IHP 2011

Properties of supergroup sigma-models and string theory

work done in collaboration with
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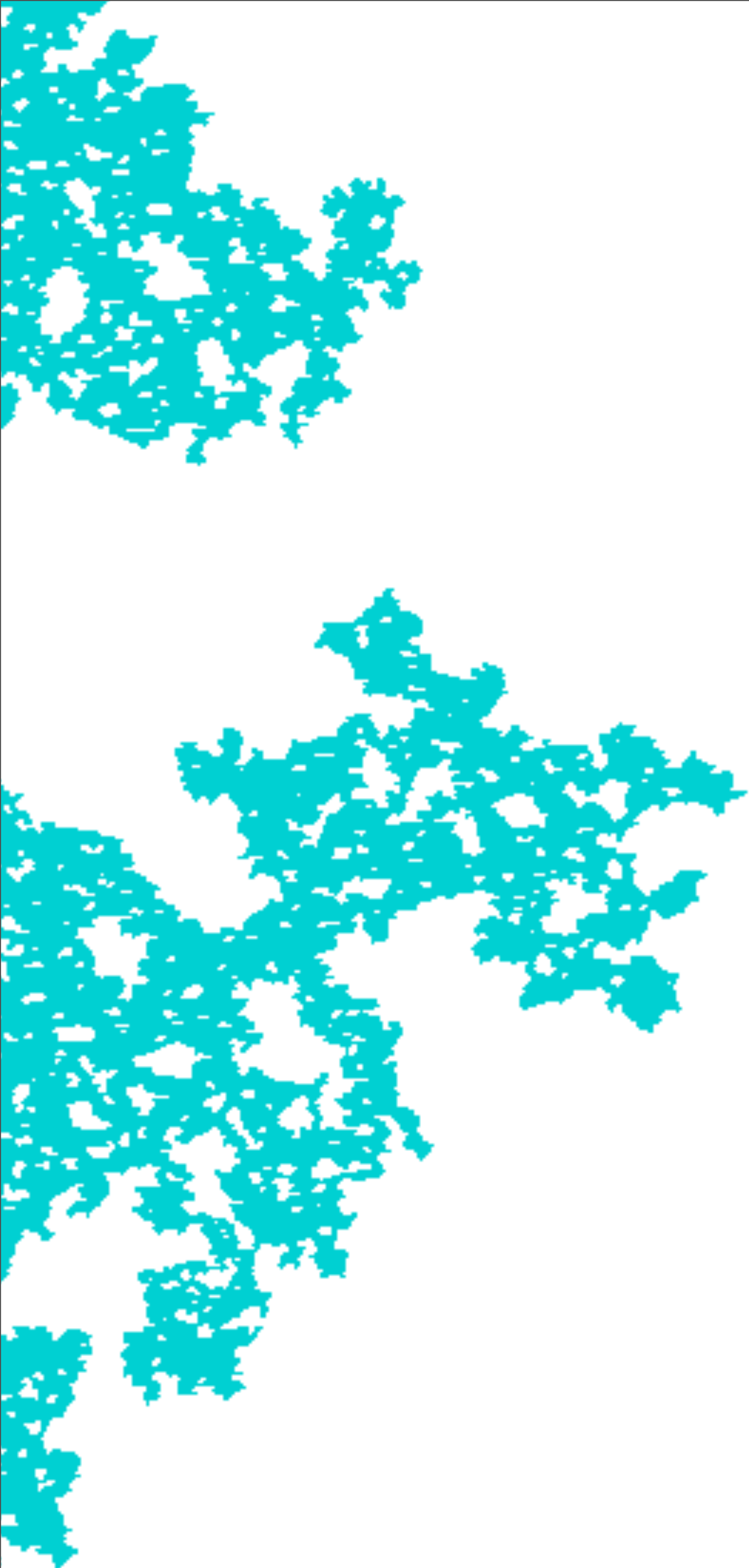


based on work by

Babichenko, Berkovits, Bernard, Bershadsky, Candu, Creutzig, Drouot, Gaberdiel, Gerigk, Germoni, Goetz, Konechny, LeClair, Quella, Read, Roehne, Saleur, Schomerus, Vafa, Vaintrob, Witten, Zhukov, ..

