Dimers, trees and loops

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Logarithmic CFT and Representation Theory
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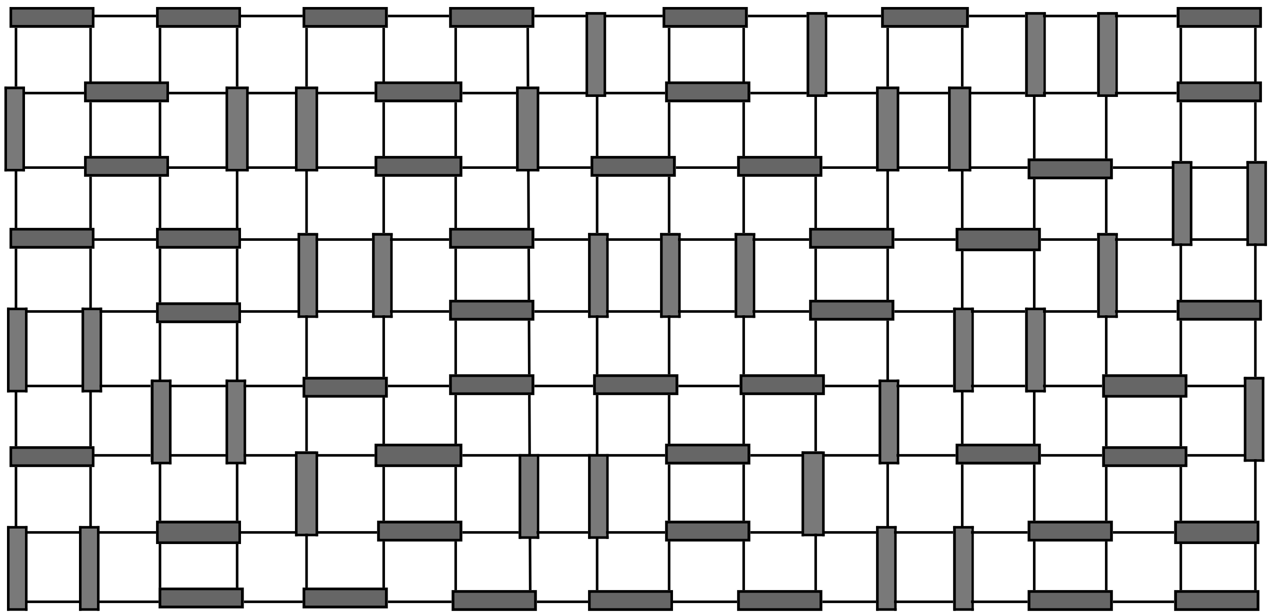
Dimers : a classical problem

Find all ways to cover a domain in $\mathbb{Z}^2$ by rods/dimers, each covering 2 sites.

Example of dimer covering of 9×18 grid :



There are 4.653×10^{18} other possible ones ...

The counting solved in 60s : Kasteleyn, Fisher, Temperley, Stephenson, Lieb, Ferdinand, Wu, Hartwig, ...

Various methods, among which Lieb's formulation in terms of Transfer Matrix.

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Solution of the Dimer Problem by the Transfer Matrix Method

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(Received 25 May 1967)

It is shown how the monomer–dimer problem can be formulated in terms of a transfer matrix, and hence in terms of simple spin operators as was originally done for the Ising problem. Thus, we rederive the solution to the pure dimer problem without using Pfaffians. The solution is extremely simple once one sees how to formulate the transfer matrix.

1. INTRODUCTION

SINCE Onsager's solution[1] of the two-dimensional Ising model in 1944, there has been a great deal of activity in the general area of nearest neighbor planar lattice problems. Basically, two approaches have been used.[2] One is the "algebraic" or "transfer matrix" method (used by Onsager) which focuses attention on the manner in which two neighboring rows are connected to each other. The second is the so-called "combinatorial method" whereby one studies graphs on the lattice as a whole. This was first used by Kac and Ward[3] for the Ising problem.

While Pfaffians have been used to rederive the solution to the Ising problem,[5] no one has yet taken the complementary step of solving the dimer problem by the transfer matrix method. The purpose of this note is to eliminate this gap. Elsewhere,[6] it has been shown how the transfer matrix method for Ising-like problems can be reduced to a few simple steps involving only fermion creation and annihilation operators. The dimer problem is likewise simple, using the transfer matrix. We also show how the more difficult and unsolved monomer–dimer problem can be formulated this way. The analogy with the problem

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Various methods, among which Lieb's formulation in terms of Transfer Matrix.

Replace dimers by arrows attached to sites :

Up ↑ arrow means presence of a dimer pointing upward
Down ↓ arrow means absence of a dimer pointing up

Transfer matrix

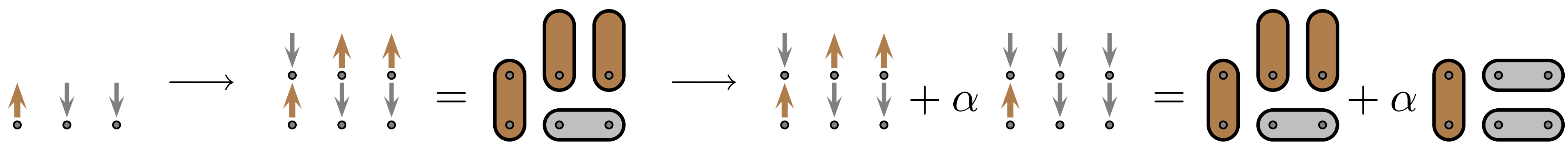
Attach each site an arrow, $\uparrow \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\downarrow \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Every site carries space $\mathbb{C}^2$:

$$\sigma_i^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \uparrow \rightleftharpoons \downarrow \,, \quad \sigma_i^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} : \downarrow \rightsquigarrow \uparrow \,, \quad \sigma_i^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} : \uparrow \rightsquigarrow \downarrow$$

Row of size N carries $(\mathbb{C}^2)^{\otimes N}$: row–to–row transfer matrix is $2N$-dimensional,

$$\boxed{V = \prod_i \left(1 + \alpha\, \sigma_i^- \sigma_{i+1}^-\right) \prod_i \sigma_i^x \qquad \text{on } (\mathbb{C}^2)^{\otimes N}}$$



Thus : $\qquad V_{\downarrow\uparrow\uparrow\,,\,\uparrow\downarrow\downarrow} = 1\,, \qquad V_{\downarrow\downarrow\downarrow\,,\,\uparrow\downarrow\downarrow} = \alpha$

Partition function

The Transfer Matrix

$$V(\alpha) = \exp\left(\alpha \sum_i \sigma_i^- \sigma_{i+1}^-\right) \prod_i \sigma_i^x$$

builds all possible arrow/dimer configs of a row from previous row; its entries are monomials in α with $V_{\text{config2,config1}} = \alpha^k$ if config2 has k horizontal dimers.

Likewise, from initial row config, V^m constructs all possible configs m rows higher, including multiplicities; entries of V^m are $\mathbb{N}$-polynomials in α.

Note : $\exp\left(\alpha \sum_{i=1}^N \ldots\right)$ and $\exp\left(\alpha \sum_{i=1}^{N-1} \ldots\right)$ mean periodic resp. open b.c. horiz.

Depending on vertical b.c.,

periodic vert. : $Z_{M,N}(\alpha) = \sum_{\text{dimer cov.}} \alpha^{\#\text{hor}} = \operatorname{Tr} V^M$ (torus/cyl)

open vert. : $Z_{M,N}(\alpha) = \sum_{\text{dimer cov.}} \alpha^{\#\text{hor}}$

$$= \sum_{|\text{in}\rangle} \langle \downarrow \ldots \downarrow | V^{M-1} | \text{in} \rangle = \langle \downarrow \ldots \downarrow | V^M | \downarrow \ldots \downarrow \rangle \qquad \text{(cyl/rect)}$$

We are in business ... for dimers

This Transfer Matrix is good starting point to study the dimer model itself, believed to be described by CFT with $c = -2$, see later.

