Logarithmic conformal field theory with boundaries

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joint work with

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Outline

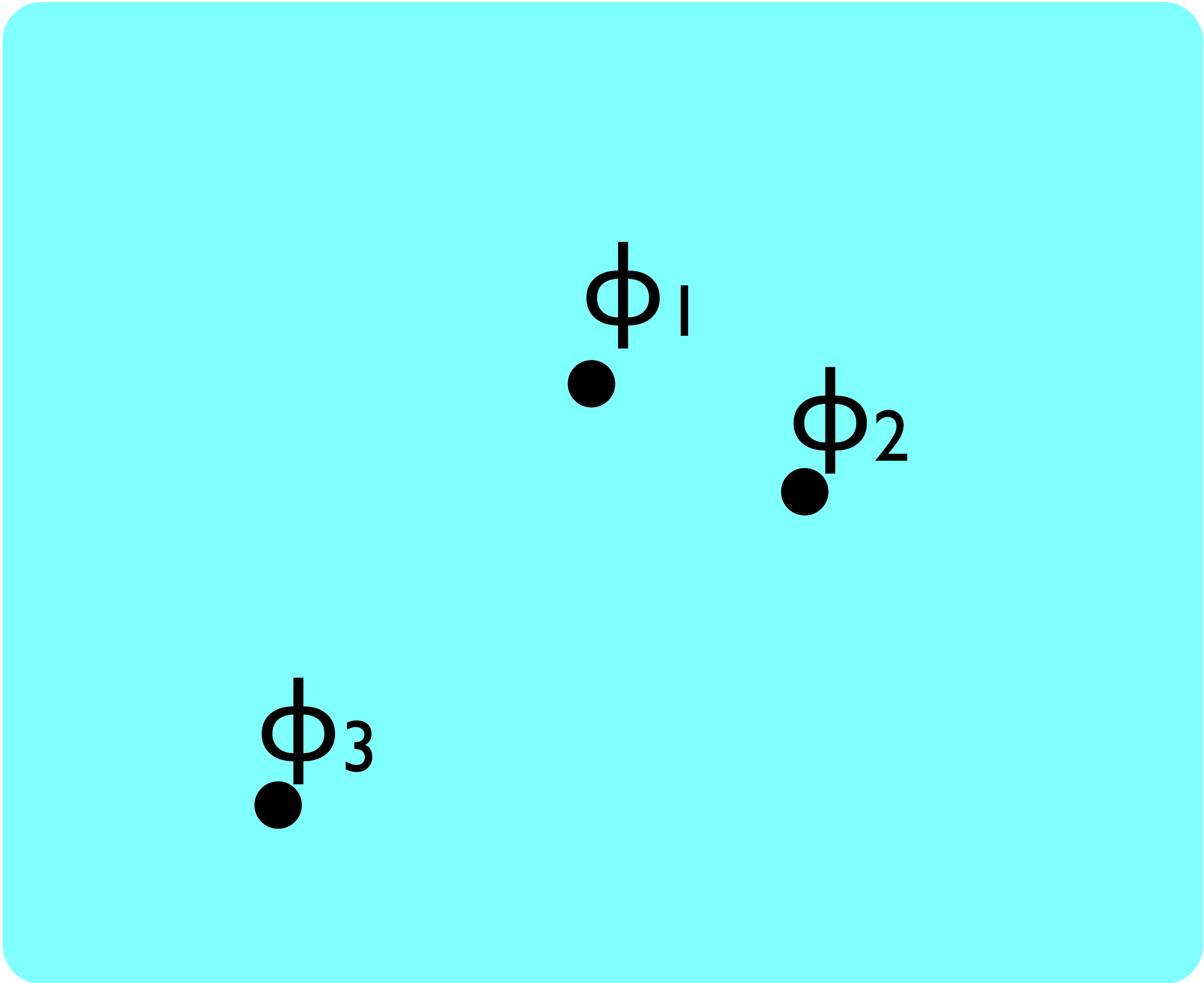
- Bulk and boundary CFT
- Algebraic reformulation
- The W_{23}-model

Bulk CFT

Vector space of fields F (typically infinite dimensional)

Correlators are smooth functions $(\mathbb{C}^n \setminus \text{diag}) \times F^n \to \mathbb{C}$ invariant under joint permutations of $\mathbb{C}$'s and F's

Notation : $\langle\, \phi_1(z_1)\ \phi_2(z_2)\ \ldots\ \phi_n(z_n)\, \rangle$

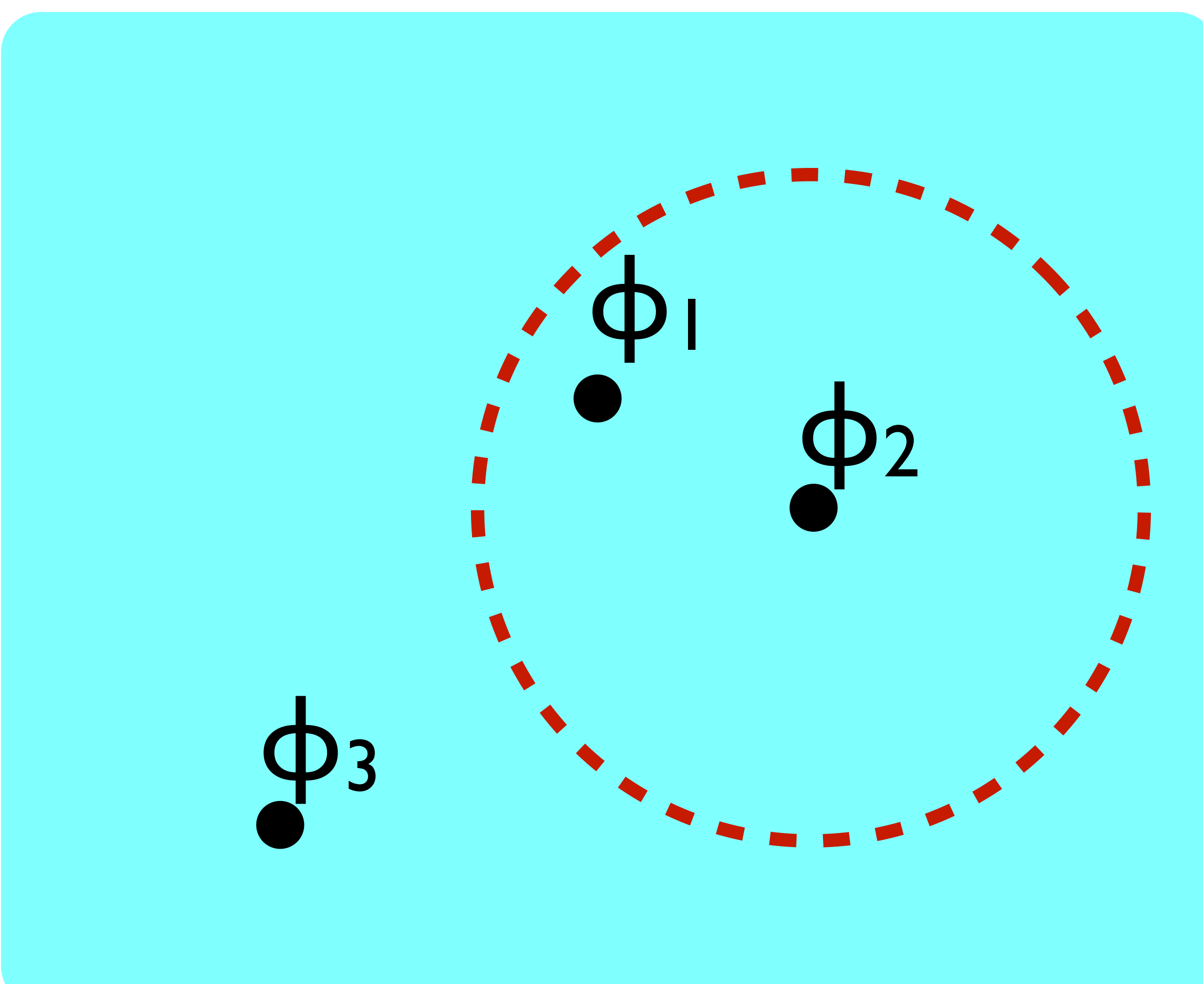


. . . bulk CFT

Short distance expansion / operator product expansion:

$$\langle\, \phi_1(z_1)\, \phi_2(z_2)\, \phi_3(z_3) \ldots \phi_n(z_n)\, \rangle$$

$$= \sum_\alpha \; f_{12,\alpha}(z_1 - z_2) \;\; \langle\, \varphi_\alpha(z_2)\, \phi_3(z_3) \ldots \phi_n(z_n)\, \rangle$$



This gives a map

$M_x : F \times F \to \overline{F}$,

the bulk OPE

(F is a direct sum of graded components, $\overline{F}$ the direct product)

. . . bulk CFT

Out-vacuum : $\Omega^* : F \to \mathbb{C}$, $\langle \Omega^*, \phi \rangle = \langle \phi(0) \rangle$

Virasoro action : Virasoro algebra Vir

sl(2,C) $\subset$ Vir , generator L_{-1}, L_0, L_1 , Vir$\oplus$Vir acts on F

Correlators are coinvariants. E.g. 2-pt correlator

$$\langle \phi(x)\, \psi(0) \rangle = \langle \Omega^*, M_x(\phi, \psi) \rangle$$

1) $d/dx \; \langle \Omega^*, M_x(\phi, \psi) \rangle = \langle \Omega^*, M_x(L_{-1}\phi, \psi) \rangle$

2) $\langle \Omega^*, M_x(L_{-1}\phi, \psi) \rangle$

$$= - x^{-1} \langle \Omega^*, M_x \circ (L_0 \otimes id + id \otimes L_0)\, (\phi, \psi) \rangle$$

(+ many more)

... bulk CFT

Logarithms: Solution to differential equation is

$$\langle\, \phi(x)\, \psi(0)\, \rangle$$

$$= \langle \Omega^* \,,\, M_1 \circ e^{-(L_0 \otimes id + id \otimes L_0)\ln x - (\bar{L}_0 \otimes id + id \otimes \bar{L}_0)\ln x^*} (\phi, \psi) \rangle$$

L_0 , $\overline{L}_0$ diagonal : $x^{-h(\phi)-h(\psi)} \big(x^*\big)^{-\bar{h}(\phi)-\bar{h}(\psi)}$

L_0 , $\overline{L}_0$ have Jordan blocks : nilpotent part gives ln(x)

. . . bulk CFT

Data

$\mathcal{F}$, the space of states, a Vir$\oplus$Vir-module

$M : \mathbb{C}^{\times} \times \mathcal{F} \times \mathcal{F} \to \overline{\mathcal{F}}$, the bulk OPE

Ω^*, the out vacuum

Conditions (for theory on complex plane):

- Existence of correlators which are coinvariants and consistent with OPE
- Non-degeneracy of 2-pt correlator

. . . bulk CFT

Non-degeneracy of 2-pt correlator : Let

$$F_0 = \{ \phi \in F \mid \langle \phi(x)\, \psi(0) \rangle = 0 \text{ for all } \psi \in F \}$$

then

- F_0 is independent of x
- F_0 is an ideal under OPE
- every correlator vanishes if a single field is from F_0

Can replace F by F / F_0 .

Modular invariance

So far : CFT on complex plane

Can demand : CFT well-defined on a torus.

- 1-point amplitude on torus

$$Z(\,\phi\,;\,\tau\,) := \mathrm{tr}_F\Big(\,e^{2\pi i\tau(L_0-c/24)}e^{-2\pi i\tau^*(\bar{L}_0-\bar{c}/24)}\,\phi(0)\Big)$$

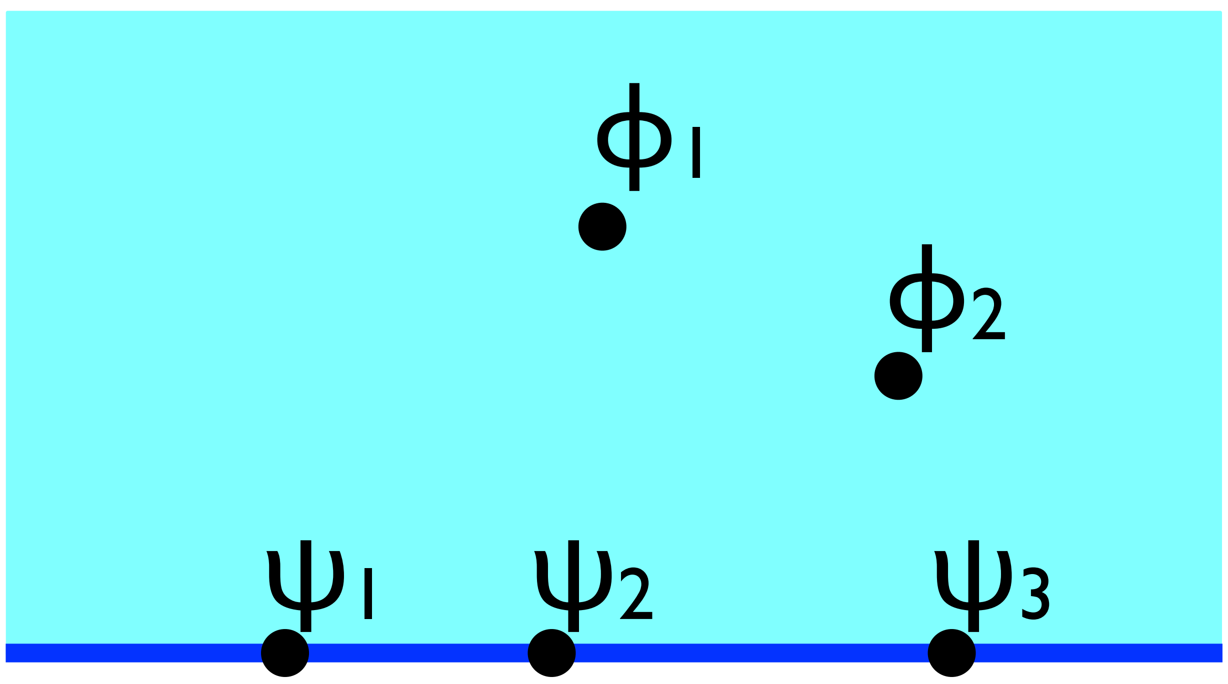
- modular invariance

$$Z(\,\phi\,;\,\tau\,) = Z(\,\phi\,;\,\tau+1) = Z(\,\phi'\,;\,-1/\tau)$$

Rule of thumb :

modular invariant bulk CFTs are all `equally big'.