and all other members of the audience

Plan

0. Motivation

1. Properties of supergroup sigma-models

   Current algebra, Integrability, Spectrum

2. Properties when embedded in string theory

   Asymptotic Symmetry Algebra, Physical States

0. Motivation

Physics

Quantum Gravity

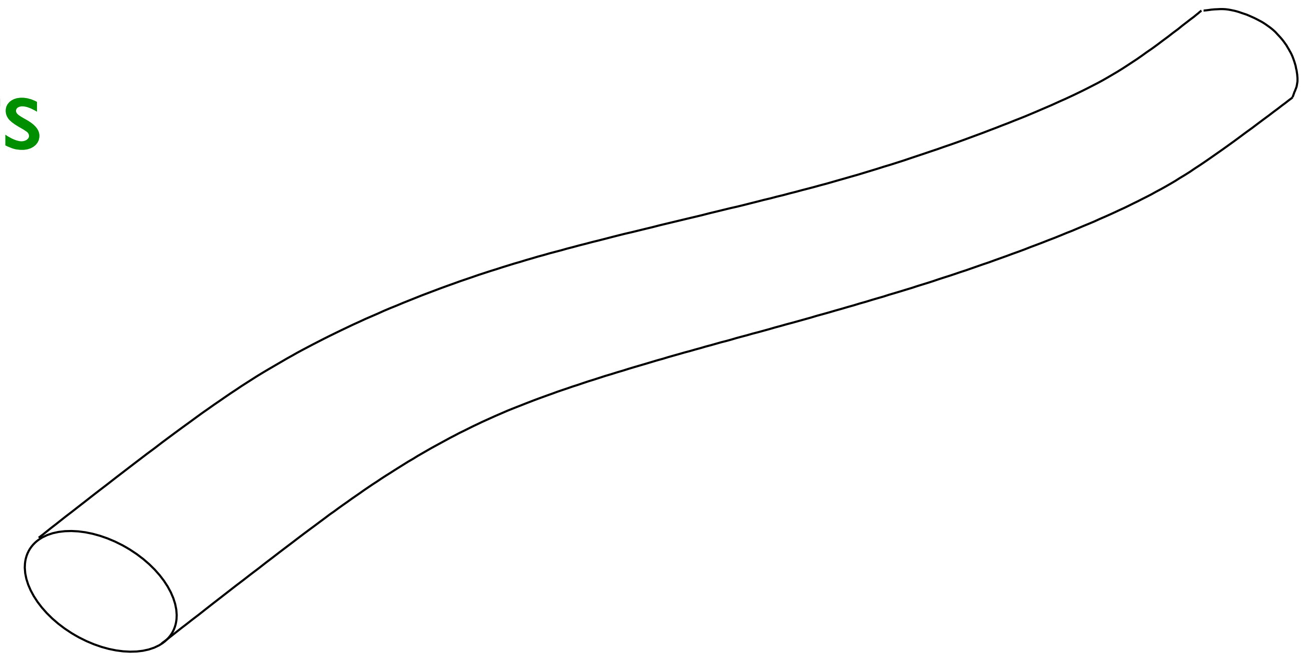
Strongly Coupled Field Theories

Holography

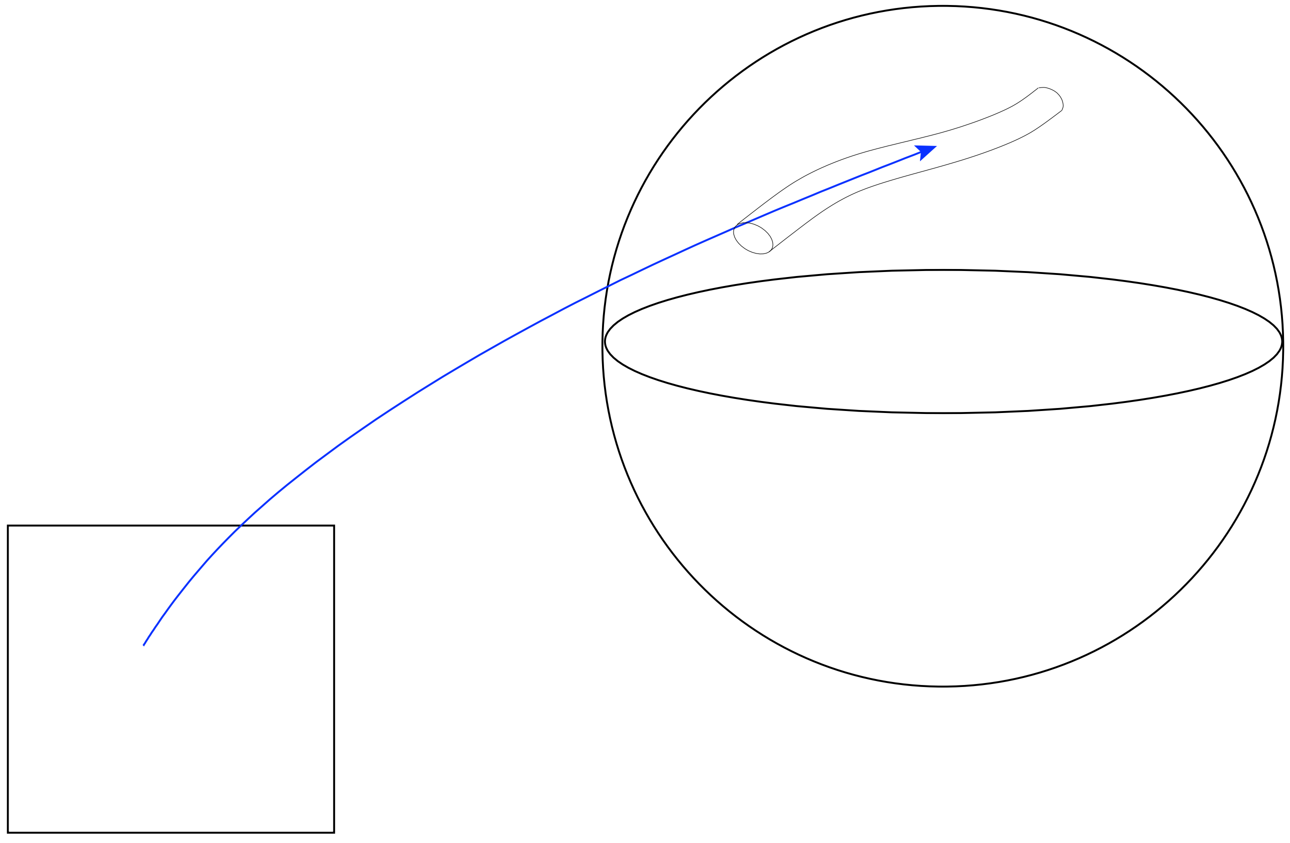
String Theory

in Anti-deSitter

Strings



correspond to Maps from Worldsheet into Space-time

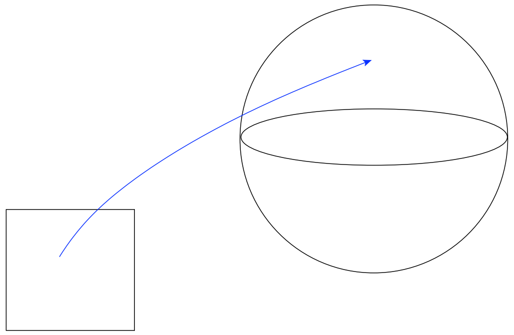


1. Properties of Supergroup Sigma-Models

1a. The Model

1b. The Current Algebra

1c. The Energy-momentum tensor and the Spectrum



Ia : The Model

$$g : \Sigma \quad \longrightarrow \quad G$$

Supergroup

The action :