Log CFT ??????

Don't know, however note that $V(\alpha)$ for general α is not hermitian (nor normal),

$$\begin{aligned} V(\alpha)^\dagger &= \left[\exp\left(\alpha \sum_i \sigma_i^- \sigma_{i+1}^-\right) \prod_i \sigma_i^x\right]^\dagger \\ &= \prod_i \sigma_i^x \exp\left(\alpha^* \sum_i \sigma_i^+ \sigma_{i+1}^+\right) \\ &= \exp\left(\alpha^* \sum_i \sigma_i^- \sigma_{i+1}^-\right) \prod_i \sigma_i^x = V(\alpha^*) \end{aligned}$$

But $V(\alpha)$ is real symmetric for real α, hence fully diagonalizable ... (L_0 may still have Jordan cells but here ??)

Note also : $[V(\alpha), V(\alpha')] \neq 0$ ...

… not for trees !

We are primarily interested in spanning trees !

Main motivation :

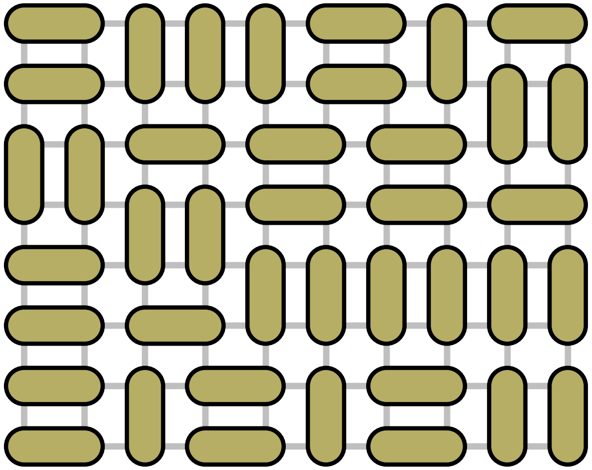
- sandpile model is usually defined in terms of height variables, but completely equivalent formulation in terms of spanning trees
- we know of lattice bulk observables which form log pairs in scaling limit; otherwise field content is poorly understood
- which types of indecomposable reps appear ?

Main question is :

Can we cook up a cylinder transfer matrix for spanning trees ?

Dimers and trees ...

Well-known relation between dimers and spanning trees (Temperley, 1974).
For simplicity, take an even–by–even grid, f.i. 8×10.



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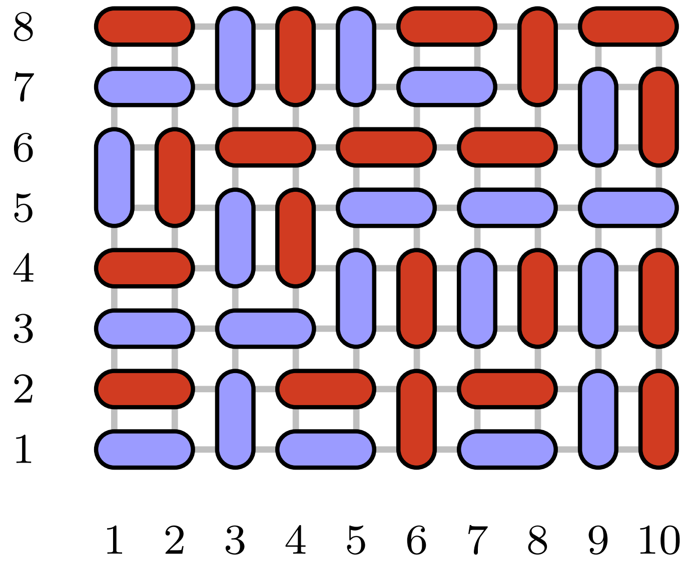

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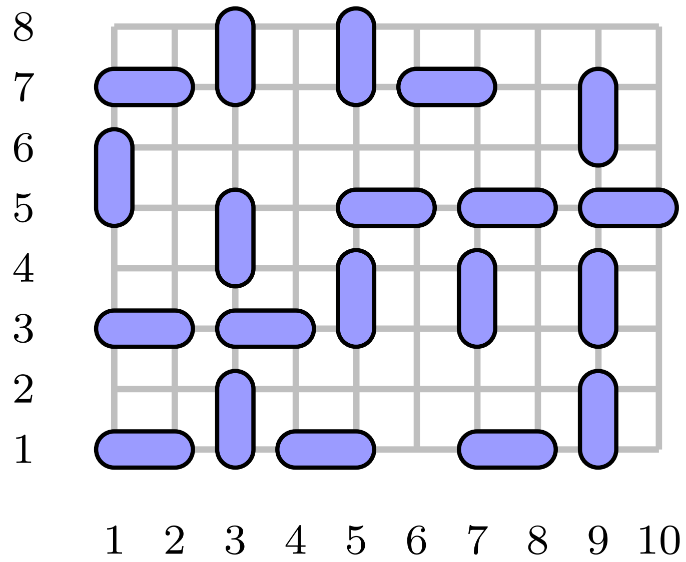

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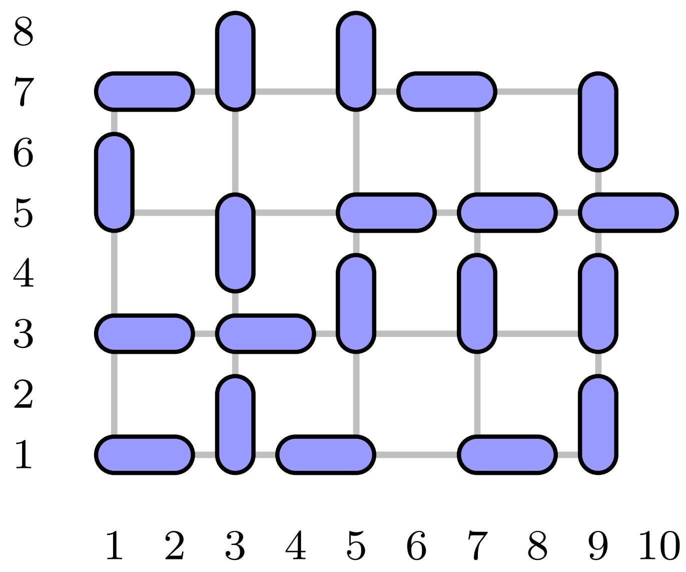

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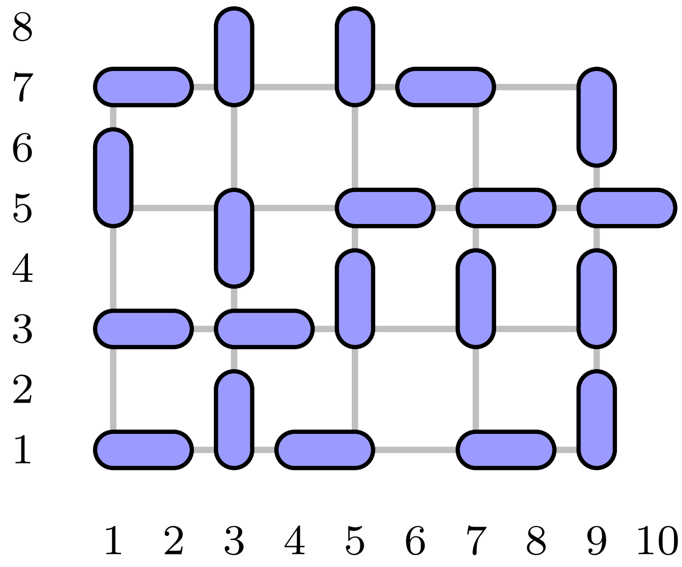

