Indecomposability parameters in LCFT

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Joint work with J.L. Jacobsen and H. Saleur at IPhT CEA Saclay and LPTENS
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ACFTA (Institut Henri Poincaré)

Indecomposability parameters $\beta_{r,s}$: a few definitions

- Original claim [Gurarie ‘93 ‘99, Gurarie & Ludwig ‘02] : SUSY approach to disordered systems yields $c = 0$ CFT with a logarithmic partner for the stress energy tensor T

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- At the time, b was thought of as an "effective central charge"

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- Using Virasoro bilinear form $L_n^\dagger = L_{-n}$, we get $\beta = \langle \psi | \phi \rangle$

- Crucial role in staggered Virasoro modules theory [Rohsiepe '96, Kytola & Ridout '09]

$$\mathcal{P} = \begin{tikzcd} & \psi \arrow[dl] \arrow[dr] & \\ \xi \arrow[dr] & & \rho \arrow[dl] \\ & \phi & \end{tikzcd}$$

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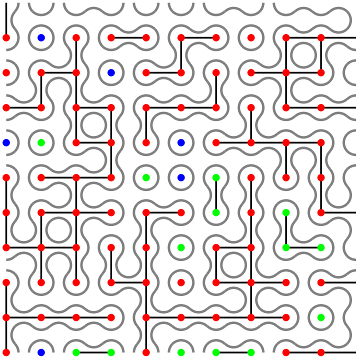
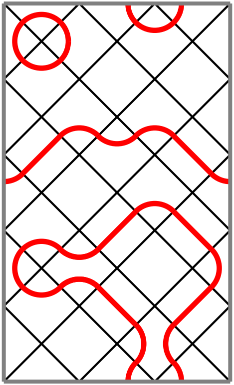
- β characterizes *uniquely* the staggered module
- Convenient to work with Virasoro bilinear form $L_n^\dagger = L_{-n}$ $\Rightarrow \beta = \langle \psi | \phi \rangle$ (given that $\langle \xi | \xi \rangle = 1$)

Questions we'd like to answer

- How to compute these (hopefully physically relevant) numbers ?
  It would also be nice to get a physical intuition of their meaning !
- Can we measure some of them on concrete lattice models (as for the central charge and the critical exponents) ? Which observables are they related to ?





A simple general formula from OPEs :
log CFTs as limits of ordinary CFTs

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- Unfortunately, these methods may be quite cumbersome ...
- $\Rightarrow$ Need of a simple explicit formula for the values of β
- $\Rightarrow$ Idea : LCFT = limit of usual (non log) CFTs

$$
\begin{aligned}
c &= 1 - \frac{6}{x(x+1)} \\
h_{r,s} &= \frac{[r(x+1) - sx]^2 - 1}{4x(x+1)}
\end{aligned}
$$

$c \to 0$ catastrophe and b number [Cardy '01, Gurarie & Ludwig '02, ...]

Conformal invariance enforces

$$\Phi_h(z)\Phi_h(0) \sim \frac{C^I_{\Phi\Phi}}{z^{2h}} \left[1 + \frac{2h}{c} z^2 T(0) + \dots \right] + \dots$$

Suppose there is another field with conformal weight $h \to 2$ as $c \to 0$ ($\phi_{1,5}$ for percolation and $\phi_{3,1}$ for polymers)

$$\Phi_h(z)\Phi_h(0) \sim \frac{C^I_{\Phi\Phi}}{z^{2h}} \left[1 + \frac{2h}{c} z^2 T(0) + \dots \right] + \frac{C^{\Phi_{15}}_{\Phi\Phi}}{z^{2h-h_{1,5}}} \left[\Phi_{1,5}(0) + \dots\right].$$

Then introduce a new field $t(z)$ through

$$\Phi_{1,5}(z) = \frac{C^I_{\Phi\Phi}}{C^{\Phi_{15}}_{\Phi\Phi}} \left(\frac{2h \langle T|T \rangle}{cb(c)} t(z) - \frac{2h}{c} T(z) \right)$$

Where $b(c) = -\frac{\langle T|T \rangle}{h_{1,5}-2}$ and $\langle T|T \rangle = \frac{c}{2}$.