$$S_{PCM+WZ} \quad = \quad \frac{1}{f^2}\int_\Sigma \langle dgg^{-1}, *dgg^{-1}\rangle + k\int_B \langle dgg^{-1}, (dgg^{-1})^2\rangle$$

Principal chiral model with Wess-Zumino term

Assume the super Lie algebra has zero Killing form

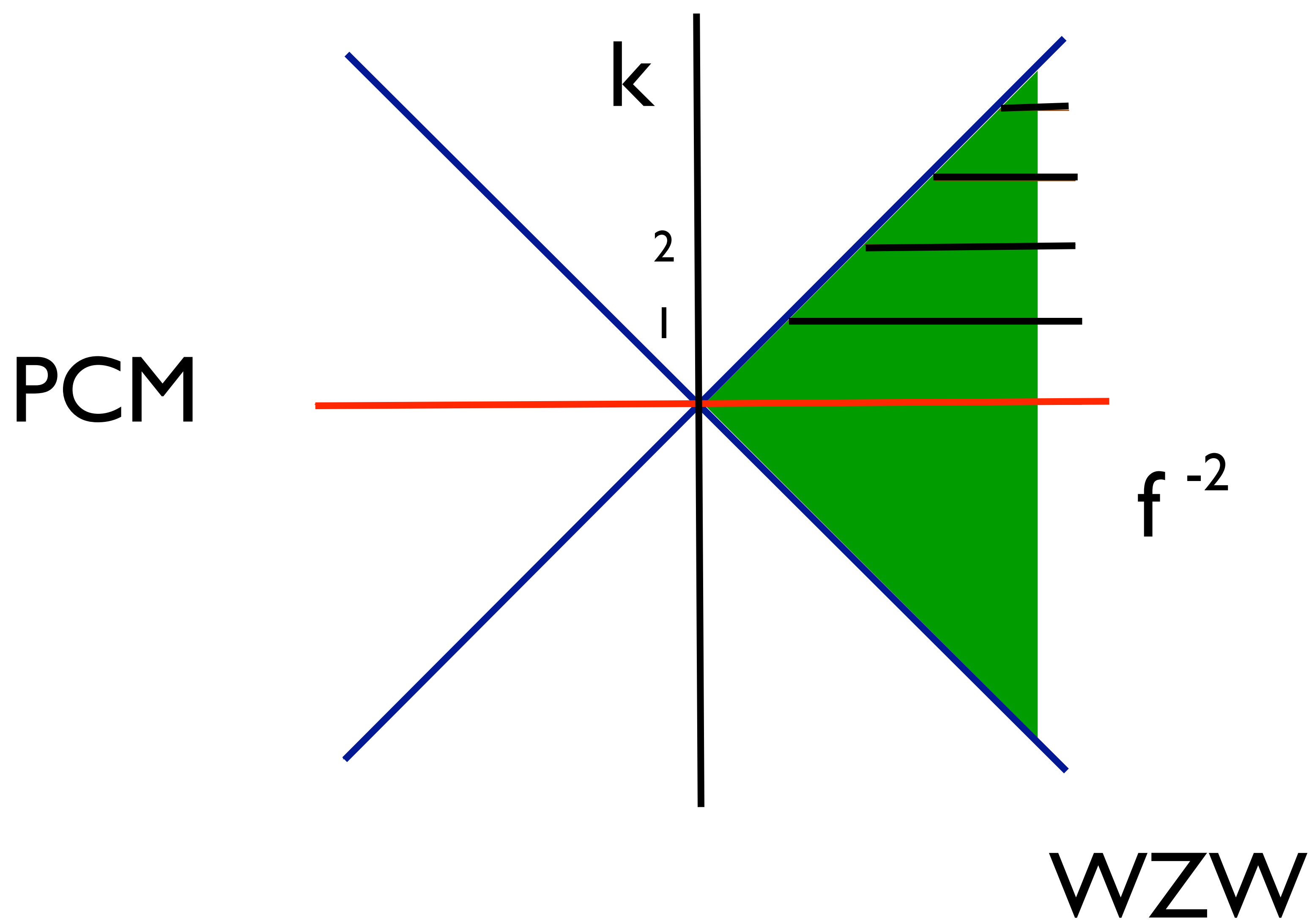
$$f^a{}_{bc} f^{cb}{}_d \quad = \quad 0$$