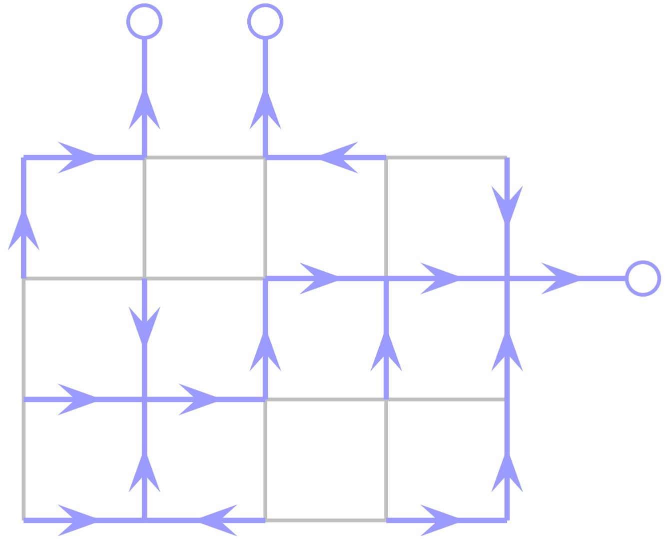

rooted spanning tree* on $\mathcal{L}_{\mathrm{odd}}$
(wired/open b.c. on top and right,
closed on bottom and left)

* Loops would encircle odd number of sites on original lattice

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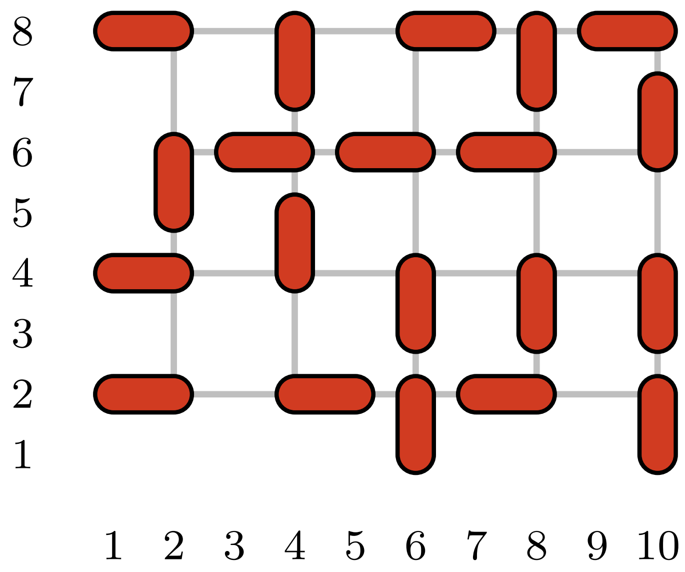

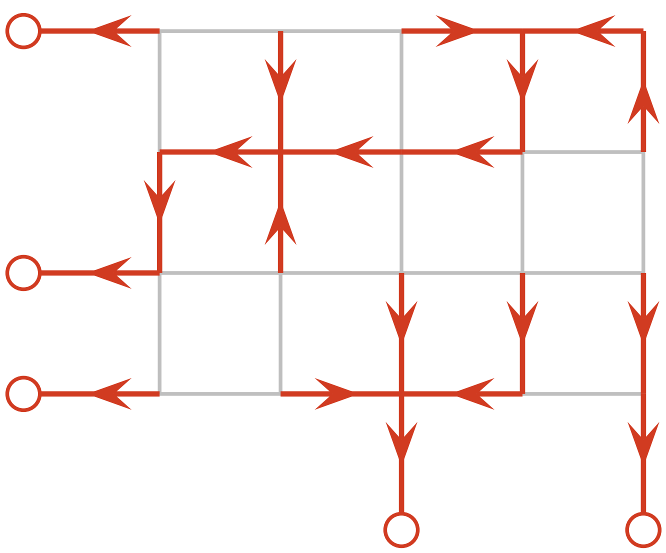

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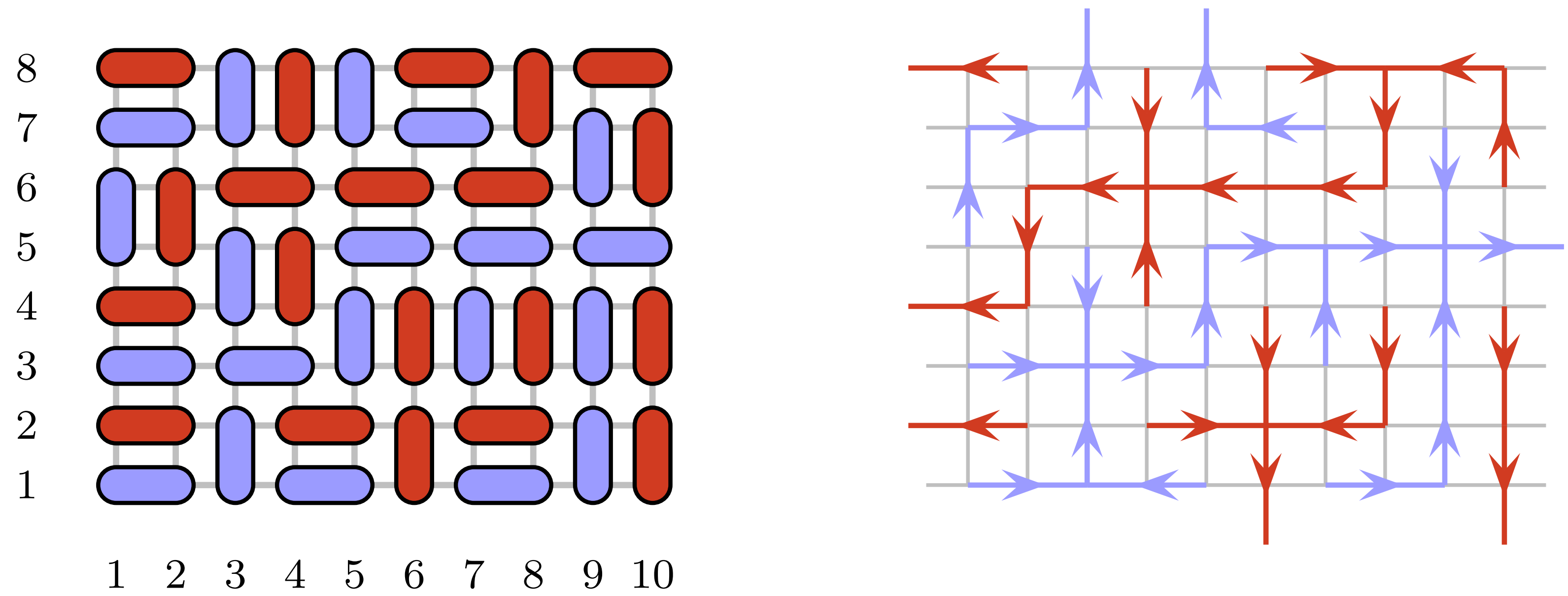
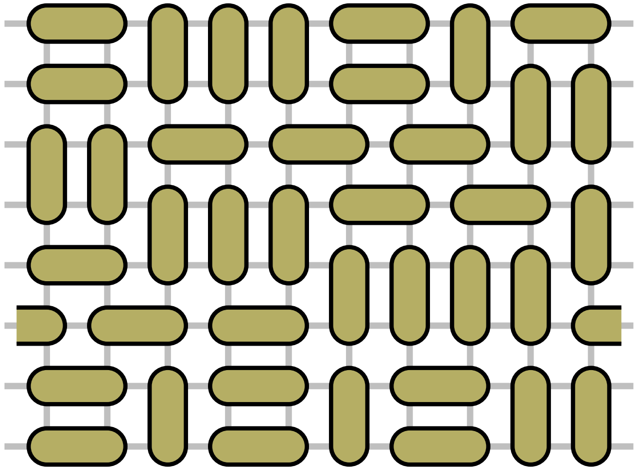


- Can use either blue or red trees/dimers : one colour completely fixes the other !
  (*blue lines and red lines cannot cross*)
- Parities of M, N determine the b.c.'s : open $\leftrightarrow$ closed and closed $\leftrightarrow$ open

What on a cylinder ?