Boundary CFT



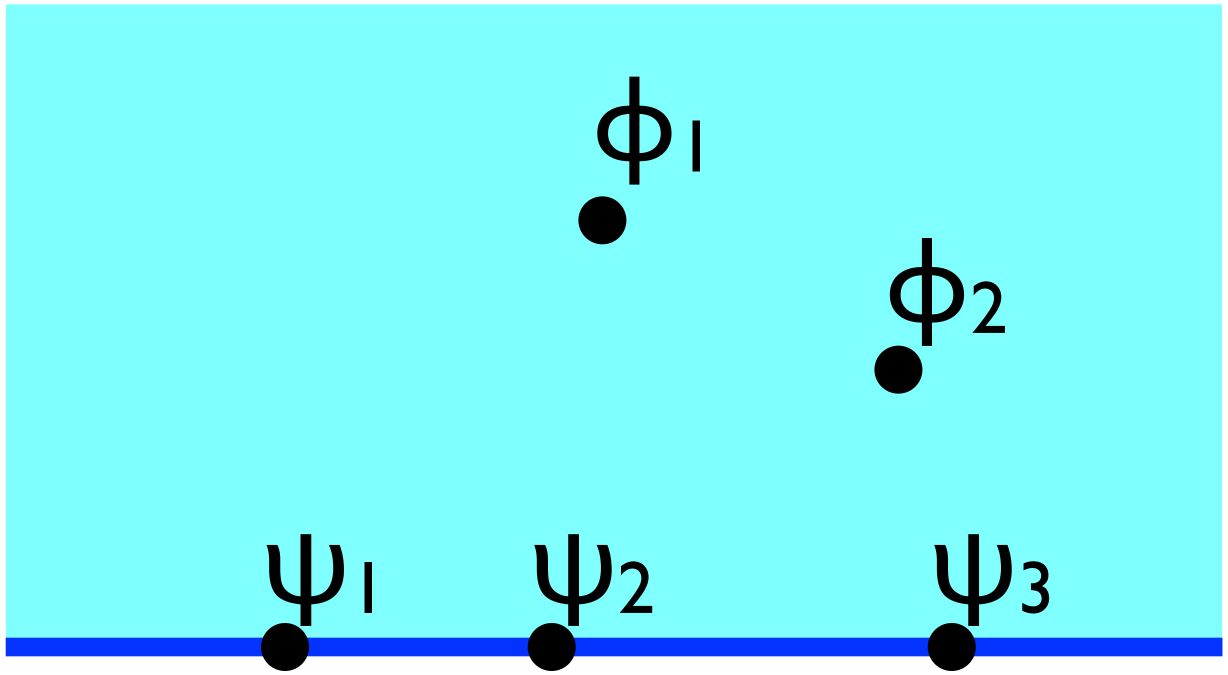
F - space of bulk fields , $\phi_1, \ldots \in F$

B - space of boundary fields, $\psi_1, \ldots \in B$

Correlators on upper half plane

$$\langle\, \phi_1(z_1)\ \phi_2(z_2) \ldots \psi_1(x_1)\ \psi_2(x_2) \ldots \rangle$$

. . . boundary CFT



Data:

(F, M, Ω^*) , a bulk CFT

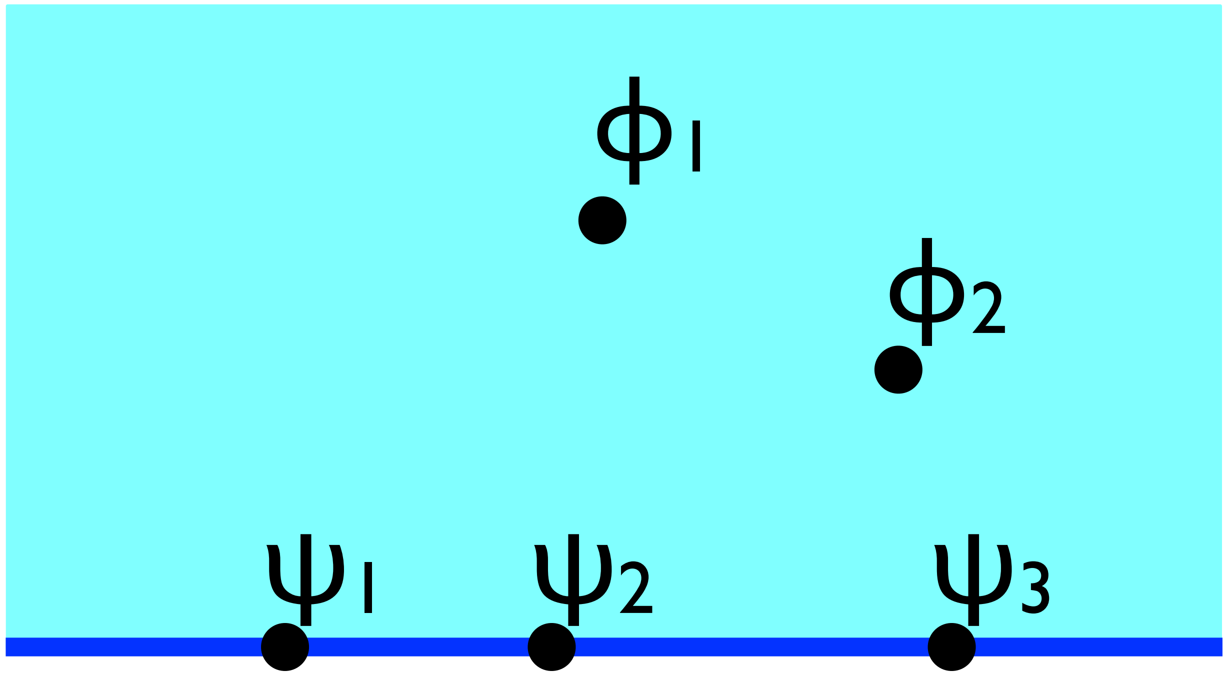
B , boundary fields (a Vir-module)

$\beta_y : F \rightarrow \overline{B}$, bulk-boundary OPE

$m_x : B \times B \rightarrow \overline{B}$, boundary OPE

$\omega^* : B \rightarrow \mathbb{C}$, out-vacuum on upper half plane

. . . boundary CFT



Conditions (for theory on upper half plane):

- Existence of correlators which are coinvariants and consistent with all three OPEs
- Non-degeneracy of 2-pt correlator

. . . boundary CFT

The basic class of examples : Virasoro minimal models

R_i : finite set of irreducible Vir-modules ($i \in I$)

R_0 : vacuum module - a vertex operator algebra

Cardy case:

$F = \bigoplus_i R_i \otimes \overline{R}_i^*$ (* not necessary in Vir)

$B = R_0$

Other choices of B are $U \otimes_f U^*$

(U is R_0-module, $\otimes_f$ is fusion tensor product)

Note: $R_0 \otimes \overline{R}_0^* \subset F$ is subtheory (closed under OPE), but it is not modular invariant.

... boundary CFT

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R_i : finite set of irreducible Vir-modules ($i \in I$)

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Cardy case:

$F = \bigoplus_i R_i \otimes \overline{R}_i^*$ (* not necessary in Vir) ← big

$B = R_0$ ← small

Other choices of B are $U \otimes_f U^*$

(U is R_0-module, $\otimes_f$ is fusion tensor product)

Note: $R_0 \otimes \overline{R}_0^* \subset F$ is subtheory (closed under OPE), but it is not modular invariant.

From boundary to bulk

Fuchs, Schweigert, IR '02
Gaberdiel, IR '07

Call (B, m, ω^*) a *theory on the boundary.*

Aim: Try to construct (F, M, Ω^*) starting from an 'easy' theory on the boundary.

Idea: Take the `biggest space' F which fits to theory on the boundary.