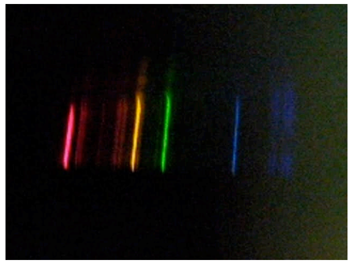
and a non-degenerate invariant metric

$$\kappa_{ab}$$

then we have

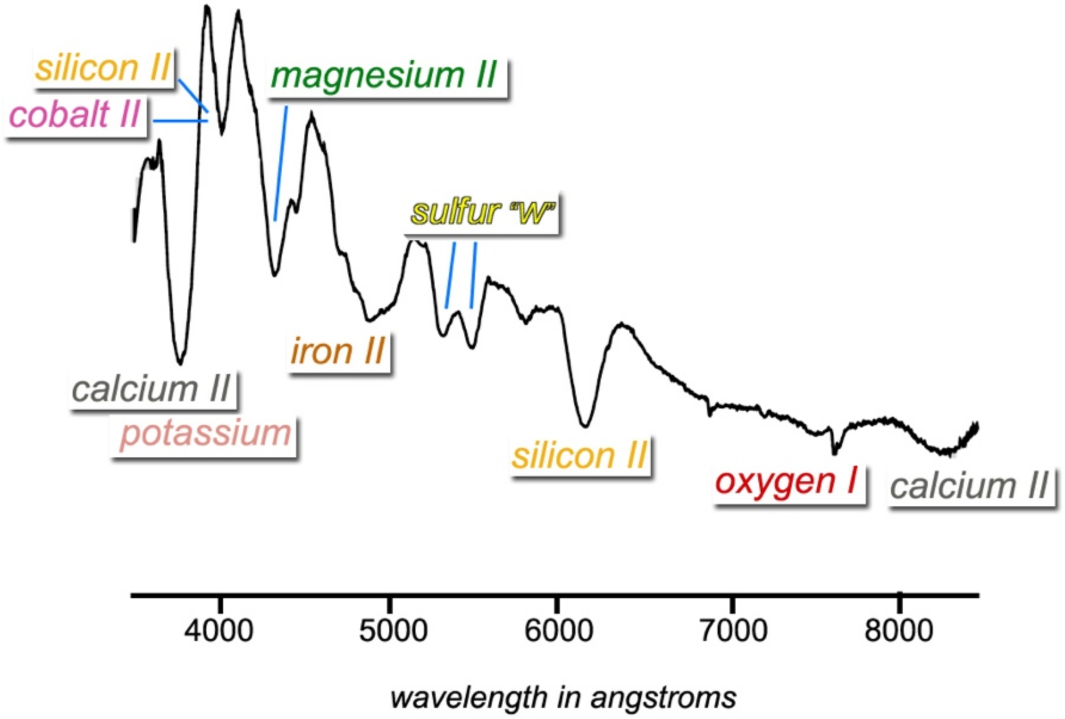
a two-dimensional group invariant moduli space of conformal field theories





a goal :
compute the spectrum

(=open problem)



Ib. Two-dimensional current algebras

Lie algebra valued
Conserved current

in Lorentz invariant
local theory

$$j_\mu = j_\mu^a t_a$$

$$t_a \in \mathrm{Lie}\, G$$

Hypothesis : the operators appearing as coefficients of singularities are the identity, the current, and its derivatives.

The current algebra contains functions of the Lorentz invariant distance between the operator insertions.

Current conservation leads to first order differential equations that can be solved under the assumption of leading powerlike singularities.

Five free parameters at first order in the currents (and zeroth order in the distance).

Eight free parameters at second order in the currents.

$$
\begin{aligned}
j^a_z(z) j^b_z(0) \sim{}& \kappa^{ab}\frac{c_1}{z^2} + f^{ab}{}_c \left[\frac{c_2}{z} j^c_z(0) + (c_2 - g)\frac{\bar z}{z^2} j^c_{\bar z}(0) \right. \\
&\left. - \frac{g}{4}\frac{\bar z}{z}\left(\partial_z j^c_{\bar z}(0) - \partial_{\bar z} j^c_z(0)\right) + \frac{c_2}{2}\partial_z j^c_z(0) + \frac{c_2 - g}{2}\frac{\bar z^2}{z^2}\partial_{\bar z} j^c_{\bar z}(0) \right] + \dots
\end{aligned}
$$

$$
\begin{aligned}
j^a_{\bar z}(x) j^b_{\bar z}(0) \sim{}& \kappa^{ab} c_3 \frac{1}{\bar z^2} + f^{ab}{}_c \left[\frac{c_4}{\bar z} j^c_{\bar z}(0) + \frac{(c_4 - g) z}{\bar z^2} j^c_z(0) \right. \\
&\left. + \frac{g}{4}\frac{z}{\bar z}\left(\partial_z j^c_{\bar z}(0) - \partial_{\bar z} j^c_z(0)\right) + \frac{c_4}{2}\partial_{\bar z} j^c_{\bar z}(0) + \frac{(c_4 - g)}{2}\frac{z^2}{\bar z^2}\partial_z j^c_z(0) \right] + \dots
\end{aligned}
$$

$$
\begin{aligned}
j^a_z(x) j^b_{\bar z}(0) \sim{}& f^{ab}{}_c \left[\frac{(c_4 - g)}{\bar z} j^c_z(0) + \frac{(c_2 - g)}{z} j^c_{\bar z}(0) + \frac{(c_4 - g) z}{\bar z}\partial_z j^c_z(0) \right. \\
&\left. - \left(c_5 + \frac{g}{4}\log \mu^2 x^2\right)\left(\partial_z j^c_{\bar z}(0) - \partial_{\bar z} j^c_z(0)\right) \right] + \dots
\end{aligned}
$$

Non-chiral current algebra with

Logarithmic term

as in Luscher+ allow parity violation

The Toolbox to Fix Coefficients

a) Semi-classical perturbation theory

b) Exactness of low order calculation (superalgebra)

c) Recursion via Maurer-Cartan (integrability)

a) Semi-classical (= large volume) perturbation theory

Develop the action near the constant map into the identity element, in an expansion in Lie algebra valued fields.

The action and the currents are an expansion in the structure constants of the Lie algebra.

Semi-classical perturbation theory is an expansion in the number of structure constants.