Take even–by–even $2M \times 2N$ grid, with horizontal periodicity.



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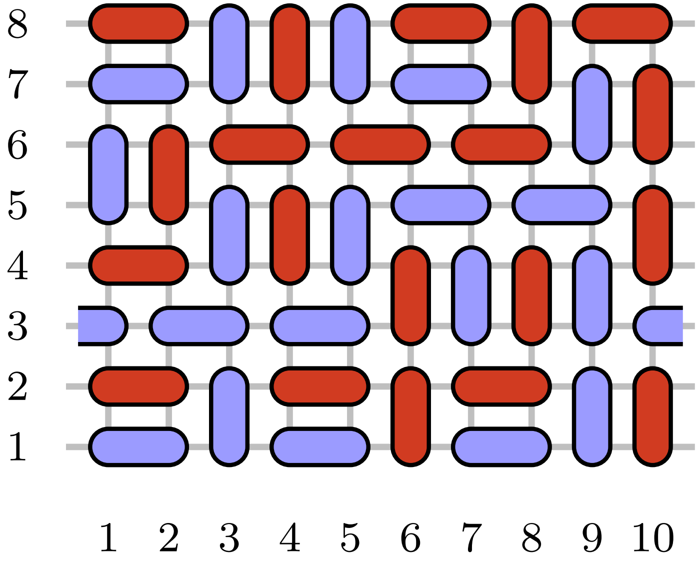

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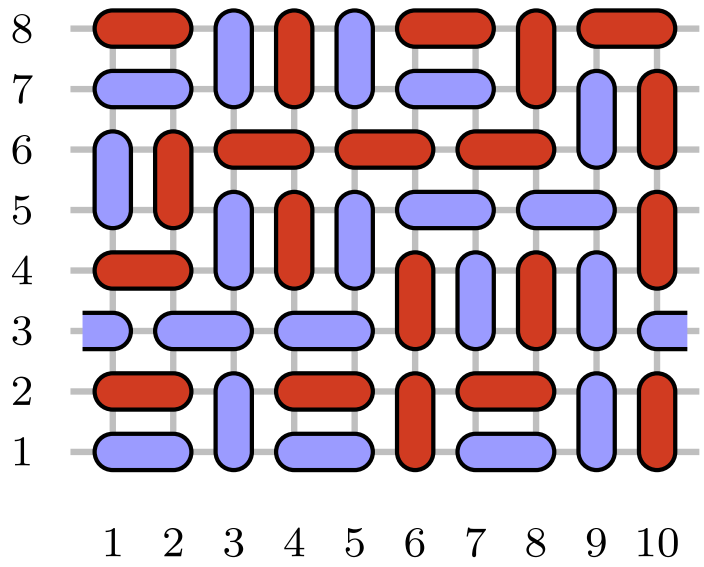

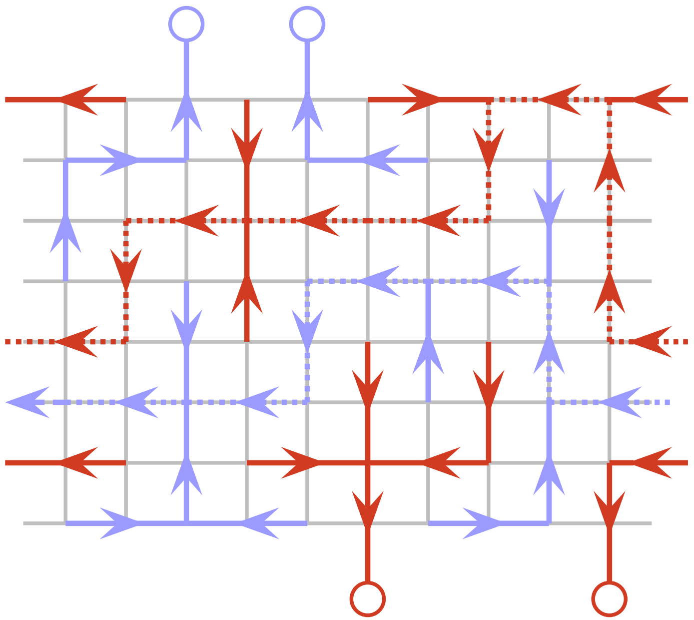

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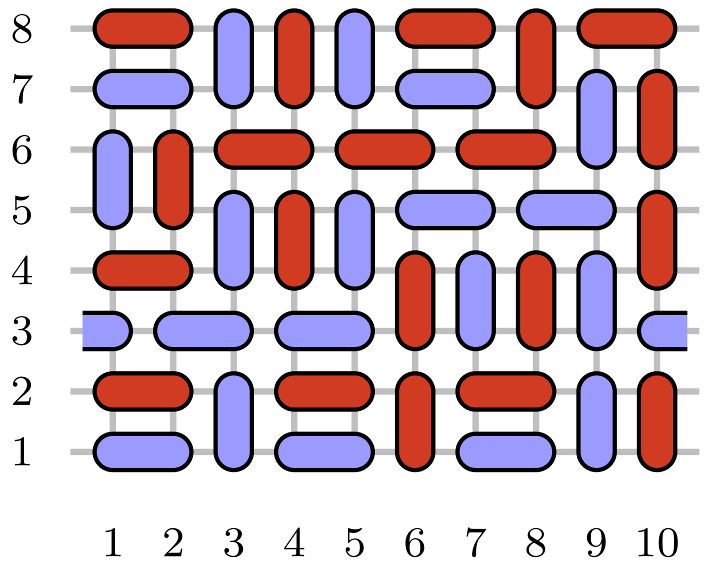

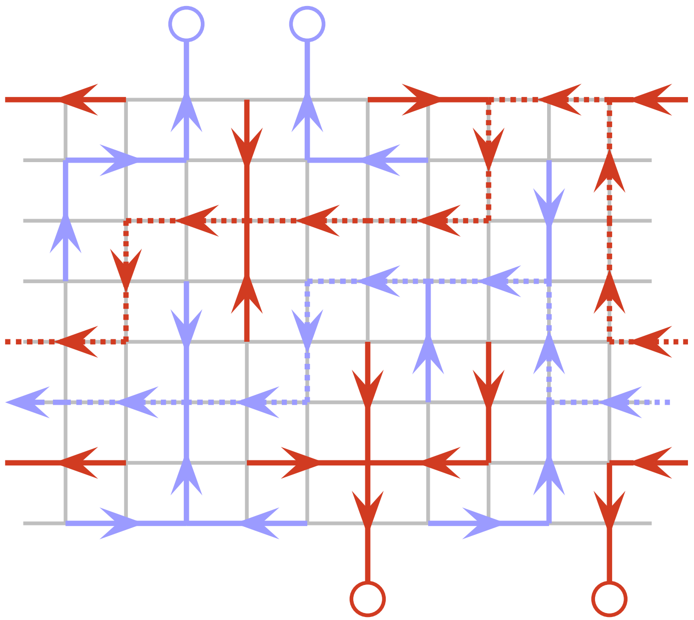

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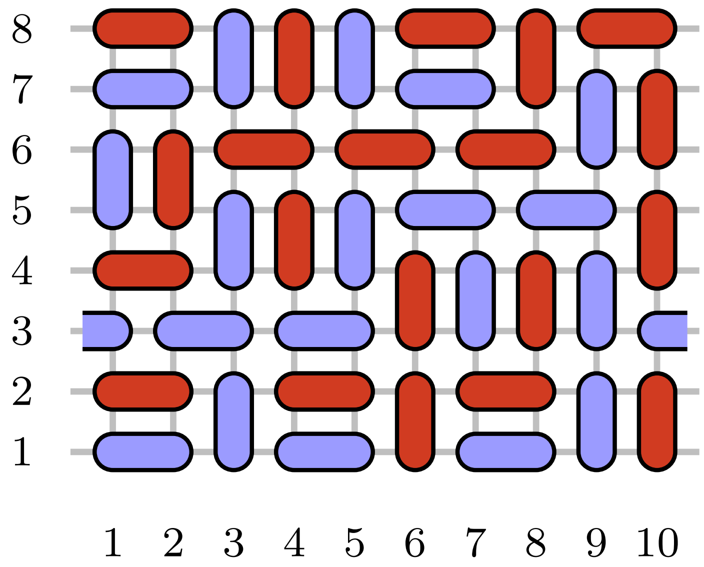

