From Fractional Quantum Hall Effect To Fractional Chern Insulator

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Motivations : Topological insulators

An insulator has a (large) gap separating a fully filled valence band and an empty conduction band

Atomic insulator : solid





How to define equivalent insulators? Find a continuous transformation from one Bloch Hamiltonian $\mathcal{H}_0(\vec{k})$ to another $\mathcal{H}_1(\vec{k})$ without closing the gap

- Vacuum is the same kind of insulator than solid argon with a gap $2m_ec^2$
- Are all insulators equivalent to the vacuum? No

Motivations : Topological insulators

What is topological order?

- cannot be described by symmetry breaking (cannot use Ginzburg-Landau theory)
- some physical quantities are given by a "topological invariant" (think about the surface genus)
- a bulk gapped system (i.e. insulator) system feeling the topology (degenerate ground state, cannot be lifted by local measurement).
- naive picture : normal strip versus Moebius strip
- a famous example : Quantum Hall Effect (QHE)
- TI theoretically predicted and experimentally observed in the past 4 years missed by decades of band theory



A rich physics emerge when turning on strong interaction in QHE

What about Topological insulators?

	no interaction	strong interaction
Time-reversal breaking	QHE (B field) - Chern Insulator -	FQHE FCI
Time reversal invariant	QSHE - 3D TI -	FQSHE ?

- Fractional Quantum Hall Effect
- Fractional Chern Insulator
- FQHE vs FCI

Fractional Quantum Hall Effect

Landau level



• Cyclotron frequency : $\omega_c = \frac{eB}{m}$

• Filling factor :
$$\nu = \frac{hn}{eB} = \frac{N}{N_{\phi}}$$

- At $\nu = n$, *n* completely filled levels and a energy gap $\hbar\omega_c$
- Integer filling : a (ℤ) topological insulator with a perfectly flat band / perfectly flat Berry curvature!
- Partial filling + interaction \rightarrow FQHE
- Lowest Landau level (u < 1) : $z^m \exp\left(-|z|^2/4l^2\right)$
- N-body wave function : $\Psi = P(z_1, ..., z_N) \exp(-\sum |z_i|^2/4)$

The Laughlin wave function

A (very) good approximation of the ground state at $\nu = \frac{1}{3}$

$$\Psi_{L}(z_{1},...z_{N}) = \prod_{i< j} (z_{i}-z_{j})^{3} e^{-\sum_{i} \frac{|z_{i}|^{2}}{4j^{2}}}$$

- The Laughlin state is the unique (on genus zero surface) densest state that screens the short range (p-wave) repulsive interaction.
- **Topological state** : the degeneracy of the densest state depends on the surface genus (sphere, torus, ...)

The Laughlin wave function : quasihole

Add one flux quantum at z_0 = one quasi-hole



- Locally, create one quasi-hole with fractional charge $\frac{+e}{3}$
- "Wilczek" approach : quasi-holes obey fractional statistics
- Adding quasiholes/flux quanta increases the size of the droplet
- For given number of particles and flux quanta, there is a specific number of qh states that one can write
- These numbers/degeneracies can be classified with respect some quantum number (angular momentum L_z) and are a fingerprint of the phase (related to the statistics of the excitations).

Fractional Chern Insulator

- A Chern insulator is a zero magnetic field version of the QHE (Haldane, 88)
- Basic building block of 2D \mathbb{Z}_2 topological insulator (half of it)
- \bullet Is there a zero magnetic field equivalent of the FQHE? \rightarrow Fractional Chern Insulator
- Several proposals for a CI with nearly flat band (K. Sun et al., Neupert et al., E. Tang et al.) that may lead to FCI
- But "nearly" flat band is not crucial for FCI like flat band is not crucial for FQHE (think about disorder)

The checkerboard lattice model



K. Sun, Z.C. Gu, H. Katsura, S. Das Sarma, Phys. Rev. Lett. 106, 236803 (2011).

• $H_1 = \sum_k (c_{kA}^{\dagger}, c_{kB}^{\dagger}) h_1(k) (c_{kA}, c_{kB})^T$

•
$$h_1(k) = \sum_i d_i(k)\sigma_i$$

• $d_x(k) = 4t_1 \cos(\phi) \cos(k_x/2) \cos(k_y/2)$

•
$$d_y(k) = 4t_1 \sin(\phi) \sin(k_x/2) \sin(k_y/2)$$

•
$$d_z(k) = 2t_2(\cos(k_x) - \cos(k_y))$$

 φ, t₁ and t₂ parameters (plus second NNN) can be optimized to get a nearly flat band