So :
$$g = e^{iA^a t_a}$$

$$S = \frac{1}{4\pi f^2}\int d^2z(\partial A^a\bar\partial A_a - \frac{1}{12}f^a{}_{bc}f_{aij}A^b\partial A^c A^i\bar\partial A^j + O(f^4))$$

$$-\frac{k}{12\pi}\int d^2z f^{abc}A_a\partial A_b\bar\partial A_c + O(f^3)$$

The action and the currents are given in terms of an expansion in the structure constants.

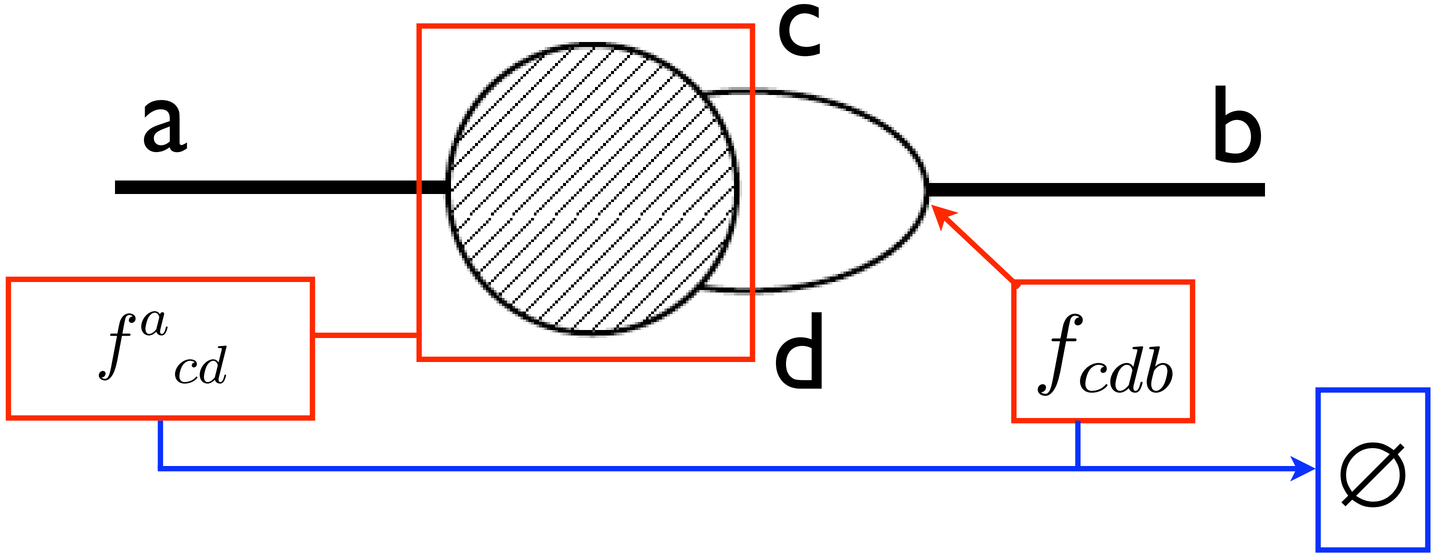
b) Superalgebra

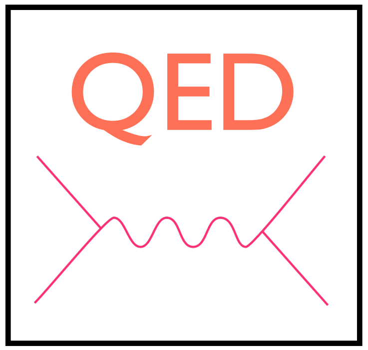
The two- and three-point functions of the currents are exact at zeroth and first order in the structure constants respectively.

Bershadsky,
Zhukov,
Vaintrob

Two-point functions $\langle j_\mu^a(z) j_\nu^b(w) \rangle$

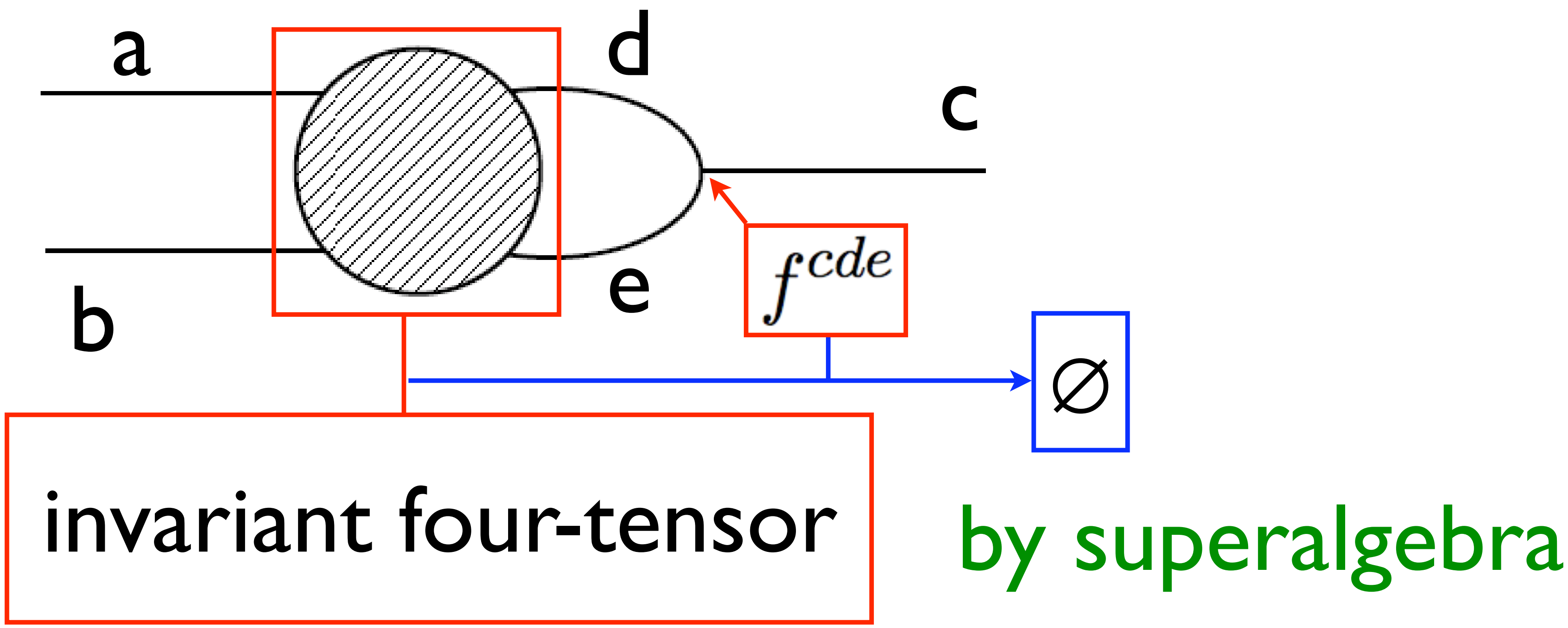
Pull out a structure constant from an arbitrary diagram:





The two-points functions take their free values.

Three-point functions $\langle j^a_\mu(z) j^b_\nu(w) j^c_\rho(y) \rangle$



There are only contributions at first order in the structure constants.

$$j_z^a(z) j_w^b(w) \approx \frac{(1+kf^2)^2 f^2 \kappa^{ab}}{(z-w)^2} + f^2(1+kf^2)\left(\frac{3}{2} - \frac{kf^2}{2}\right) i f^{ab}{}_c \frac{j^c(w)}{z-w} + f^2(1+kf^2)^2 \frac{1}{2} i f^{ab}{}_c \frac{\bar{z}-\bar{w}}{(z-w)^2} j^c_{\bar{w}}(w) + \ldots$$

$$j_{\bar{z}}^a(z) j_{\bar{w}}^b(w) \approx \frac{f^2(1-kf^2)^2 \kappa^{ab}}{(\bar{z}-\bar{w})^2} + f^2(1-kf^2)\left(\frac{3}{2} + \frac{kf^2}{2}\right) i f^{ab}{}_c \frac{j^c_{\bar{w}}(w)}{\bar{z}-\bar{w}} + f^2 \frac{1}{2}(1-kf^2)^2 i f^{ab}{}_c \frac{z-w}{(\bar{z}-\bar{w})^2} j^c(w) + \ldots.$$

$$j_z^a(z) j_{\bar{w}}^b(w) \approx -2\pi f^2(1+kf^2)(1-kf^2)\kappa^{ab}\delta^{(2)}(z-w) + f^2 \frac{1}{2}(1+kf^2)^2 i f^{ab}{}_c \frac{j^c_{\bar{w}}(w)}{z-w}$$
$$+ f^2 \frac{1}{2}(1-kf^2)^2 i f^{ab}{}_c \frac{1}{\bar{z}-\bar{w}} j^c(w) + \ldots$$

5 coefficients fixed exactly by low order perturbation theory

c) Recursion via Maurer-Cartan equation

In the classical theory, the Maurer-Cartan equation reads:

$$d(dgg^{-1}) = dgg^{-1} \wedge dgg^{-1}$$

The Maurer-Cartan equation contains a composite operator which potentially needs UV regularization.

Assume MC validity in the quantum theory (a strong assumption, implying integrability).

By exploiting the MC assumption:

$$\partial j^a_{\bar{z}} - \bar{\partial} j^a_z + i f^2 f^a{}_{bc} : j^b_z j^c_{\bar{z}} : \quad = \quad 0$$

We can recursively compute higher order terms in the current-current operator product expansion.

Result for the current-current OPE up to zeroth order in the distance, and second order in the current operators:

$$
\begin{aligned}
j^a_z(z) j^b_z(0) \sim\ & \kappa^{ab} \frac{c_1}{z^2} + f^{ab}{}_c \left[\frac{c_2}{z} j^c_z(0) + (c_2 - g) \frac{\bar{z}}{z^2} j^c_{\bar{z}}(0) \right] \\
& + f^{ab}{}_c \left[-\frac{g}{4} \frac{\bar{z}}{z} \left(\partial_z j^c_{\bar{z}}(0) - \partial_{\bar{z}} j^c_z(0) \right) + \frac{c_2}{2} \partial_z j^c_z(0) + \frac{c_2 - g}{2} \frac{\bar{z}^2}{z^2} \partial_{\bar{z}} j^c_{\bar{z}}(0) \right] \\
& + : j^a_z j^b_z : (0) \\
& + \underline{A^{ab}{}_{cd}} \frac{1}{2} \frac{\bar{z}^2}{z^2} : j^c_{\bar{z}} j^d_{\bar{z}} : (0) + B^{ab}{}_{cd} \frac{\bar{z}}{z} : j^c_z j^d_{\bar{z}} : (0) - C^{ab}{}_{cd} \log |z|^2 : j^c_z j^e_z : (0) \\
& + \ldots
\end{aligned}
$$

$$
\begin{aligned}
\underline{A^{ab}{}_{cd}} &= -f^2 (1 + k f^2) \frac{1}{2} \left(f^b{}_{cg} f^{ag}{}_d (-1)^{cd} + f^b{}_{dg} f^{ag}{}_c \right) + \mathcal{O}(f^4) \\
B^{ab}{}_{cd} &= f^2 (1 - k^2 f^4) \left(f^b{}_{cg} f^{ag}{}_d (-1)^{cd} + f^b{}_{dg} f^{ag}{}_c \right) + \mathcal{O}(f^4) \\
C^{ab}{}_{cd} &= -f^2 (1 - k f^2) \frac{1}{2} \left(f^b{}_{cg} f^{ag}{}_d (-1)^{cd} + f^b{}_{dg} f^{ag}{}_c \right) + \mathcal{O}(f^4)
\end{aligned}
$$

The fact that a solution exists depends on tensor identities, valid in these supergroups.