. . . from boundary to bulk

Fix a theory on the boundary (B, m, ω^*).

Consider pairs (F,' β') such that

0) F' is Vir⊕Vir module (later : V⊗V-module for V a VOA)

1) $\beta'_y : F \to \overline{B}$ satisfies coinv. condition (for Vir, later for V)

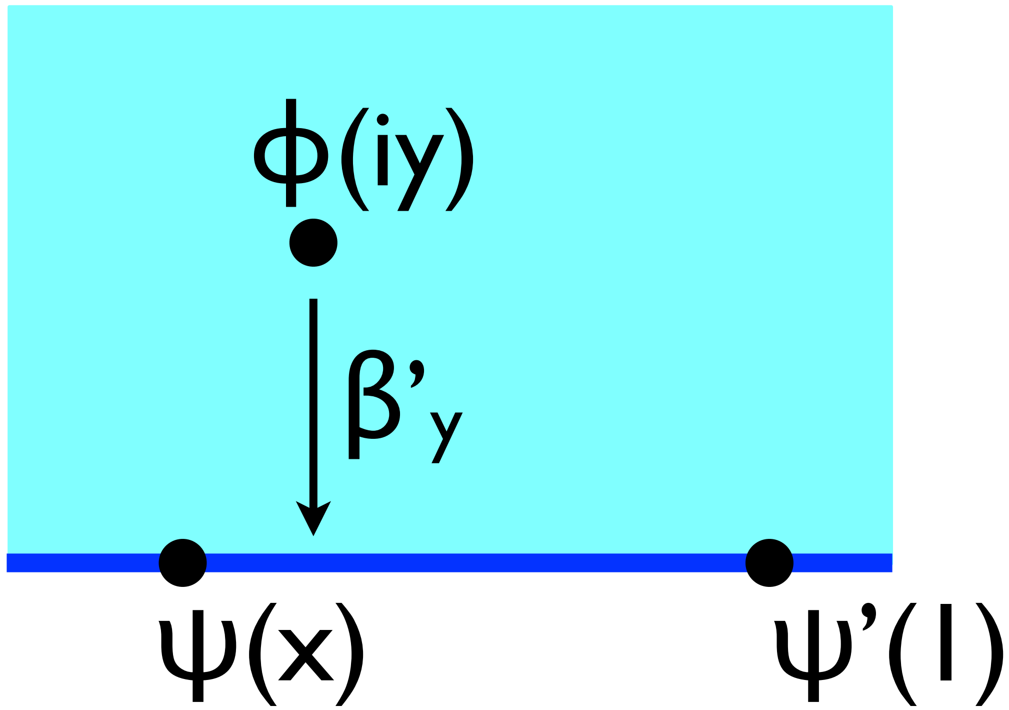
2) β' is central

. . . from boundary to bulk

Lewellen '92

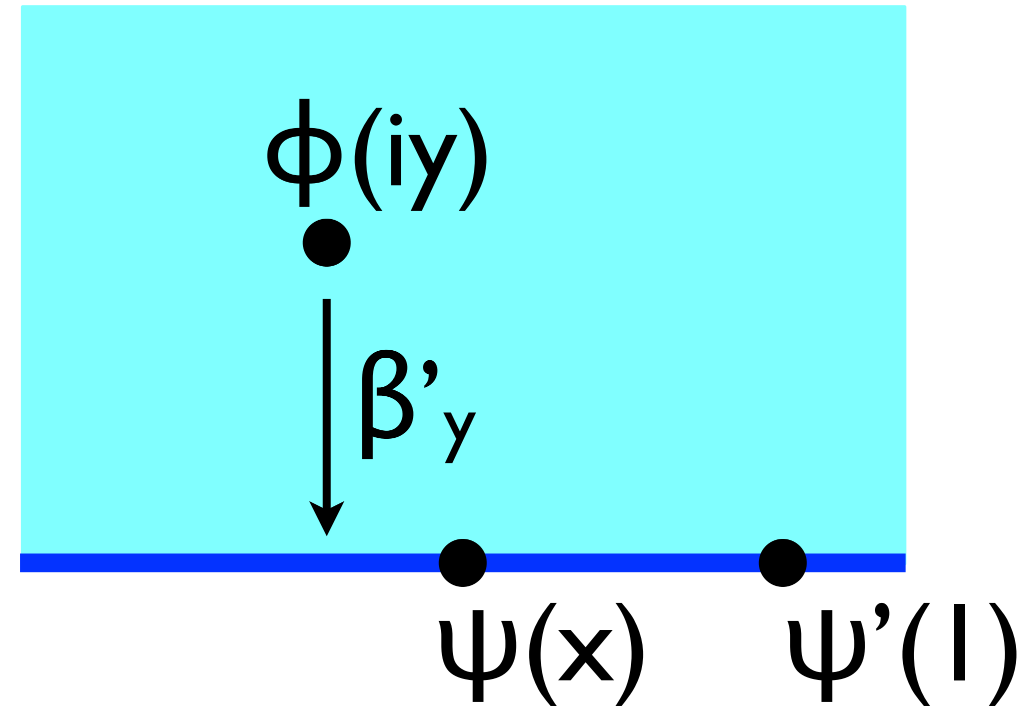
For $x<0$:

$L(x) =$



For $x>0$:

$R(x) =$

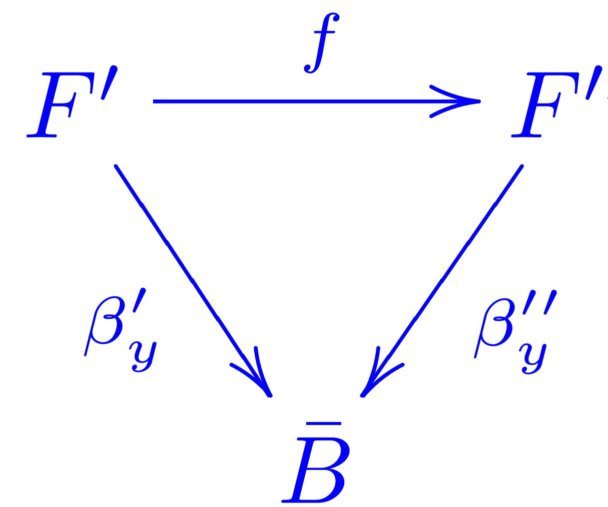


β' is *central* if for all $\psi,\psi' \in B$, $\phi \in F'$ and $y>0$:

$$\lim_{x \nearrow 0} L(x) = \lim_{x \searrow 0} R(x)$$

... from boundary to bulk

A morphism of pairs (F', β')→(F'', β'') is a Vir⊕Vir intertwiner f : F'→F'' such that the diagram on the right commutes.



We make the ansatz that the space of bulk fields (F(B), β(B)) induced by the theory B on the boundary is the terminal object in the category of such pairs.

If it exists, (F(B), β(B)) is unique up to unique isomorphism.

