs,r's'}{}^{r''s''} N_{r'',s''}, \quad N_{r,s} = N_{p-r,p'-s} = d_{r,s}T_r(\tfrac{X}{2})T_s(\tfrac{Y}{2}), \quad 0 \le r \le p; \quad 0 \le s \le p'$$

where $T_{p+1}(\frac{X}{2}) - T_{p-1}(\frac{X}{2}) = T_{p'+1}(\frac{Y}{2}) - T_{p'-1}(\frac{Y}{2}) = T_p(\frac{X}{2}) - T_{p'}(\frac{Y}{2}) = 0$.

- Explicitly

$$N_{rs,r's'}{}^{r''s''} = \frac{d_{r,s}d_{r',s'}}{4d_{r'',s''}}\Big(\delta_{r'',|r-r'|} + \delta_{r'',p-|p-r-r'|}\Big)\Big(\delta_{s'',|s-s'|} + \delta_{s'',p'-|p'-s-s'|}\Big)$$

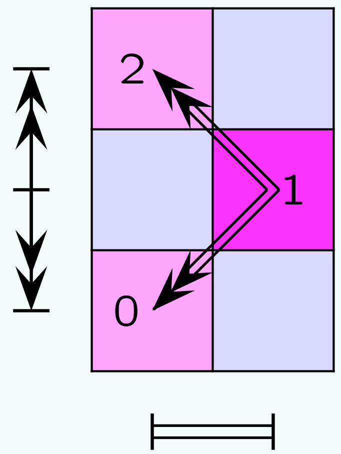
- The coset graphs $A^{(2)}_{p,p'}$ are symmetrizable, with degrees

$$d_{r,s} = d^{(p)}_r d^{(p')}_s = \begin{cases} 1, & (r,s) \text{ is a corner} \\ 2, & (r,s) \text{ is on an edge} \\ 4, & (r,s) \text{ is in the interior} \end{cases} \qquad D = \text{diag}(\ldots, d_{r,s}, \ldots) = \text{graph valencies}$$

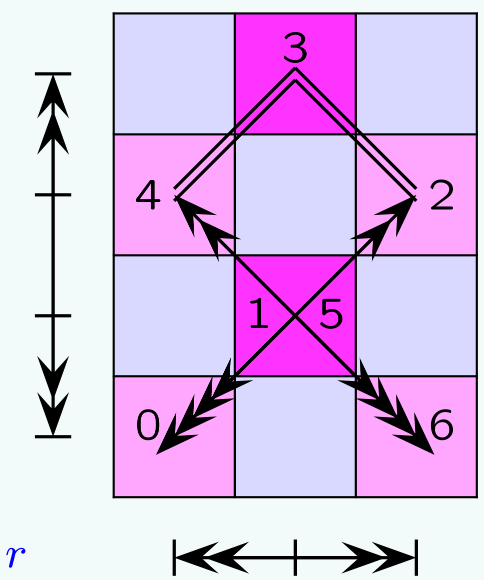
- The ranks of projective representations are thus related (through the Coxeter labels) to data of twisted affine Dynkin graphs.