Open problem :
complete the algebraic bootstrap.

Ic : Applications

i) The Sugawara energy-momentum tensor.

ii) Current algebra primaries
and their conformal dimension.

i) The Sugawara energy-momentum tensor

for non-chiral currents

Use

- the current algebra
- the vanishing of self-contracted structure constants $\kappa_{ab} f^{ab}{}_{c} \quad = \quad 0$
- vanishing of the Killing metric $f^{abc} f^{d}{}_{bc} = 0$
- the Maurer-Cartan equation

In detail :

- current / current bi-linear OPE :

$$
\begin{aligned}
j_z^a(z) :j_z^b j_z^b:(w) &= \lim_{:x\to w:} j_z^a(z)\left[j_z^b(x) j_z^b(w)\right] \\
&= \lim_{:x\to w:} \left[\left(\frac{c_1\kappa^{ab}}{(z-x)^2} + f^{ab}{}_c\left(\frac{c_2}{z-x} j_z^c(x) + (c_2-g)\frac{\bar z-\bar x}{(z-x)^2} j_{\bar z}^c(x)\right)\right) j_z^b(w)\right. \\
&\qquad \left. + j_z^b(x)\left(\frac{c_1\kappa^{ab}}{(z-w)^2} + f^{ab}{}_c\left(\frac{c_2}{z-w} j_z^c(w) + (c_2-g)\frac{\bar z-\bar w}{(z-w)^2} j_{\bar z}^c(w)\right)\right)\right]
\end{aligned}
$$

- most of the terms vanish by the above rules :

$$
j_z^a(z) :j_z^b j_z^b:(w) = 2c_1\frac{j_z^a(w)}{(z-w)^2} + c_2\boxed{\frac{f^a{}_{bc}}{z-w}\left(:j_z^b j_z^c:(w) + :j_z^c j_z^b:(w)\right)} + (c_2-g)\boxed{f^a{}_{bc}\frac{\bar z-\bar w}{(z-w)^2}\left(:j_z^b j_{\bar z}^c:(w) + :j_{\bar z}^c j_z^b:(w)\right)}
$$

$\varnothing$

Use the Maurer-Cartan equation to simplify this term.

- The current component has dimension one for an appropriate choice of energy momentum tensor :

$$
T(w) = \gamma :j_z^b j_z^b:(w) \qquad \gamma = \frac{1}{2c_1} \quad \Rightarrow \quad \boxed{T(w) j_z^a(z) = \frac{j_z^a(z)}{(w-z)^2} + \frac{\partial j_z^a(z)}{w-z}}
$$

- One computes the energy-momentum tensor OPE similarly :

$$
T(z)T(w) = \frac{1}{2c_1}\lim_{:x\to w:} T(z)\left[j_z^a(x) j_z^a(w)\right]
$$

$$
\boxed{T(z)T(w) = \frac{dim\ G}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}}
$$

a zero Killing metric
leads to
a (non-chiral) conformal current algebra.

ii) Current algebra primaries and their conformal dimensions

We define current algebra primaries :

$$c_{\pm} = -\frac{(1 \pm k f^2)}{2 f^2}$$

$$j^a_{L,z}(z,\bar{z})\phi_{\mathcal{R}}(w,\bar{w}) = -\frac{c_+}{c_+ + c_-} t^a_R \frac{\phi_{\mathcal{R}}(w,\bar{w})}{z - w} + \text{order zero}$$

$$j^a_{L,\bar{z}}(z,\bar{z})\phi_{\mathcal{R}}(w,\bar{w}) = -\frac{c_-}{c_+ + c_-} t^a_R \frac{\phi_{\mathcal{R}}(w,\bar{w})}{\bar{z} - \bar{w}} + \text{order zero}$$

Their OPE with the current algebra can be completed recursively.

We find:

$$j^a_{L,z}(z,\bar{z})\phi(w,\bar{w}) = -\frac{c_+}{c_+ + c_-}t^a\frac{\phi(w,\bar{w})}{z-w} + :j^a_{L,z}\phi:(w,\bar{w})$$

$$+ \underline{A^a{}_c} \log|z-w|^2 :j^c_{L,z}\phi:(w,\bar{w}) + B^a{}_c\frac{\bar{z}-\bar{w}}{z-w} :j^c_{L,\bar{z}}\phi:(w,\bar{w}) + \ldots$$

$$j^a_{L,\bar{z}}(z,\bar{z})\phi(w,\bar{w}) = -\frac{c_-}{c_+ + c_-}t^a\frac{\phi(w,\bar{w})}{\bar{z}-\bar{w}} + :j^a_{L,\bar{z}}\phi:(w,\bar{w})$$

$$- A^a{}_c\frac{z-w}{\bar{z}-\bar{w}} :j^c_{L,z}\phi: - B^a{}_c \log|z-w|^2 :j^c_{L,\bar{z}}\phi:(w,\bar{w}) + \ldots$$

$$\underline{A^a{}_c} = \frac{c_-}{(c_+ + c_-)^2} i f^a{}_{cb} t^b + \mathcal{O}(f^4) \qquad ; \qquad B^a{}_c = \frac{c_+}{(c_+ + c_-)^2} i f^a{}_{cb} t^b + \mathcal{O}(f^4)$$

Current primaries
(large radius) conformal dimension:

$$\Delta_\phi = \bar{\Delta}_\phi = \frac{f^2}{2} t^a t^b \kappa_{ba} + \mathcal{O}(f^4)$$

An overcomplete basis of operators is:

:j:j:j:j ... phi ::::

Compute their conformal dimension perturbatively.