The OPE then becomes well defined at $c = 0$

$$\Phi_h(z)\Phi_h(0) \sim \frac{C^I_{\Phi\Phi}}{z^{2h}} \left[1 + \frac{h}{b} z^2 (T(0) \log z + t(0)) + \dots\right] + \dots,$$

One can show that $L_0 t = 2t + T$ at $c = 0$, correlation functions can also be calculated in a similar fashion.
This yields

$$\begin{aligned} b_{\text{percolation}} &= -\lim_{c \to 0} \frac{c/2}{h_{1,5} - 2} = -\frac{5}{8} \\ b_{\text{polymers}} &= -\lim_{c \to 0} \frac{c/2}{h_{3,1} - 2} = \frac{5}{6} \end{aligned}$$

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Generalization to a more complicated case

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$$\boxed{\beta = -\lim_{\epsilon\to 0} \frac{\langle\phi|\phi\rangle}{h_\psi - h_\xi - n} = -\frac{\frac{\mathrm{d}}{\mathrm{d}\epsilon}\langle\phi|\phi\rangle\big|_{\epsilon=0}}{\frac{\mathrm{d}}{\mathrm{d}\epsilon}(h_\psi - h_\xi)\big|_{\epsilon=0}}}$$

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- One needs to properly identify the conformal weights h_ξ and h_ψ in the spectrum (quite easy)

Application to Symplectic fermion theory [Kausch '95]
General structure of the Jordan cells

$$\begin{aligned} L_0\phi^{(j)} &= h_{1,1+2j}\phi^{(j)} \\ L_0\psi^{(j)} &= h_{1,1+2j}\psi^{(j)} + \phi^{(j)} \\ \phi^{(j)} &= A_j\xi^{(j)} \\ A_j^\dagger\psi^{(j)} &= \beta_{1,1+2j}\xi^{(j)} \end{aligned}$$

associated with the staggered structure

$$\mathcal{P}_j = \begin{tikzcd} & \psi^{(j)} \arrow[dl] \arrow[dr] & \\ \xi^{(j)} \arrow[dr] & & \rho^{(j)} \arrow[dl] \\ & \phi^{(j)} & \end{tikzcd}$$

This OPE formula allows us to conjecture the general expression

$$\beta_{1,1+2j} = -\lim_{x\to1}\frac{\langle\xi|A^\dagger A|\xi\rangle}{h_{1,1+2j}-h_{1,1+2(j-1)}-(j-1)} = -\frac{[(2j-3)!]^2}{4^{j-2}}(j-1)$$

Lattice models, indecomposability and scaling limit

- The Temperley-Lieb algebra is behind all the models we shall consider

$$\begin{aligned} [e_i, e_j] &= 0 \ (|i-j| \geq 2) \\ e_i^2 &= (q+q^{-1})e_i \\ e_i e_{i\pm 1} e_i &= e_i \end{aligned}$$

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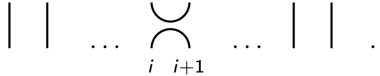

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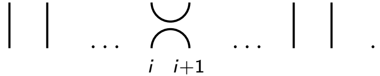
$e_i =$ 

- Standard (cell) modules $\mathcal{V}_j$: $\mathcal{V}_2 = \{ \,|\,|\,|\,|\, \}$, $\mathcal{V}_1 = \{ \,| \cup |\,, \cup |\,|\,, |\,| \cup \}$, and $\mathcal{V}_0 = \{$ , $\cup \cup \}$.

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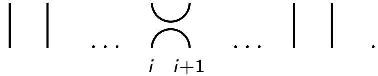
$e_i =$ 

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$\mathcal{V}_1 = \{ \,|\cup|\, , \cup||\, , ||\cup \}$, and $\mathcal{V}_0 = \{ \,\cup\!\!\!\cup\, , \cup\,\cup \}$.
- In a given module, we define our models by
Transfer Matrix $T = \prod_{\mathrm{i\ even}} (1+e_i) \prod_{\mathrm{i\ odd}} (1+e_i)$
Hamiltonian $H = -\sum_i e_i$

6-Vertex or XXZ representation $\mathcal{H}_{\mathrm{XXZ}} = (\mathbb{C}^2)^{\otimes 2N}$
Representation of the TL algebra on this space

$$e_i = \mathbb{I} \otimes \mathbb{I} \otimes \cdots \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & -1 & 0 \\ 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \cdots \otimes \mathbb{I},$$

The Hamiltonian reads

$$H = \frac{1}{2} \sum_{i=1}^{2N-1} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \frac{q+q^{-1}}{2} \sigma_i^z \sigma_{i+1}^z \right) + \frac{q-q^{-1}}{4} (\sigma_1^z - \sigma_{2N}^z)$$

$U_q(\mathfrak{sl}_2)$ symmetry $[H, U_q(\mathfrak{sl}_2)] = 0$

$$\boxed{\text{TL} \longleftrightarrow \text{Virasoro}}$$

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- The standard TL module $\mathcal{V}_j$ has a structure that mimics that of $\mathcal{V}_{h_{1,1+2j}}/\mathcal{V}_{h_{1,-1-2j}}$ on the Virasoro side

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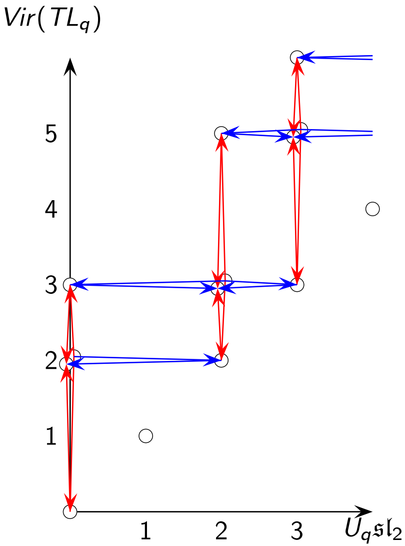
- For q root of unity, we can deduce which indecomposable $TL_q(2N)$ modules occur in the decomposition using the indecomposable structure of $V_{\frac{1}{2}}^{\otimes 2N}$ for $U_q\mathfrak{sl}_2$ (*cf.* lectures of A. Gaynutdinov, G. Lehrer ...)