. . . from boundary to bulk

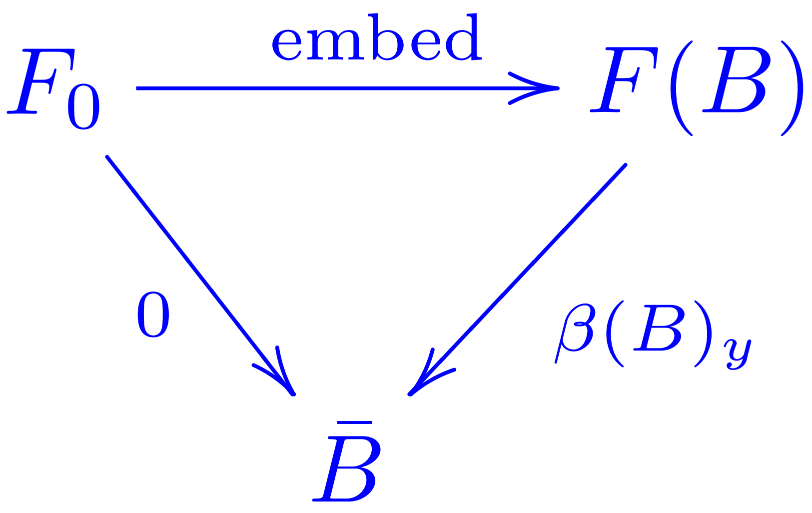
Remark: $\beta(B)_y : F(B) \to \bar{B}$ is injective

Let F_0 be its kernel. By coinv. condition, F_0 is Vir⊕Vir module.

Then the embedding $F_0 \subset F$ is arrow in the category of pairs.

But also $0 : F_0 \to F$ is an arrow.

By uniqueness, the embedding map of the kernel is 0.



. . . from boundary to bulk

Let (F’, β’) be another pair with β’ injective.
Then F’→F(B) is injective.

In this sense F(B) is *maximal* space of bulk fields that fits to (B, m, ω^*)

What about M and Ω^* in data (F, M, Ω^*) for bulk theory?
Is the resulting bulk theory modular invariant?

→ algebraic reformulation

Algebra in braided monoidal categories

Let V be a VOA such that Huang-Lepowski-Zhang tensor product theory applies. Then $C = \mathrm{Rep}\, V$ is a $\mathbb{C}$-linear abelian braided monoidal category.

Work with category C (not necessarily Rep V) such that
(automatic for C=RepV?)

- braided abelian monoidal $\mathbb{C}$-linear
- right exact tensor product
- finite # of isocl. of simple obj., finite dim. Hom
- simple objects have projective covers
- $(-)^*$ involutive functor together with a natural iso.
  $\mathrm{Hom}(A, B) \rightarrow \mathrm{Hom}(A \otimes B^*, I^*)$

. . . algebra in braided monoidal categories

Aside: For $C = \text{Rep}\, V$,

$(-)^*$ is contragredient representation

$\text{Hom}(A, B) \to \text{Hom}(A \otimes B^*, I^*)$ comes from isomorphism of 3pt blocks on P^2 :

- A at 0 , V at x , B^* at ∞
- A at 0 , V at ∞ , B^* at x

. . . algebra in braided monoidal categories

Do *not* assume (all properties below fail in W_{23})

- semi-simple
- tensor unit simple
- rigid
- exact tensor product

Fuchs, Schweigert, IR '02
Kong '06
Gaberdiel, IR '07

. . . algebra in braided monoidal categories

Translate theory on boundary (B, m, ω^*) to C :

- $B \in C$,
- $m : B \otimes B \rightarrow B$ associative product,
- $\varepsilon : B \rightarrow I^*$ such that $\varepsilon \circ m$ is a non-degenerate pairing

  (its preimage under $\mathrm{Hom}(B , B^*) \rightarrow \mathrm{Hom}(B \otimes B , I^*)$
  is an isomorphism)
- may demand in addition choice of unit $\eta : I \rightarrow B$

  (C = Rep V : boundary condition respects V-symmetry)

. . . algebra in braided monoidal categories

Translate pairs (F', β') to C :

Notation:

- $\overline{C}$ is C with inverse braiding
- $C \boxtimes \overline{C}$ is Deligne product
- $T : C \boxtimes \overline{C} \to C$ is induced by tensor product
  (takes $U \boxtimes V$ to $U \otimes V$ - uses right exactness of $\otimes$)

Then $F' \in C \boxtimes \overline{C}$, $\beta' : T(F') \to B$.

Kong '06
Gaberdiel, IR '07

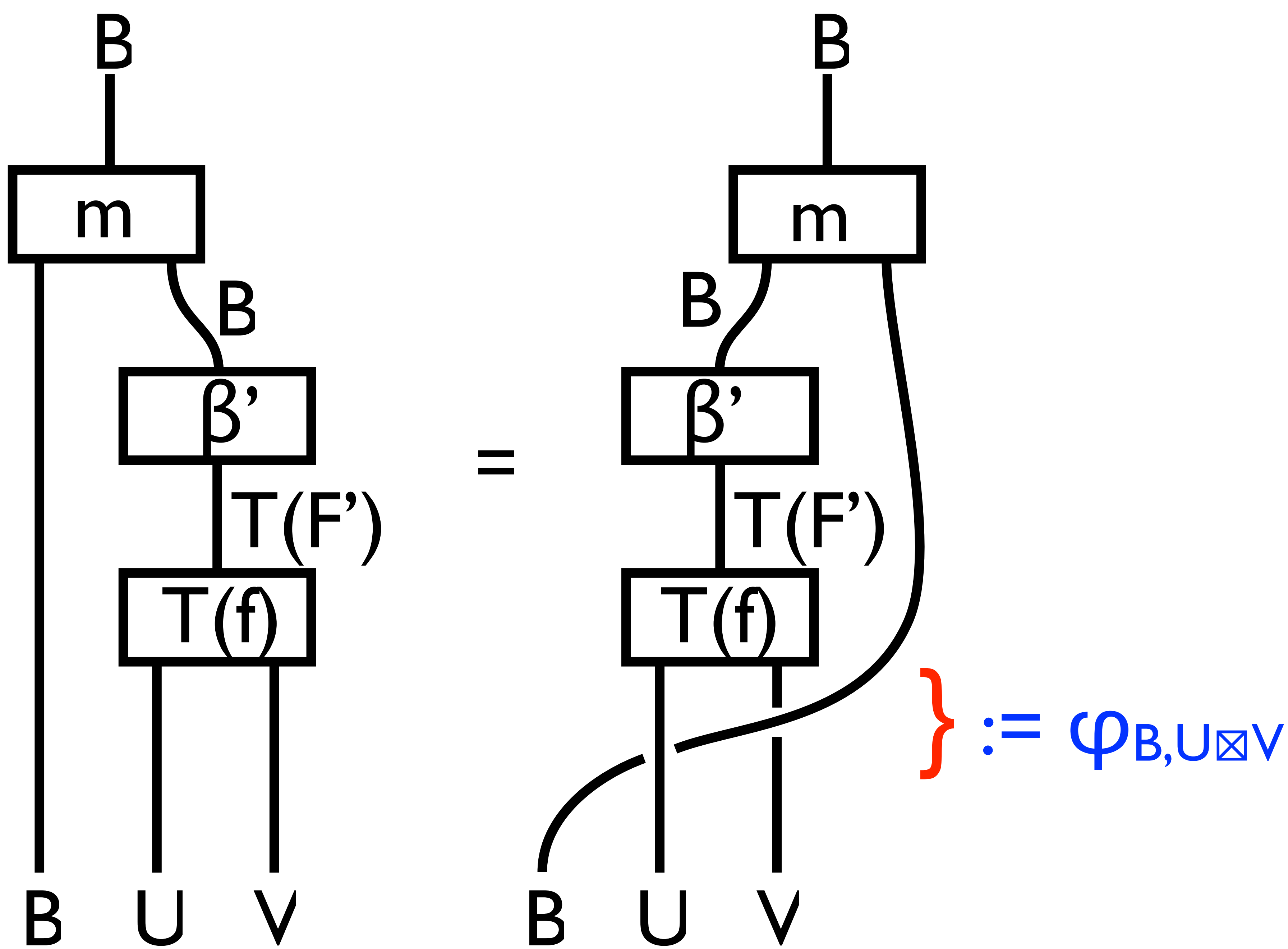
... algebra in braided monoidal categories

Translate centrality condition for $\beta' : T(F') \to B$:

For all

$f : U \boxtimes V \to F'$

have



Fjelstad, Fuchs, Schweigert, IR '06
Davydov '09

. . . algebra in braided monoidal categories

Def: Let A be an algebra in C.