Example:

:j phi:

Perturbative conformal dimension:

$$h\left(\left[:j^a_{L,z}\phi:\right]_{\tilde{\mathcal{R}}}\right) \quad = \quad \frac{f^2}{2}c^{(2)}_{\mathcal{R}} + 1 + \frac{f^2}{2}(1-kf^2)(c^{(2)}_{\tilde{\mathcal{R}}} - c^{(2)}_{\mathcal{R}}) + \mathcal{O}(f^4)$$

Interpolation

between WZW value
and particle approximation

see also

First order perturbative prove of the Hirota equation for cosets (e.g. string on AdS5 x S5).

Raphael Benichou

2. Properties in String Theory

2a : The Physical String States

2b : An Asymptotic Symmetry Group

2a : The Physical String States

Simplify : A Particle on a Supergroup

$$L = \int d\tau e \langle \partial_\tau g g^{-1}, \partial_\tau g g^{-1} \rangle$$

Worldsheet reparameterization invariance leads to physical states in the cohomology

$$Q = c\, C_2 = c\, L_0 = c\, \Delta$$

$$C_2 \quad = \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$ not closed (in Siegel gauge)

One obtains that L_0 becomes diagonal in cohomology.

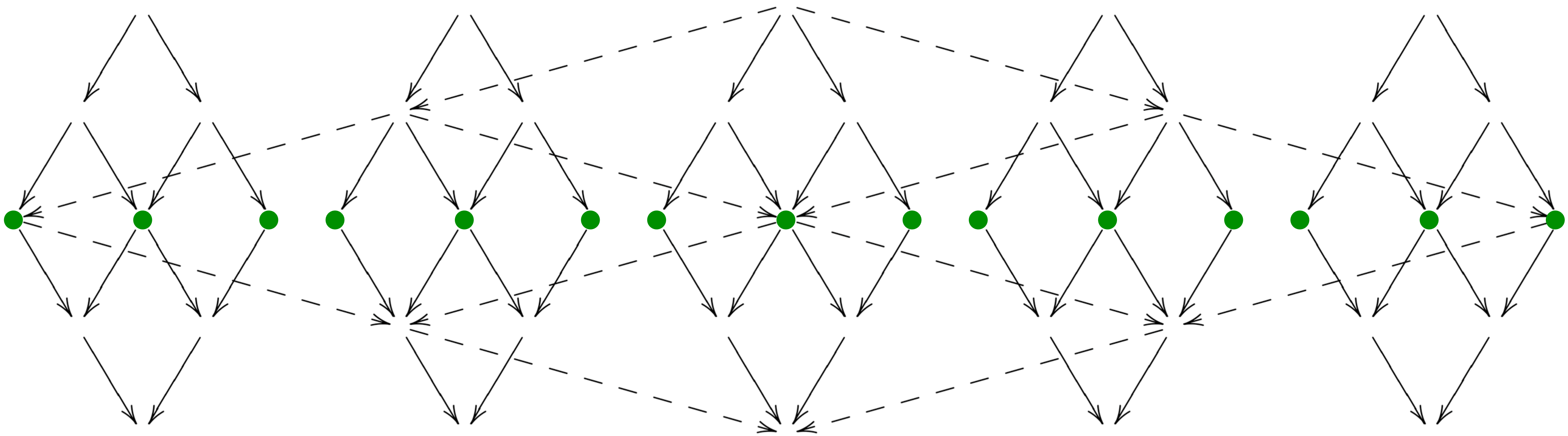
The signature of logarithmicity disappears on physical string states.

Superstring on a Supergroup

Local Fermionic Invariance of (unknown) ancestor of Berkovits formalism

Leads to cohomological problem of the type

$$Q \quad = \quad K_{ab} F^a F^b \quad \text{in} \quad \mathcal{L}^2(PSU(2|2))$$



covariant calculation

$$\mathcal{H} \;=\; \sum_{j=0}^{\infty} \left([j+1]_L \otimes [j]_R \oplus 2[j]_L \otimes [j]_R \oplus [j]_L \otimes [j+1]_R \right)$$

equal to the supergravity KK spectrum

$$\mathcal{L}^2(PSU(N|N))$$

Analogue of Peter-Weyl theorem ?

Gaberdiel-Gerigk

Solvable Subsector of Log CFT ?

2b. The asymptotic symmetry algebra of string theory on AdS3 x S3 with Ramond-Ramond flux

Space-time currents are functionals of worldsheet currents and an (almost) primary field with logarithmic correlators.

Space-time UV singularities arise from worldsheet UV singularities

The worldsheet current OPEs lead to : two copies of the extended supersymmetric space-time Virasoro algebra centrally extended.

Construction of the space-time conformal symmetry of the theory dual to quantum gravity in the bulk of AdS3 (with RR-flux). I.e. the D1-D5 system.

Open problems

The complete algebraic bootstrap (for currents and conformal dimensions).

Quadratically integrable functions on (non-)compact supergroups

Cohomology for massive string states

Consequence of logarithmicity for string ?

Identify solvable subsector of hard logarithmic CFTs