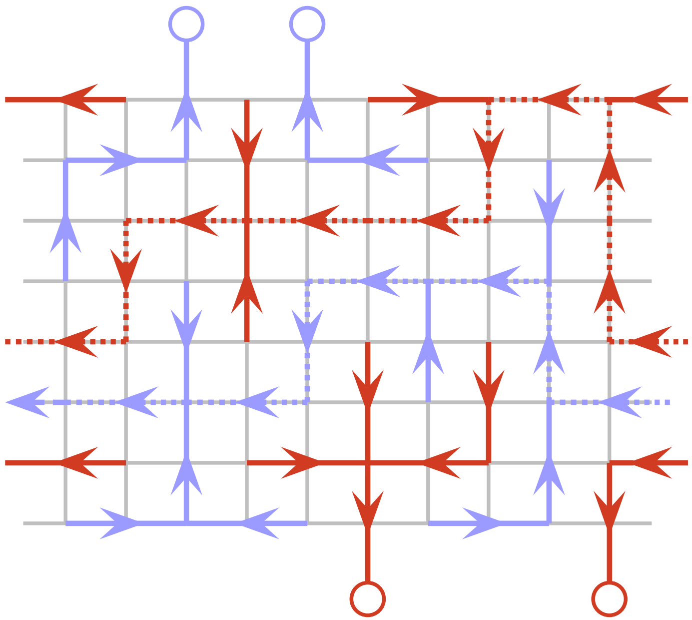

- the roots are on top and bottom
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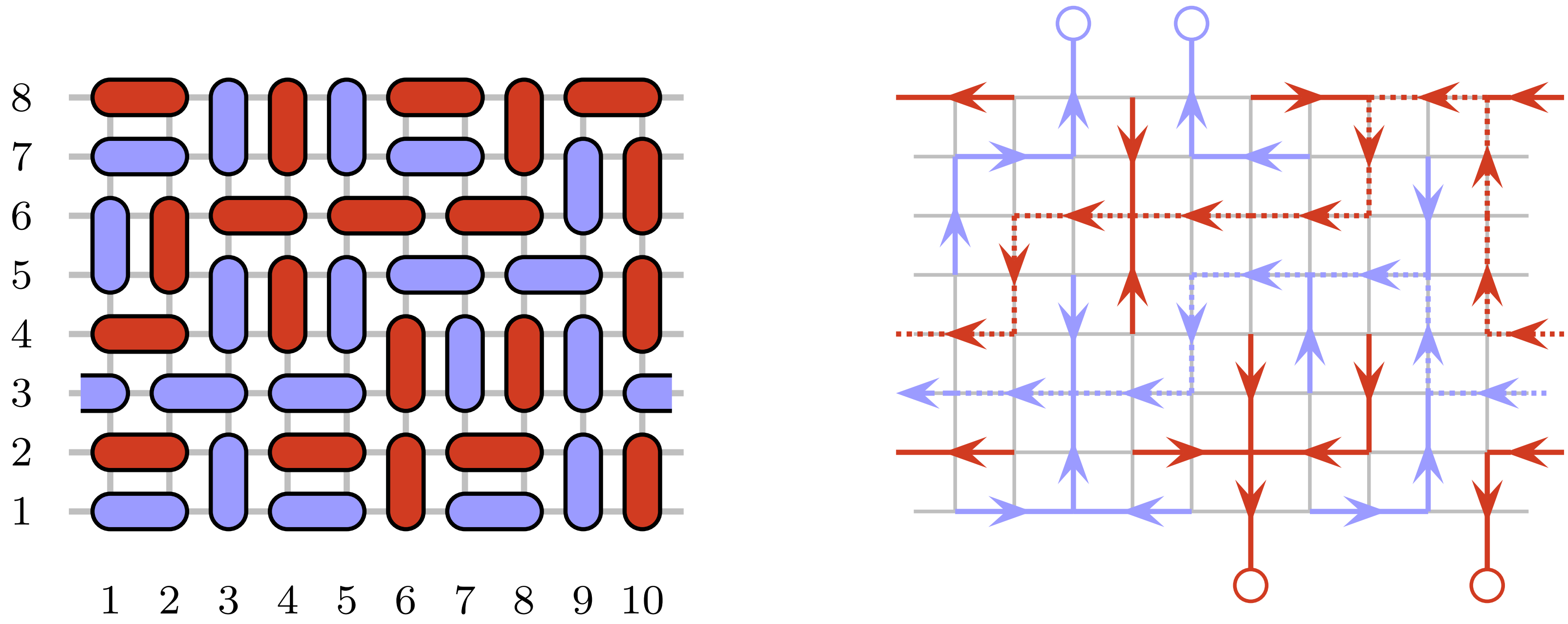

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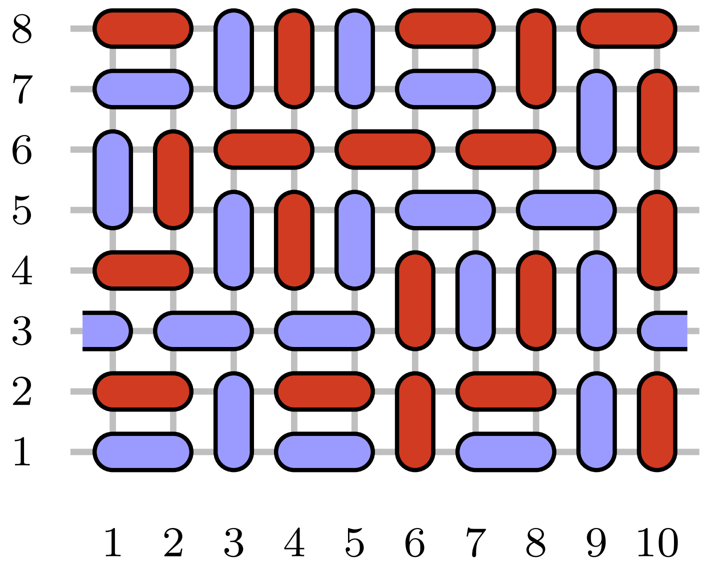