The *full centre of* A *in* C⊠$\overline{\text{C}}$ is a pair (Z(A) , z), which is terminal among pairs (Y∈C⊠$\overline{\text{C}}$, u:T(Y)→A) such that

$$
\begin{array}{ccccc}
T(Y)\otimes A & \xrightarrow{u\otimes id} & A\otimes A & & \\
\Big\downarrow{\scriptstyle \varphi_{T(Y),A}} & & & \searrow^{m} & \\
 & & & & A \\
 & & & \nearrow_{m} & \\
A\otimes T(Y) & \xrightarrow{id\otimes u} & A\otimes A & &
\end{array}
$$

commutes.

... algebra in braided monoidal categories

Thm:

- $Z(A)$ exists.
- There exists a unique algebra structure on $Z(A)$ such that $z : T(Z(A)) \to A$ is an algebra map.
- With this algebra structure, $Z(A)$ is commutative. If A is unital, so is $Z(A)$.

Non-logarithmic rational CFT

Huang '05

$\mathcal{C} = \mathrm{Rep}\,V$ is a modular category.

Fuchs, Schweigert, IR '02
Fjelstad, Fuchs, Schweigert, IR '06
Kong, IR '08

Theory on the boundary is a symmetric Frobenius algebra A in $\mathcal{C}$.

The bulk theory is the full centre $Z(A)$, a commutative symmetric Frobenius algebra in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$.

$Z(A)$ is modular invariant.

E.g. : $Z(1) = \sum_i U_i \boxtimes U_i^*$

. . . non-logarithmic rational CFT

Thm:

All modular invariant commutative Frobenius algebras Z with $\dim \mathrm{Hom}(1,Z)=1$ are given by $Z(A)$ for some Frobenius algebra A.

Furthermore, $Z(A) = Z(B)$ iff A and B are Morita equivalent.

Kausch '91, Gaberdiel, Kausch '96
Fuchs, Hwang, Semikhatov, Tipunin '03
Adamovic, Milas '07
. . .

W_{1p} models

$C = \mathrm{Rep}\, V$ is (conjecturally) a finite tensor category.

Quella, Schomerus '07
Gaberdiel, IR '07

- $Z(1) = \sum_i P_i \boxtimes P_i^* / N$
- as $\mathbb{R} \times \mathbb{R}$ graded vector space have $Z(1) = \sum_i U_i \boxtimes P_i^*$
  (graded by generalised L_0 and $\overline{L}_0$ eigenspaces)
- $Z(1)$ gives a modular invariant torus partition function.

The W_{23} model

Feigin, Gainutdinov, Semikhatov, Tipunin '06

- Virasoro Verma module for h=0 and c=0: two independent null vectors

$$N_1 = L_{-1}\Omega \qquad\qquad N_2 = (L_{-2} - \tfrac{3}{2}L_{-1}L_{-1})\Omega$$

- Divide by N_1 and N_2 : get

$$\mathcal{V}(0) = \mathbb{C} \cdot \Omega$$

- Divide by N_1 but <u>not</u> by N_2 : get quasi-rational, but not rational theory

. . . the W_{23} model

- Extend by 3 fields with h=15 , get VOA W

- W is indecomposable but not irreducible

$$0 \longrightarrow \mathcal{W}(2) \longrightarrow \mathcal{W} \longrightarrow \mathcal{W}(0) \longrightarrow 0$$

$\mathcal{W}(2)$: irreducible sub-representation

$\mathcal{W}(0)$: irreducible quotient

C = Rep V supposedly

- abelian, braided, ⊗ right exact, $(-)^*$, finiteness...

but not

- semi-simple, rigid, simple 1, ⊗ exact

. . . the W_{23} model

Feigin, Gainutdinov, Semikhatov, Tipunin '06
Adamovic, Milas '09

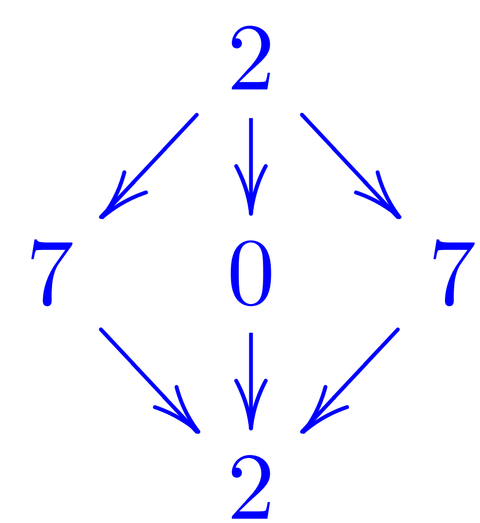
Irreducible representations are:

	$s = 1$	$s = 2$	$s = 3$
$r = 1$	$0,\ 2,\ 7$	$0,\ 1,\ 5$	$\frac{1}{3},\ \frac{10}{3}$
$r = 2$	$\frac{5}{8},\ \frac{33}{8}$	$\frac{1}{8},\ \frac{21}{8}$	$-\frac{1}{24},\ \frac{35}{24}$

Gaberdiel, Wood, IR '09

The tensor unit W does not have a non-degenerate pairing. The simplest theory on the boundary is:

B := W(5/8) $\otimes_f$ W(5/8) =



. . . the W_{23} model

Gaberdiel, Wood, IR '10

- We could not compute Z(B) for this algebra. Instead, we compute (without full proof) $Z(W^*)$ for the commutative algebra W^*.

- $Z(W^*)$ is (conjecturally) a commutative algebra. But it has no unit (just as W^*).

- As $\mathbb{R} \times \mathbb{R}$ graded vector space: $Z(W^*) = \sum_i U_i \boxtimes P_i^*$

  Gives modular invariant partition function, can be expressed (up to a constant) via c=1 free boson partition functions (Pearce, Rasmussen '10) and has appeared in context of dilute polymers (Saleur '91) .

Composition series of Z(W*) in integer weight-sector

	0	1	2	5	7
0	1				
1		1			
2			1		
5				1	
7					1

	0	1	2	5	7
0		1	1		
1	1			2	2
2	1			2	2
5		2	2		
7		2	2		

	0	1	2	5	7
0	1			2	2
1		2	4		
2		4	2		
5	2			2	4
7	2			4	2

	0	1	2	5	7
0		1	1		
1	1			2	2
2	1			2	2
5		2	2		
7		2	2		

	0	1	2	5	7
0	1				
1		1			
2			1		
5				1	
7					1

.