Projective Grothendieck Kac Tables as Coset Graphs

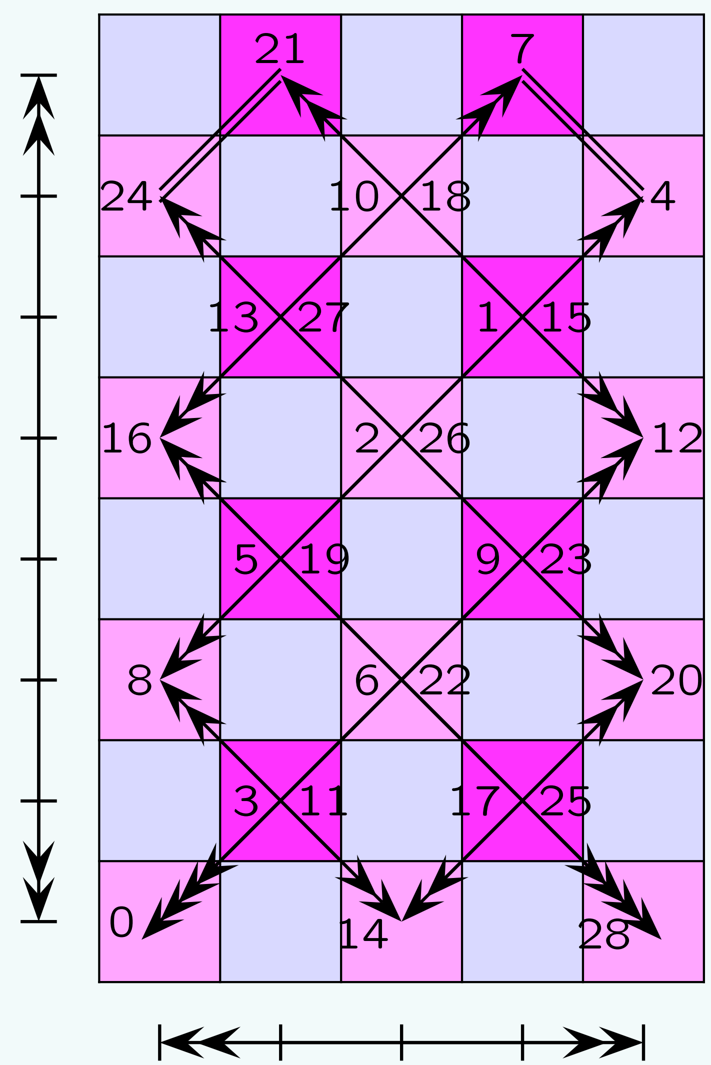
s \ r	0	1
2	$\frac{3}{8}$	$-\frac{1}{8}$
1	0	0
0	$-\frac{1}{8}$	$\frac{3}{8}$



s \ r	0	1	2
3	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$
2	$\frac{5}{8}$	0	$\frac{1}{8}$
1	$\frac{1}{8}$	0	$\frac{5}{8}$
0	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$



s \ r	0	1	2	3	4
7	$\frac{775}{112}$	$\frac{27}{7}$	$\frac{187}{112}$	$\frac{5}{14}$	$-\frac{9}{112}$
6	$\frac{81}{16}$	$\frac{5}{2}$	$\frac{13}{16}$	0	$\frac{1}{16}$
5	$\frac{391}{112}$	$\frac{10}{7}$	$\frac{27}{112}$	$-\frac{1}{14}$	$\frac{55}{112}$
4	$\frac{247}{112}$	$\frac{9}{14}$	$-\frac{5}{112}$	$\frac{1}{7}$	$\frac{135}{112}$
3	$\frac{135}{112}$	$\frac{1}{7}$	$-\frac{5}{112}$	$\frac{9}{14}$	$\frac{247}{112}$
2	$\frac{55}{112}$	$-\frac{1}{14}$	$\frac{27}{112}$	$\frac{10}{7}$	$\frac{391}{112}$
1	$\frac{1}{16}$	0	$\frac{13}{16}$	$\frac{5}{2}$	$\frac{81}{16}$
0	$-\frac{9}{112}$	$\frac{5}{14}$	$\frac{187}{112}$	$\frac{27}{7}$	$\frac{775}{112}$



Building Coset Graphs from $A_n^{(2)}$ Graphs

- The coset graph $A_{p,p'}^{(2)}$ can be built from the linear $A_{pp'}^{(2)}$ graph by folding and gluing pairs of nodes together. [cf. p.0-14]

Intertwining similarity relation

$$A_{p,p'}^{(2)} = A_p^{(2)} \otimes A_{p'}^{(2)}/\mathbb{Z}_2 = 2C_L A_{pp'}^{(2)} C_R, \qquad C_L C_R = I$$

- The **intertwining matrices** C_L and C_R are **rectangular**, not square. It is an intertwining relation since the common eigenvalues of $A_{p,p'}^{(2)}$ and $2A_{pp'}^{(2)}$ are intertwined. It is a similarity in the sense that C_L and C_R are generalized inverses.

Eigenvalues of the coset graph $A_{p,p'}^{(2)}$

$$\lambda_{r,s}^{p,p'} = 4\cos\frac{r\pi}{p}\cos\frac{s\pi}{p'} = 2\cos\frac{(rp'-sp)\pi}{pp'} + 2\cos\frac{(rp'+sp)\pi}{pp'} = \lambda_{rp'-sp}^{pp'} + \lambda_{rp'+sp}^{pp}$$

- A node is labelled by a pair of Bezout conjugate integers associated to $\varkappa_{k(p'-p)}^n(q)$ and $\varkappa_{\omega_0 k(p'-p)}^n(q)$, $k = 0, 1, \ldots, n = pp'$, or equivalently to $\lambda_{rp'-sp}^{pp'}$ and $\lambda_{rp'+sp}^{pp}$.