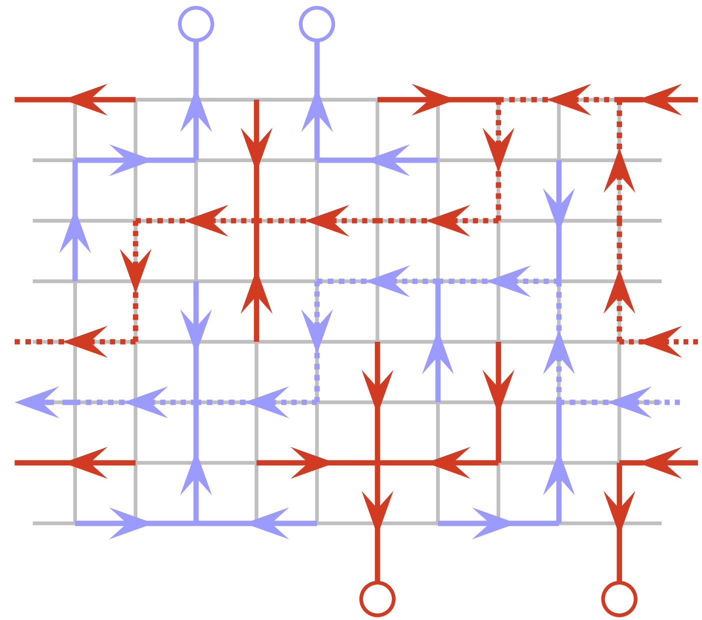

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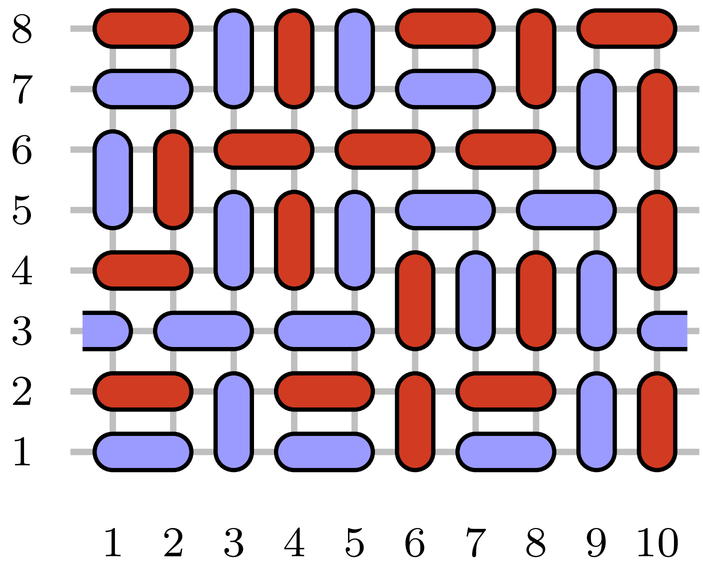

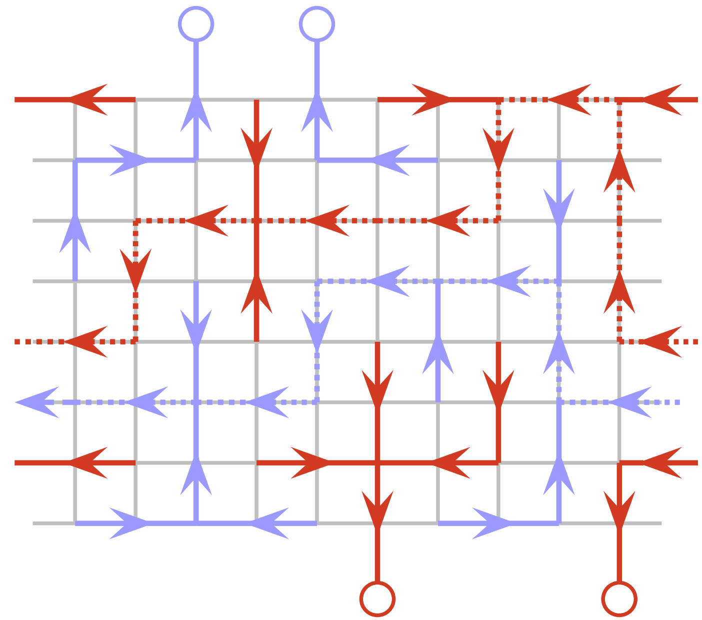

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What on a cylinder ?

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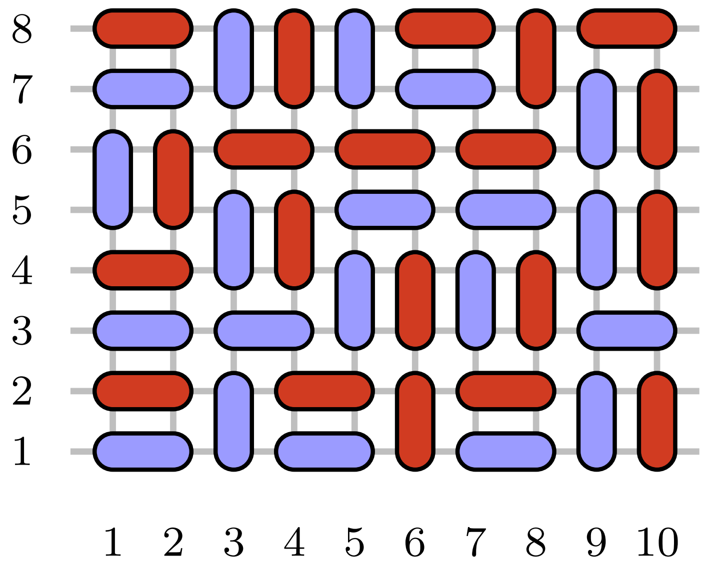

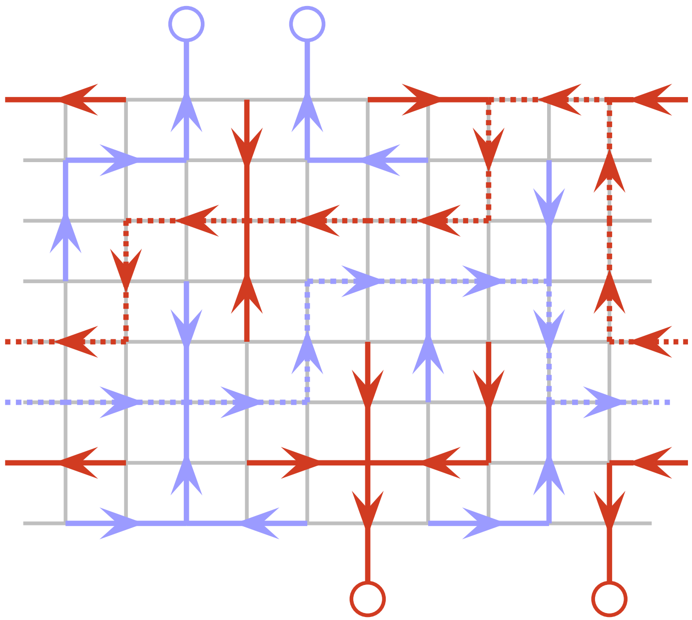

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Counting the loops

Dimers on $2M \times 2N$ lattice $\mathcal{L} \longrightarrow$ blue spanning web on $\mathcal{L}_{\rm odd}$ (or red on $\mathcal{L}_{\rm even}$).

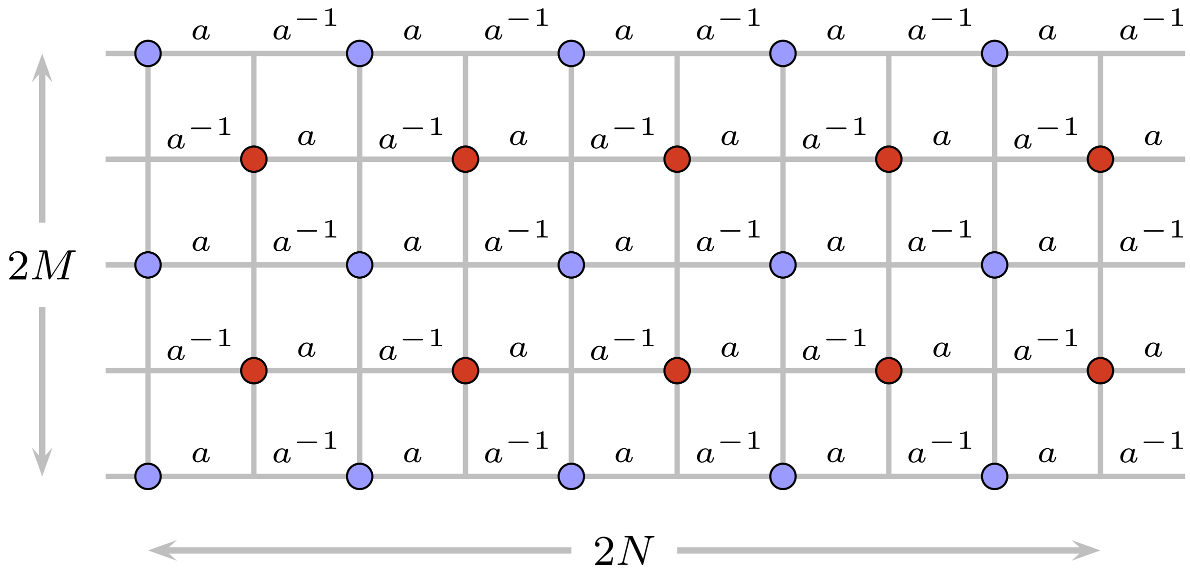
Group configs according to number of loops $2L$ (L blue and L red) :

$$
\begin{aligned}
\mathcal{Z}^{\rm dimers}_{2M,2N} &= \text{total number of dimer configs} \\
&= \sum_{L=0}^{M} \#\{\text{dimer configs on } 2M \times 2N, \text{ with } 2L \text{ loops}\} \\
&= \sum_{L=0}^{M} 2^L \cdot \#\{\text{blue spanning webs on } M \times N, \text{ with } L \text{ loops}\} \\
&= \sum_{L=0}^{M} 2^L \cdot Z^{\rm SW}_{M,N}(L)
\end{aligned}
$$

Need to disantengle the various contributions for fixed L ...