Composition series of Z(W*) in integer weight-sector

L_0

η

	0	1	2	5	7
0	1				
1		1			
2			1		
5				1	
7					1

$\longrightarrow$

	0	1	2	5	7
0		1	1		
1	1			2	2
2	1			2	2
5		2	2		
7		2	2		

$\longrightarrow$

ω

	0	1	2	5	7
0	1			2	2
1		2	4		
2		4	2		
5	2			2	4
7	2			4	2

L_0

	0	1	2	5	7
0		1	1		
1	1			2	2
2	1			2	2
5		2	2		
7		2	2		

$\longrightarrow$

Ω

	0	1	2	5	7
0	1				
1		1			
2			1		
5				1	
7					1

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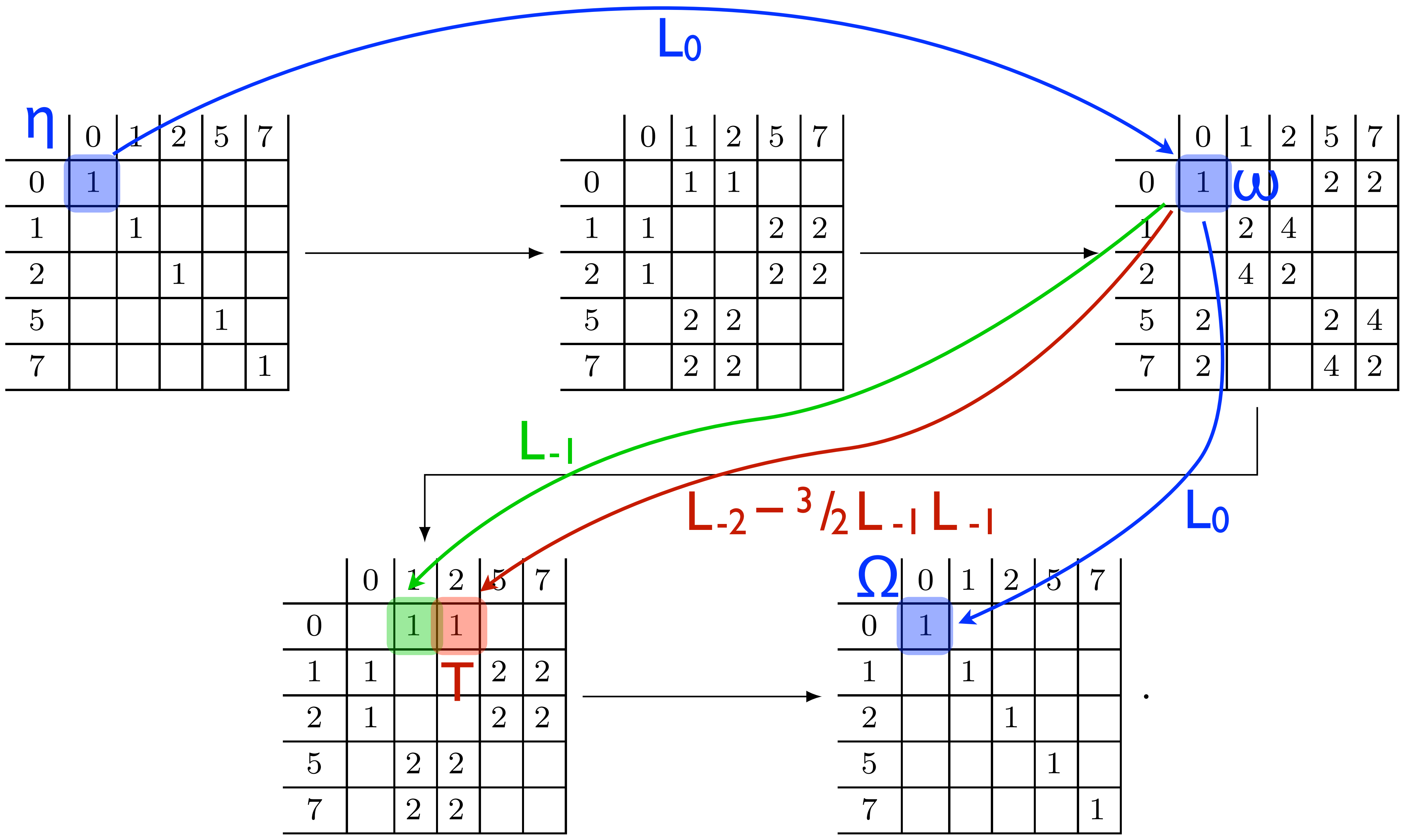
Composition series of Z(W*) in integer weight-sector



	0	1	2	5	7
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1		1			
2			1		
5				1	
7					1

	0	1	2	5	7
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1	1			2	2
2	1			2	2
5		2	2		
7		2	2		

	0	1	2	5	7
0	1			2	2
1		2	4		
2		4	2		
5	2			2	4
7	2			4	2

	0	1	2	5	7
0		1	1		
1	1			2	2
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7		2	2		

	0	1	2	5	7
0	1				
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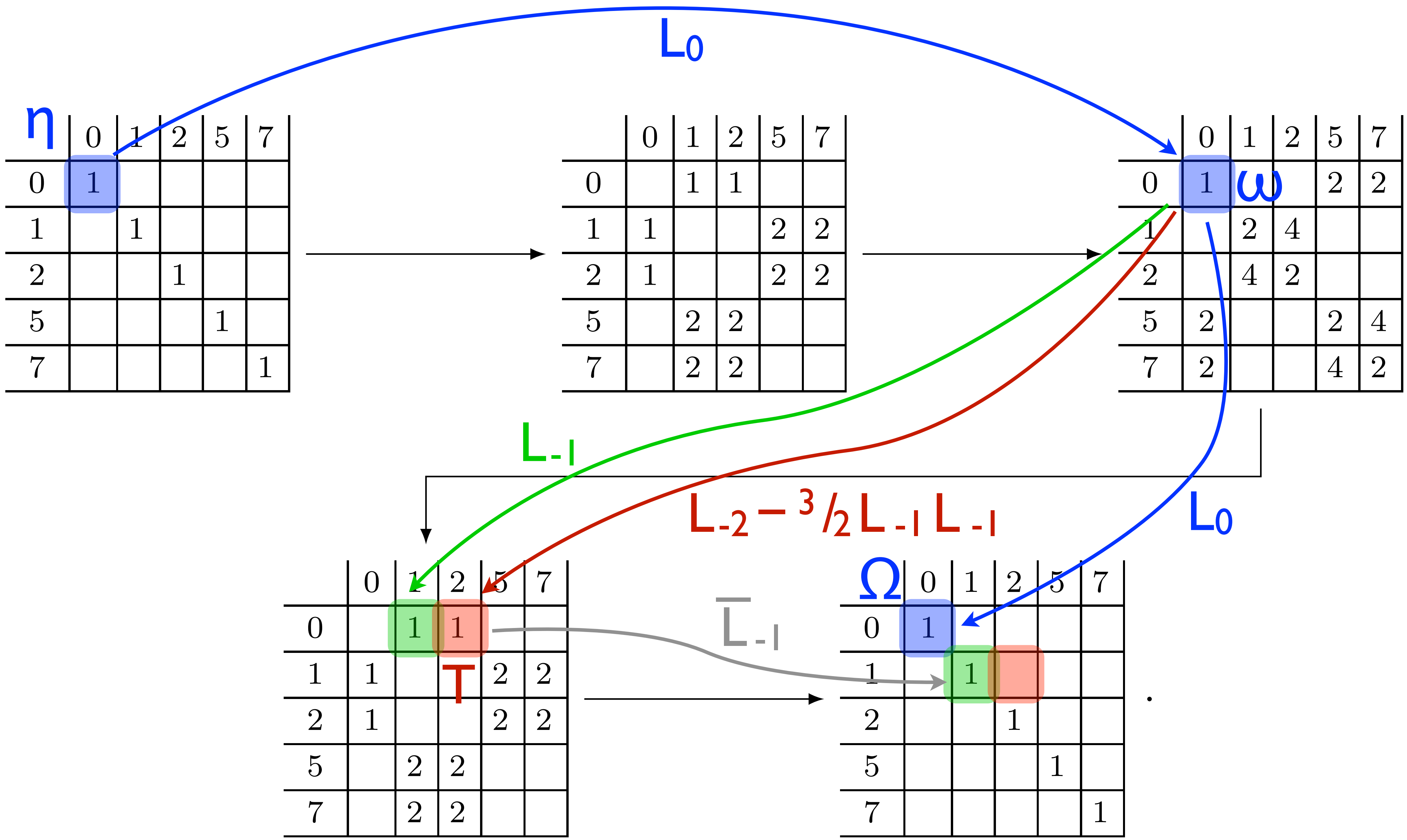
2			1		
5				1	
7					1

.