The flat band limit



- The checkerboard lattice model has a nearly flat valence band.
- δ ≪ E_c ≪ Δ (E_c being the interaction energy scale)
- We can deform continuously the band structure to have a perfectly flat valence band
- and project the system onto the lowest band, similar to the projection onto the lowest Landau level

Two body interaction and the checkerboard lattice

Our goal : stabilize a Laughlin-like state at $\nu = 1/3$. **A key property** : the Laughlin state is the unique densest state that screens the short range repulsive interaction.



$$H_{\text{int}} = \sum_{\langle i,j \rangle} n_i n_j$$

- A nearest neighbor repulsion should mimic the FQH interaction.
- We give the same energy penalty to any of these Red-Blue clusters
- Sheng et al. Nature Communications 2, 389 (2011), Neupert et al. Phys. Rev. Lett. 106, 236804 (2011)

The $\nu = 1/3$ filling factor

An almost threefold degenerate ground state as you expect for the Laughlin state on a torus (here lattice with periodic BC)



But 3fold degeneracy is not enough to prove that you have Laughlin-like physics there.



FCI ($\nu = 1/3$, N = 8) The degeneracy is not perfect. The Berry curvature density is not flat and thus we don't have an exact magnetic translation algebra (see S. A. Parameswaran, R. Roy, S. L. Sondhi, arXiv :1106.4025)



- Many-body gap can actually increase with the number of particles due to aspect ratio issues.
- Finite size scaling not and not monotonic reliable because of aspect ratio in the thermodynamic limit.
- The 3-fold degeneracy at filling 1/3 in the continuum exists for any potential and is not a hallmark of the FQH state. On the lattice, 3-fold degeneracy at filling 1/3 means more than in the continuum, but still not much





Quasihole excitations

- The form of the groundstate of the Chern insulator at filling 1/3 is not exactly Laughlin-like. However, the universal properties SHOULD be.
- The hallmark of FQH effect is the existence of fractional statistics quasiholes.
- In the continuum FQH, Quasiholes are zero modes of a model Hamiltonians - they are really groundstates but at lower filling. In our case, for generic Hamiltonian, we have a gap from a low energy manifold (quasihole states) to higher generic states.



N = 9, $N_x = 5$, $N_y = 6$ The number of states below the gap matches the one of the FQHE !

Question : *"Why not using overlaps ?"* **Possible answers :**

- high overlaps do not guarantee that you are in the same phase
- we don't know how to write the Laughlin state on such a lattice
- we don't know if there is an exact hamiltonian for the Laughlin state on such a lattice

we cannot compute an overlap but we can compute an entanglement spectrum...

Particle entanglement spectrum

Particle cut : start with the ground state Ψ for N particles, remove $N - N_A$, keep N_A

$$\rho_{A}(x_{1},...,x_{N_{A}};x'_{1},...,x'_{N_{A}})$$

$$= \int ... \int dx_{N_{A}+1}...dx_{N} \qquad \Psi^{*}(x_{1},...,x_{N_{A}},x_{N_{A}+1},...,x_{N})$$

$$\times \quad \Psi(x'_{1},...,x'_{N_{A}},x_{N_{A}+1},...,x_{N})$$

"Textbook expression" for the reduced density matrix.



Laughlin
$$\nu = 1/3$$
 state $N = 8$,
 $N_A = 4$

- The particle ES reveals the fingerprint of the phase.
- This information that comes from the bulk excitations is encoded within the groundstate !
- If the FCI at $\nu = 1/3$ is a Laughlin-like phase, we should observe the Laughlin counting.

Particle entanglement spectrum



PES for N = 12, $N_A = 4$, 58905 states below the gap as expected

From topological to trivial insulator

One can go to a trivial insulator, adding a +M potential on A sites and -M potential on B sites. A perfect (atomic) insulator if $M \to \infty$.



Sudden change in both gaps at the transition $(M = 4t_2)$

Emergent Symmetries in the Chern Insulator

- In FCIs, there is in principle no exact degeneracy (apart from the lattice symmetries).
- But both the low energy part of the energy and entanglement spectra exhibit an emergent translational symmetry.
- The momentum quantum numbers of the FCI can be deduced by folding the FQH Brillouin zone.