Projection

- The **$\mathcal{W}$-projective characters** are expressible in terms of the $c\!=\!1$ $u(1)$ characters $[n = pp']$

$$\chi[\mathcal{G}_{r,s}](q) = d_{r,s}\varkappa_{r,s}^n(q), \qquad \varkappa_{r,s}^n(q) = \tfrac{1}{2}\Big[\varkappa_{rp'-sp}^n(q) + \varkappa_{rp'+sp}^n(q)\Big], \qquad 0 \le r \le p; \quad 0 \le s \le p'$$

- The intertwining similarity implements a change of basis to symmetric and anti-symmetric combinations of $\varkappa_{rp'-sp}^n(q)$ and $\varkappa_{rp'+sp}^n(q)$, **projecting out** the anti-symmetric combinations

$$pp' + 1 - \tfrac{1}{2}(p-1)(p'-1) = \tfrac{1}{2}(p+1)(p'+1)$$

Verlinde Formula

- The anti-symmetric combinations are ordinary minimal Virasoro characters

$$\mathrm{ch}_{r,s}(q) = \varkappa^n_{rp'-sp}(q) - \varkappa^n_{rp'+sp}(q)$$

- They **do** appear in the modular invariant partition functions.

Modular transformations

- The modular matrix $S = S^{p,p'}$ $(S^2 = I, S \neq S^T)$ of the $\mathcal{W}$-projective characters forms a representation of the modular group. [Feigin-Gainutdinov-Semikhatov-Tipunin (2006)]

Standard Verlinde algebra

$$N_i N_j = \sum_{k=0}^{\frac{1}{2}(p+1)(p'+1)-1} N_{ij}{}^k N_k$$

where i, j, k run over allowed pairs (r, s) in the projective Grothendieck Kac table, while

$$N_{ij}{}^k = (N_i)_j{}^k = \sum_{m=0}^{\frac{1}{2}(p+1)(p'+1)-1} \frac{S_{im}S_{jm}S_{mk}}{S_{0m}} \in \mathbb{N}_0$$

- This is **precisely** the graph algebra of the twisted coset graph $A^{(2)}_{p,p'} = A^{(2)}_p \otimes A^{(2)}_{p'}/\mathbb{Z}_2$.
- The modular matrix **diagonalizes** the multiplication rules of the $\mathcal{W}$-projective Grothendieck ring.

Conformal Partition Functions Revisited

- The conformal partition functions for $\mathcal{W}$-projective boundary conditions are given by

$$Z_{i|j}(q) = \sum_{k=0}^{\frac{1}{2}(p+1)(p'+1)-1} N_{ij}{}^{k} (F\boldsymbol{\chi}[\mathcal{G}])_k(q)$$

where

$$F = \begin{cases} \displaystyle\sum_{r,\,s\ \text{odd}} N_{r,s}, & p+p'\ \text{odd} \\ \displaystyle\sum_{\substack{r+s\ \text{even} \\ s\,\le\,(p'-1)/2}} N_{r,s}, & p+p'\ \text{even} \end{cases}$$

acts on the column of characters $\boldsymbol{\chi}[\mathcal{G}] = \{\chi[\mathcal{G}_{r,s}]\}$.

- Explicitly, the 'block characters' are

$$(F\boldsymbol{\chi}[\mathcal{G}])_{r,s}(q) = \sum_{r''=\epsilon(p+r+1),\,\text{by } 2}^{p-\epsilon(r+1)} \;\; \sum_{s''=\epsilon(p'+s+1),\,\text{by } 2}^{p'-\epsilon(s+1)} d_{r,s}\, \chi[\mathcal{G}_{r'',s''}](q)$$

where ϵ is the parity

$$\epsilon(r) = r \pmod 2$$

Bulk Modular Invariants in $\mathcal{WLM}(p,p')$

Sesquilinear form in $\mathcal{W}$-irreducible characters

$$Z = \sum_{i,j \in \mathrm{Irr}} M_{ij}\chi_i(q)\chi_j(\bar{q}), \qquad |\mathrm{Irr}| = 2pp' + \tfrac{1}{2}(p-1)(p'-1)$$

Proposition: An S-invariant sesquilinear form in $\mathcal{W}$-irreducible characters can be expressed as a sesquilinear form in $\mathcal{W}$-projective and minimal characters.