Composition series of Z(W*) in integer weight-sector



	0	1	2	5	7
0	1				
1		1			
2			1		
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7					1

	0	1	2	5	7
0		1	1		
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7		2	2		

	0	1	2	5	7
0	1			2	2
1		2	4		
2		4	2		
5	2			2	4
7	2			4	2

	0	1	2	5	7
0		1	1		
1	1			2	2
2	1			2	2
5		2	2		
7		2	2		

	0	1	2	5	7
0	1				
1		1			
2			1		
5				1	
7					1

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Composition series of Z(W*) in integer weight-sector

η

	0	1	2	5	7
0	1				
1		1			
2			1		
5				1	
7					1

L_0

	0	1	2	5	7
0		1	1		
1	1			2	2
2	1			2	2
5		2	2		
7		2	2		

ω

	0	1	2	5	7
0	1			2	2
1		2	4		
2		4	2		
5	2			2	4
7	2			4	2

$L_{-2} - {}^3/_2 L_{-1} L_{-1}$

L_0

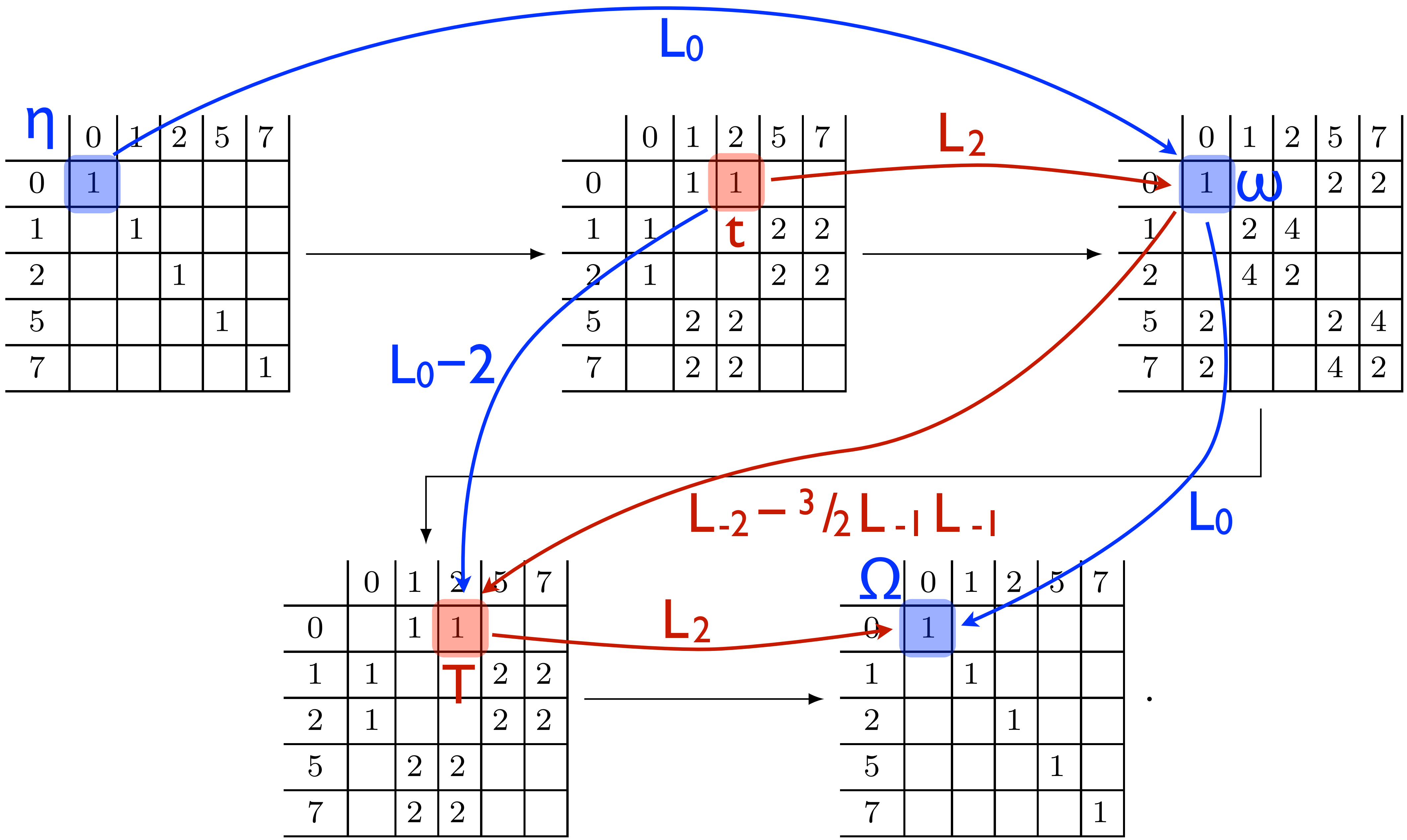
T

	0	1	2	5	7
0		1	1		
1	1			2	2
2	1			2	2
5		2	2		
7		2	2		

Ω

	0	1	2	5	7
0	1				
1		1			
2			1		
5				1	
7					1	.

Composition series of $Z(W^*)$ in integer weight-sector



	0	1	2	5	7
0	1				
1		1			
2			1		
5				1	
7					1

→

	0	1	2	5	7
0		1	1		
1	1			2	2
2	1			2	2
5		2	2		
7		2	2		

→

	0	1	2	5	7
0	1			2	2
1		2	4		
2		4	2		
5	2			2	4
7	2			4	2

→

	0	1	2	5	7
0		1	1		
1	1			2	2
2	1			2	2
5		2	2		
7		2	2		

→

	0	1	2	5	7
0	1				
1		1			
2			1		
5				1	
7					1

.

OPEs: $\Omega\,\Omega \sim 0$ (!) $\quad \eta\,\Omega \sim \Omega \quad T\,T \sim 0$ (!)

$$t\,T \sim z^{-4} L_2 T + z^{-2} L_0 T + z^{-1} L_{-1} T$$

So that's your theory?

- $Z\left(\begin{array}{ccc} & 2 & \\ \swarrow & \downarrow & \searrow \\ 7 & 0 & 7 \\ \searrow & \downarrow & \swarrow \\ & 2 & \end{array}\right)$ will have better properties, in particular an embedding of $W \boxtimes W$ (and a vacuum and a stress tensor). Please compute it for us.

- $Z(W^*)$ as some properties not seen before, e.g. an OPE preserving projection

  $$Z(W^*) \to Z(W(0)) = W(0) \boxtimes W(0)$$

  All correlators transform under conformal maps, yet the theory has no stress tensor.

- The space of modular invariant bilinear combinations of the 13 characters of irreducibles is 2-dimensional. It is spanned by the characters of $Z(W^*)$ and $Z(W(0))$.