Managing the loops

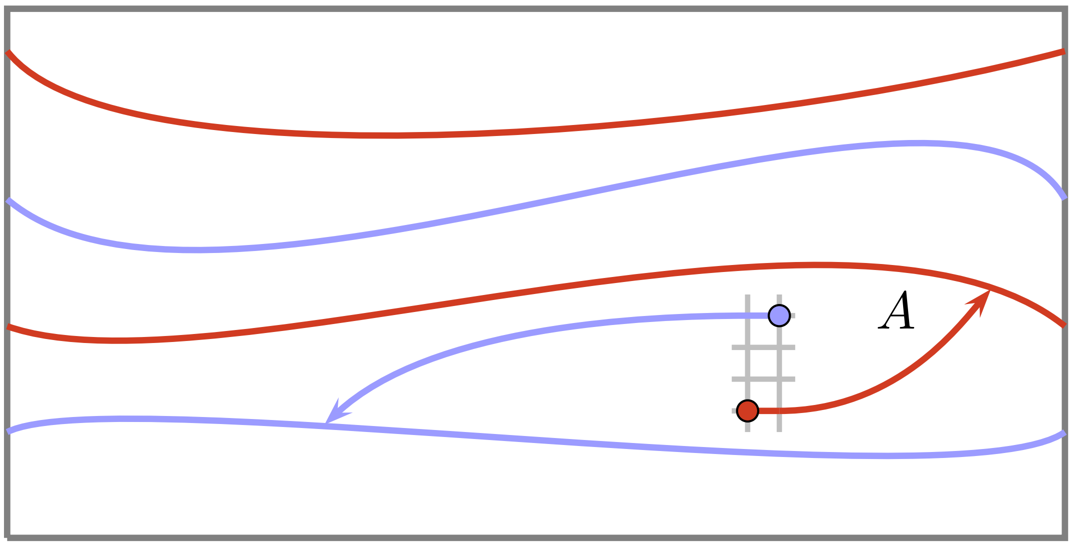
Assign alternating weights $a = w^{1/N}$ and $a^{-1} = w^{-1/N}$ to horizontal dimers, as :



Dimer/arrow pointing to $\left\{\begin{matrix}\text{right}\\ \text{left}\end{matrix}\right\}$ gets weight $\left\{\begin{matrix}w^{1/N}\\ w^{-1/N}\end{matrix}\right\}$, whether blue or red

$\Longrightarrow$ a loop oriented $\left\{\begin{matrix}\text{left} \rightarrow \text{right}\\ \text{left} \leftarrow \text{right}\end{matrix}\right\}$ gets a weight $\left\{\begin{matrix}w\\ w^{-1}\end{matrix}\right\}$!

Moreover, horizontal dimers/arrows not in loops, bring total contribution equal to 1 !



Left blue arrows and right red arrows must alternate vertically (and vice-versa) :
as they carry inverse weights, their contributions cancel out.

Counting the loops (cont'd)

Dimers on $2M \times 2N$ lattice $\mathcal{L} \longrightarrow$ blue spanning web on $\mathcal{L}_{\text{odd}}$ (or red on $\mathcal{L}_{\text{even}}$).

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Group configs according to number of loops $2L$ (L blue and L red) :

$$\mathcal{Z}^{\rm dimers}_{2M,2N}(w) = \text{total number of dimer configs appropr. weighted}$$

$$= \sum_{L=0}^{M} \#\{\text{weighted dimer configs, with } 2L \text{ loops}\}$$

$$= \sum_{L=0}^{M} \left\{ \#\begin{bmatrix}\text{configs } 2L \text{ loops,} \\ \text{all left-right}\end{bmatrix} \cdot w^{2L} + \#\begin{bmatrix}\text{configs } 2L \text{ loops,} \\ \text{all but 1 left-right}\end{bmatrix} \cdot w^{2L-2} + \#\begin{bmatrix}\text{configs } 2L \text{ loops,} \\ \text{all but 2 left-right}\end{bmatrix} \cdot w^{2L-4} + \ldots \right\}$$

$$= \sum_{L=0}^{M} \#\begin{bmatrix}\text{configs } 2L \text{ loops,} \\ \text{all left-right}\end{bmatrix} \cdot \left\{ w^{2L} + \binom{2L}{1} w^{2L-2} + \binom{2L}{2} w^{2L-4} + \ldots \right\}$$

$$= \sum_{L=0}^{M} \frac{\#[\text{configs } 2L \text{ loops}]}{2^{2L}} \cdot (w+w^{-1})^{2L} = \sum_{L=0}^{M} \#\begin{bmatrix}\text{blue SW with} \\ L \text{ loops}\end{bmatrix} \cdot \left(\frac{w+w^{-1}}{\sqrt{2}}\right)^{2L}$$

Partition functions for loops

Gives the partition functions for spanning webs with fixed number loops in terms of dimer configurations on (two times) denser grid:

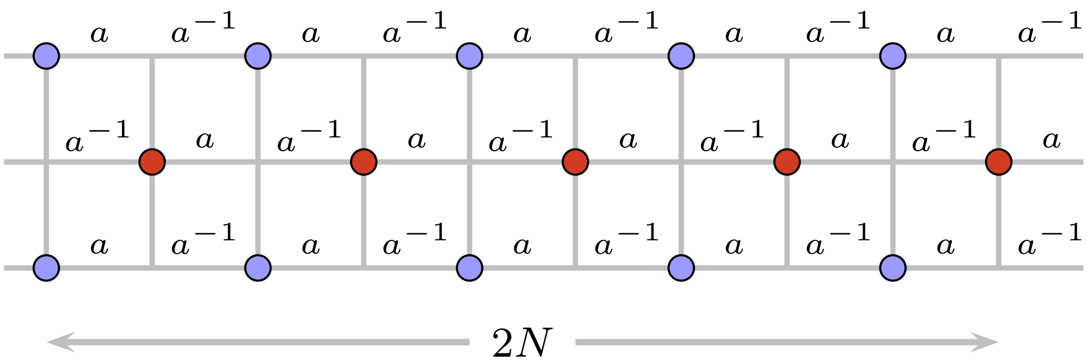
$$\mathcal{Z}^{\rm dimers}_{2M,2N}(w) = \sum_{L=0}^{M} \#\left[\begin{matrix}\text{blue SW with}\\ L \text{ loops}\end{matrix}\right] \cdot \left(\frac{w+w^{-1}}{\sqrt{2}}\right)^{2L} = \sum_{L=0}^{M} Z^{\rm SW}_{M,N}(L) \cdot \left(\frac{w+w^{-1}}{\sqrt{2}}\right)^{2L}$$