- This is equi-numerous with the linearly independent $u(1)$ characters $[c=1,\ R=\sqrt{2p'/p}]$

$$pp' + 1 = \tfrac{1}{2}(p+1)(p'+1) + \tfrac{1}{2}(p-1)(p'-1)$$

Conjecture: A modular invariant sesquilinear form in $\mathcal{W}$-projective and minimal characters decomposes into a sum of separate modular invariant sesquilinear forms in $\mathcal{W}$-projective and minimal characters

$$Z = Z^{\mathrm{Proj}} + Z^{\mathrm{Min}}$$

Evidence: Verified for all $p < p'$ coprime satisfying $pp' \le 225$.

Projective and Minimal A-Type Modular Invariants

Projective part

- Our coset graphs provide new expressions for the diagonal A-type modular invariants in $\mathcal{W}$-projective characters considered by FGST (2006) and Wood (2010)

$$Z^{\mathsf{Proj}}_{p,p'}(q) = \tfrac{1}{2}\sum_{r=0}^{p}\sum_{s=0}^{p'} d_{r,s}\,|\varkappa^n_{r,s}(q)|^2 = \tfrac{1}{2}\sum_{r=0}^{p}\sum_{s=0}^{p'} \frac{1}{d_{r,s}}\Big|\chi[\mathcal{G}_{r,s}](q)\Big|^2 = \sum_{\hat{r}=0}^{2p-1}\sum_{\hat{s}=0}^{p'-1}\frac{1}{d^2_{\hat{r},\hat{s}}}|\chi[\widehat{\mathcal{P}}_{\hat{r},\hat{s}}](q)|^2$$

The factors of $\frac{1}{2}$ in the first two double sums reflect the $\mathbb{Z}_2$ Kac-table symmetry, $|\varkappa_{0,0}(q)|^2$ appears with multiplicity 1 and all multiplicities are non-negative integers.

- The modular invariance of $Z^{\mathsf{Proj}}_{p,p'}(q)$ follows from the identities

$$Z^{\mathsf{Proj}}_{p,p'}(q) = \tfrac{1}{2}\Big[Z^{\mathsf{Circ}}_{1,pp'}(q) + Z^{\mathsf{Circ}}_{p,p'}(q)\Big], \qquad Z^{\mathsf{Proj}}_{1,p'}(q) = Z^{\mathsf{Circ}}_{1,p'}(q)$$

Minimal part

- The coset graphs also encode the rational minimal A-type modular invariants

$$Z^{\mathsf{Min}}_{p,p'}(q) = \tfrac{1}{2}\sum_{r=1}^{p-1}\sum_{s=1}^{p'-1}|\mathsf{ch}_{r,s}(q)|^2 = \tfrac{1}{2}\sum_{r=0}^{p}\sum_{s=0}^{p'}\Big|\varkappa^n_{rp'-sp}(q) - \varkappa^n_{rp'+sp}(q)\Big|^2$$

where the factors of $\frac{1}{2}$ reflect a $\mathbb{Z}_2$ Kac-table symmetry. In terms of the $c=1$ boson

$$Z^{\mathsf{Min}}_{p,p'}(q) = \tfrac{1}{2}\Big[Z^{\mathsf{Circ}}_{1,pp'}(q) - Z^{\mathsf{Circ}}_{p,p'}(q)\Big], \qquad Z^{\mathsf{Min}}_{1,p'}(q) = 0$$

$\mathcal{A}$-Type $\mathcal{WLM}(p,p')$ Modular Invariants

- Assuming $Z^{\text{Proj}} \neq 0$ and that the operator with minimal conformal weight enters exactly once, the $\mathcal{A}$-type $\mathcal{WLM}(p,p')$ modular invariant partition functions must be of the form

$$\boxed{Z_{p,p'}(q) = Z^{\text{Proj}}_{p,p'}(q) + n_{p,p'} Z^{\text{Min}}_{p,p'}(q), \qquad n_{p,p'} \in \mathbb{Z}}$$

- For $p = 1$

$$\boxed{Z^{\text{Min}}_{1,p'}(q) = 0 \quad \Rightarrow \quad Z_{p,p'}(q) = Z^{\text{Proj}}_{p,p'}(q)}$$