• FQH :
$$m = GCD(N, N_{\phi} = N_x \times N_y)$$
,
FCI : $n_x = GCD(N, N_x)$,
 $n_y = GCD(N, N_y)$



FQHE vs FCI

- What is specific about the checkerboard lattice model?
- What are the important ingredients?
- Any evidence for other fractions?
 - p/2p + 1 series? Composite fermions?
 - Moore-Read / Read-Rezayi states
- Is there any FQHE feature missing on the FCI side?

Four other models



 $1/2~\mbox{HgTe}$ (2 bands)



FCI on the Haldane model

 $\nu=1/3$ on the Haldane model with short range repulsion.



We can tune t_1 , t_2 and Φ to make a flat band model (Neupert et al.)

Exactly the same features than $\nu = 1/3$ on the checkerboard lattice (but ...)

FCI on the Haldane model

How does the flatness of the Berry curvature affects the FCI spectrum ?

Look at $N = 8, \nu = 1/3$, tuning Φ tracking σ_B , the deviation to the average Berry curvature $(1/2\pi)$



The flatness of the Berry curvature helps but not as much as expected

Benchmarking the FCIs







Gaps for the Kagome model

- both the Haldane and 1/2 HgTe models show a weak FCI phase at $\nu = 1/3$
- Checkerboard, Kagome and ruby models exhibit a clear Laughlin like phase.
- Two versions of the Kagome model : the one with the flattest band (with NNN) is the weakest FCI.
- A flat band model is not a guarantee to get a good FCI



PES Kagome $N = 12, N_A = 5$

The Moore-Read state

Prototype of the non-abelian state that may describe the incompressible fraction $\nu = 5/2$. In the FQHE, the MR state can be exactly produced using a three-body interaction. What about the FCI?



Two types of nearest neighbor 3 particle cluster :

- Blue-Blue-Red
- Red-Red-Blue

We give the same energy penalty to any of these clusters

Energy spectrum at $\nu = 1/2$ for N = 12, $N_x = 6$, $N_y = 4$



6-fold almost degenerate groundstate (2 at $k_x = k_y = 0$, 4 at $k_x = 0$, $k_y = 4$), as predicted by the FQHE to FCI mapping !

Particle entanglement spectrum at $\nu = 1/2$ for $N = 12, N_x = 6, N_y = 4, N_A = 5$



1277 states in each momentum sector below the dotted line.

The Moore-Read state

Energy spectrum at $\nu = 1/2$ + one flux quantum for $N = 12, N_x = 5, N_y = 5$



7 states in each momentum sector below the dotted line. Can we stabilize the MR state with a two-body interaction? Work in progress...

The Read-Rezayi states



We can cook-up an interaction for each RR states.

- works like a charm for the RR k = 3.
- not completely clear for the RR k = 4. Finite size effect ?
- MR do not show up in the Haldane and 1/2 HgTe models
- Strong MR phase in the Kagome and ruby models.

Particle-hole symmetry

An important feature of the FQHE (with a two body interaction) : $u = 1/3 \leftrightarrow \nu = 2/3$



 $N = 6, N_x = 6, N_y = 3\nu = 1/3$

 $N = 12, N_x = 6, N_y = 3\nu = 2/3$

The p-h symmetry is absent from the FCI model, it might be restored at large NFCI are not as boring as expected

- What is specific about the checkerboard lattice model? not really
- What are the important ingredients? Don't know
- Any evidence for other fractions?
 - p/2p + 1 series? Composite fermions? TODO
 - Moore-Read / Read-Rezayi states yes !
- Is there any FQHE feature missing on the FCI side? particle-hole symmetry

Conclusion

- Fractional topological insulator at zero magnetic field exists as a proof of principle.
- There is a counting principles for the groundstate and the excitations.
- Beyond $\nu = 1/3$: $\nu = 1/5$, " $\nu = 5/2$ " (at least with three body interaction). Other fractions?
- What are the good ingredients for an FCI?
- First time entanglement spectrum is used to find information about a new state of matter whose ground-state wavefunction is not known.
- Entanglement spectrum powerful tool to understand strongly interacting phases of matter.
- **Roadmap** : find one such insulator experimentally, 2D-3D fractional topological insulators?