- Dimer partition function (lhs) can be computed in terms of a transfer matrix.
- Allows to compute all fixed-number-of-loops partition functions for spanning webs, but no specific transfer matrix for fixed L.
- The choice $w = \mathrm{i}$ kills all loops !!
  Transfer matrix for weighted dimers becomes transfer matrix for spanning trees ...
- Keep general weight w and set $w = \exp \mathrm{i}\pi z$:

$$z = 0 \quad \longrightarrow \quad \text{usual dimer model}$$

$$z = \tfrac{1}{2} \quad \longrightarrow \quad \text{spanning trees}$$

Transfer matrix for weighted dimers



$$a = w^{1/N} = e^{\mathrm{i}\pi z/N}$$

Two-step transfer matrix is $4^N \times 4^N$: $\quad T(w) = V(w^{-1})\, V(w) \quad$ with

$$V(w) = \exp\left[a\,\sigma_1^-\sigma_2^- + a^{-1}\,\sigma_2^-\sigma_3^- + \ldots\right] \prod_{i=1}^{2N} \sigma_i^x = \exp\left(\sum_{i=1}^{2N} a^{(-1)^{i+1}}\,\sigma_i^-\sigma_{i+1}^-\right) \prod_{i=1}^{2N} \sigma_i^x$$

Hence

$$\boxed{T(w) = \exp\left(\sum_{i=1}^{2N} a^{(-1)^i}\,\sigma_i^-\sigma_{i+1}^-\right) \exp\left(\sum_{i=1}^{2N} a^{(-1)^{i+1}}\,\sigma_i^+\sigma_{i+1}^+\right)}$$

Spectrum

Transfer matrix $T(w) = V(w^{-1})\,V(w)$ Hermitian (diagonalizable) for w on unit circle

$$T^\dagger(w) = [V(w^{-1})\,V(w)]^\dagger = V^\dagger(w)\,V^\dagger(w^{-1}) = V(w^*)\,V(w^{*-1})$$

Away from unit circle, no longer Hermitian nor even normal, but fully diagonalizable, except at a finite set of isolated points.

Standard techniques to diagonalize (Jordan-Wigner, see Lieb):

$$\lambda^{\text{odd}} = \prod_{k=0}^{N-1} \left[1 \ \underline{\text{or}}\ 1 \ \underline{\text{or}}\ \left(\alpha_k + \sqrt{1+\alpha_k^2}\right)^2 \ \underline{\text{or}}\ \left(\alpha_k - \sqrt{1+\alpha_k^2}\right)^2 \right]\Big|_{\text{odd}}$$

$$\lambda^{\text{even}} = \prod_{k=0}^{N-1} \left[1 \ \underline{\text{or}}\ 1 \ \underline{\text{or}}\ \left(\beta_k + \sqrt{1+\beta_k^2}\right)^2 \ \underline{\text{or}}\ \left(\beta_k - \sqrt{1+\beta_k^2}\right)^2 \right]\Big|_{\text{even}}$$

with $\alpha_k = \sin\frac{\pi(k+z)}{N}$ and $\beta_k = \sin\frac{\pi(k+z+\frac{1}{2})}{N}$. (Remember $a = w^{1/N}$ and $w = e^{\mathrm{i}\pi z}$.)

OK for all w, z provided $\alpha_k^2 \neq -1$ and $\beta_k^2 \neq -1$.

Spectrum generating functions

Want to form : $\mathcal{Z}(z) = \sum \lambda^M = \sum e^{-ME} = \sum \left(\lambda^{\text{odd}}\right)^M + \sum \left(\lambda^{\text{even}}\right)^M$

and look at universal part when $M, N \to \infty$.

Odd part reads ($\alpha_k = \sin \frac{\pi(k+z)}{N}$)

$$\begin{aligned} \mathcal{Z}^{\text{odd}}(z) &= \prod_{k=0}^{N-1} \left[1 + 1 + \left(\alpha_k + \sqrt{1+\alpha_k^2}\right)^{2M} + \left(\alpha_k - \sqrt{1+\alpha_k^2}\right)^{2M} \right]\Big|_{\text{odd}} \\ &\to \exp\left(\frac{4\mathrm{G}MN}{\pi}\right) q^{z^2} \frac{\theta_1^2(q^z|q) + \theta_2^2(q^z|q)}{\eta^2(q)} \qquad (q = e^{-2\pi M/N}). \end{aligned}$$

Even part follows from $z \to z + \frac{1}{2}$

$$\mathcal{Z}^{\text{even}}(z) \to \exp\left(\frac{4\mathrm{G}MN}{\pi}\right) q^{z^2} \frac{\theta_3^2(q^z|q) + \theta_4^2(q^z|q)}{\eta^2(q)}$$

Where $\theta_1(y|q) = -\mathrm{i}\sqrt{y}\, q^{1/8} \prod_{n\geq 1}(1-q^n) \prod_{n\geq 0}(1-yq^{n+1})(1-y^{-1}q^n) \ldots$

Spectrum generating functions

Full conformal spectrum generating function thus reads

$$\mathcal{Z}(z|q) = q^{z^2} \, \frac{\theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2}{2\eta^2}(q^z|q).$$

Expanding

$$\mathcal{Z}(z|q) = \sum_{L=0}^{\infty} Z^{\mathrm{SW}}(L;q) \left(\frac{w+w^{-1}}{\sqrt{2}}\right)^{2L} = \sum_{L=0}^{\infty} Z^{\mathrm{SW}}(L;q) \left(\sqrt{2}\,\cos \pi z\right)^{2L}$$

allows to compute

$Z^{\mathrm{SW}}(L;q) =$ spectrum generating function for spanning webs containing exactly L non-contractible loops, wrapping once around perimeter of cylinder.

Two simple cases: $z = 0$ for dimers, and $z = \frac{1}{2}$ for spanning trees.

Dimers

One recovers well-known result (Ferdinand 1967)

$$Z^{\text{dimers}}(q) = \frac{\theta_2^2 + \theta_3^2 + \theta_4^2}{2\eta^2}(q).$$

- Fully modular invariant: Z^{dimers} is the partition function for dimer model on torus with module $\tau = \mathrm{i}M/N$.
- Reproduces partition function for symplectic fermions (Gaberdiel-Kausch 1996)

$$Z^{\text{dimers}}(q) = \chi_{(-1/8,-1/8)} + \chi_{(3/8,3/8)} + \chi_{\mathcal{R}}.$$

But not clear that equivalent : no trace here of Jordan ?

Lieb's transfer matrix is rich enough ?

Spanning trees

Spectrum generating function reads

$$Z^{\text{trees}}(q) = \frac{\theta_2^2 + \theta_3^2 - \theta_4^2}{2\eta^2}(q).$$

- Not modular invariant, as expected. Loops have been killed in one direction but not in the other. Not a torus partition function for spanning trees (cannot be). Partition function for something else ...
- In terms of W-characters at $c = -2$

$$Z^{\text{trees}}(q) = \chi_{(-1/8,3/8)} + \chi_{(3/8,-1/8)} + \chi_{\mathcal{R}}$$

appears as Z_2-twisted partition function of Z^{dimers} ... ??

$$Z_{PP} = \chi_{(-1/8,-1/8)} + \chi_{(3/8,3/8)} + \chi_{\mathcal{R}}, \qquad Z_{PA} = \chi_{(-1/8,-1/8)} + \chi_{(3/8,3/8)} - \chi_{\mathcal{R}}$$

$$Z_{AP} = \chi_{(-1/8,3/8)} + \chi_{(3/8,-1/8)} + \chi_{\mathcal{R}}, \qquad Z_{AA} = -\chi_{(-1/8,3/8)} - \chi_{(3/8,-1/8)} + \chi_{\mathcal{R}}$$

Conclusion

- Lieb's old transfer matrix revisited and adapted to keep track of loops
- Weighted TM shows no trace of Jordan cell, despite degeneracies at finite size match expected degeneracies from logCFT (for dimers)
- Spanning trees ($\equiv$ sandpile): no modular invariance, as expected, but surprising appearance of Z_2-twisted partition function Z_{AP}. Physical meaning ??
- Weighted TM useful on cylinder: allows to compute partition functions for fixed number of loops, but complicated expressions.