Conjecture: For $p > 1$, the physical modular invariants of $\mathcal{A}$-type $\mathcal{WLM}(p,p')$ are given by

$$\boxed{n_{p,p'} = 2}$$

- Examples of modular invariant partition functions

$$Z_{1,2}(q) = |\varkappa_{-\frac{1}{8}}(q)|^2 + 2|\varkappa_0(q)|^2 + |\varkappa_{\frac{3}{8}}(q)|^2$$

$$Z_{2,3}(q) = |\varkappa_{-\frac{1}{24}}(q)|^2 + |\varkappa_0(q) + \varkappa_1(q)|^2 + 2|\varkappa_{\frac{1}{8}}(q)|^2 + 2|\varkappa_{\frac{1}{3}}(q)|^2 + 2|\varkappa_{\frac{5}{8}}(q)|^2 + |\varkappa_{\frac{35}{24}}(q)|^2 + 2|\text{ch}_0(q)|^2$$

$$Z_{3,4}(q) = |\varkappa_{-\frac{1}{48}}(q)|^2 + |\varkappa_0(q) + \varkappa_1(q)|^2 + |\varkappa_{\frac{1}{16}}(q) + \varkappa_{\frac{33}{16}}(q)|^2 + 2|\varkappa_{\frac{1}{6}}(q)|^2 + 2|\varkappa_{\frac{5}{16}}(q)|^2$$
$$+ |\varkappa_{\frac{1}{2}}(q) + \varkappa_{\frac{5}{2}}(q)|^2 + 2|\varkappa_{\frac{35}{48}}(q)|^2 + 2|\varkappa_{\frac{21}{16}}(q)|^2 + 2|\varkappa_{\frac{5}{3}}(q)|^2 + |\varkappa_{\frac{143}{48}}(q)|^2$$
$$+ 2\left\{|\text{ch}_0(q)|^2 + |\text{ch}_{\frac{1}{16}}(q)|^2 + |\text{ch}_{\frac{1}{2}}(q)|^2\right\}$$

Supporting Evidence

- For $n_{2,3} = 2$, we recover the $\mathcal{WLM}(2,3)$ modular invariant partition function of Gaberdiel-Runkel-Wood (2011).
- As in the $\mathcal{WLM}(2,3)$ case of GRW 2011, we find generally that for $n_{p,p'} = 2$

$$\boxed{Z_{p,p'}(q) = \sum_{i \in \text{Irr}} \chi_i(q)\, \chi[\mathcal{P}_i](\bar{q})}$$

where the sum is over all $\mathcal{W}$-irreducible representations. As demonstrated above, this partition function is in fact left-right symmetric when expanded in $u(1)$ characters.

- In terms of the $c = 1$ boson

$$\boxed{Z_{p,p'}(q) = Z^{\text{Circ}}_{1,pp'}(q) + (n_{p,p'} - 1) Z^{\text{Min}}_{p,p'}(q)}$$

Our conjecture thus yields a **minimal extension** of the compactified boson $Z^{\text{Circ}}_{1,pp'}(q)$ by adding the partition function for the **rational minimal model** with coefficient

$$\boxed{n_{p,p'} - 1 = 1}$$

as encoded in the coset graph viewed as a folded $A^{(2)}_{pp'}$ graph.

Summary and Open Questions

- Infinite series of Yang-Baxter integrable lattice models of non-local statistical mechanics.
- Description in terms of planar Temperley-Lieb algebras.
- Logarithmic CFTs with infinitely many indecomposable (higher-rank) representations.
- **$\mathcal{W}$-projective** representations emerge as **building blocks** akin to the role played by irreducible representations in rational CFTs.
- The $\mathcal{W}$-projective Grothendieck ring leads to a **standard** Verlinde-like formula involving **twisted** affine coset graphs.
- Compact formulas for the conformal partition functions with $\mathcal{W}$-projective boundary conditions.
- A-type $\mathcal{WLM}(p,p')$ modular invariants encoded by twisted affine coset graphs.
- The **boundary and bulk** A-type logarithmic minimal models are 'classified' by the **same** twisted affine coset graphs.

- Will an extension of Gaberdiel-Runkel-Wood for $c = 0$ confirm $n_{p,p'} = 2$?
- A-D-E classification of the logarithmic Verlinde graph fusion algebras à la Behrend-Pearce-Petkova-Zuber?
- A-D-E classification of the logarithmic modular invariant partition functions à la Cappelli-Itzykson-Zuber?
- Logarithmic coset construction à la Goddard-Kent-Olive?
- D- and E-type logarithmic minimal models on the lattice?