Bimodule

[Read & Saleur '07]



- The indecomposable diamond TL modules ‘tend’ to several staggered Virasoro modules in the limit

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- $\Rightarrow$ We know exactly from the representation theory of associative algebras the structure of the TL modules (and of the Jordan cells) on the lattice (finite size). The TL indecomposable modules mimic the staggered Virasoro modules that occur in the continuum limit. We can thus study the lattice models as ‘regularizations’ of LCFTs built out of such staggered modules. Most of the continuum features can be recovered in finite size (subquotient structure, fusion rules, . . .) and we’ll see that this is also true for indecomposability parameters.

Measure of indecomposability parameters from lattice models

From lattice Jordan cells to the continuum

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- We normalize our states $\{|\phi^{(L)}\rangle, |\psi^{(L)}\rangle\}$ to prepare the comparison with CFT

$$H^{(L)} - E_0(L)\mathbf{1} = \frac{\pi v_F}{L} \begin{pmatrix} h^{(L)} & 1 \\ 0 & h^{(L)} \end{pmatrix},$$

with $h^{(L)} = \frac{L}{\pi v_F}(E(L) - E_0(L))$.

Lattice scalar product

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- OK, so let's compute $\langle \psi^{(L)} | \phi^{(L)} \rangle$ then ...

- …but the Jordan cell is invariant under a global rescaling of the basis states $\left|\phi^{(L)}\right\rangle \to \alpha\left|\phi^{(L)}\right\rangle$ and $\left|\psi^{(L)}\right\rangle \to \alpha\left|\psi^{(L)}\right\rangle$ while $\left\langle\psi^{(L)}|\phi^{(L)}\right\rangle \to |\alpha|^2\left\langle\psi^{(L)}|\phi^{(L)}\right\rangle$!

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- We can construct $A^{(L)}$ on the lattice. We expect the state $A^{(L)}\left|\xi^{(L)}\right\rangle$ to correspond to the operator $\phi = A\xi$ in the limit $L \to \infty$.

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- Compute $\beta^{(L)} = \dfrac{\left|\left\langle\psi^{(L)}|A^{(L)}\xi^{(L)}\right\rangle\right|^2}{\left\langle\psi^{(L)}|\phi^{(L)}\right\rangle}$

Numerics for the Symplectic fermion theory

$L = 2N$	$\beta_{1,5}$	$\beta_{1,7}$	$\beta_{1,9}$
8	-0.937759	-13.3574	
10	-0.959708	-14.8908	-1518.37
12	-0.971844	-15.7936	-1805.43
14	-0.979236	-16.3612	-2013.66
16	-0.984064	-16.7384	-2157.86
18	-0.987388	-17.0006	-2262.59
20	-0.989771	-17.1898	-2340.51
22	-0.991539	-17.3304	-2399.80
24			-2445.81
∞	-1.0000 $\pm$ 0.0002	-18.0(0) $\pm$ 0.05	-27(00) $\pm$ 25
Exact	-1	-18	-2700

Some results (cont'd)

Percolation ($q = e^{i\pi/3}$)			
L	$\beta_{1,4}$	L	$\beta_{1,5}$
7	-0.471874	8	-0.609088
9	-0.476386	10	-0.605858
11	-0.479983	12	-0.606403
13	-0.482724	14	-0.607775
15	-0.484837	16	-0.609226
17	-0.486503	18	-0.610561
19	-0.487845	20	-0.611738
21	-0.488946	22	-0.612764
∞	-0.5000 $\pm$ 0.0001	∞	-0.6249 $\pm$ 0.0005
Exact	-1/2	Exact	-5/8 = 0.625

A example of new result : Logarithmic Ising model ($q = e^{i\pi/4}$)

$$L_0 = \begin{pmatrix} 5/2 & 1 \\ 0 & 5/2 \end{pmatrix}, \quad \phi(z) = (L_{-2} - \frac{3}{4}L_{-1}^2)\xi(z).$$

$L = 2N$	$\beta_{1,5}$
8	-1.26986
10	-1.29548
12	-1.31743
14	-1.33489
16	-1.34876
18	-1.35993
20	-1.36905
22	-1.37663
∞	-1.4582(8) $\pm$ 0.0001
Exact	$-35/24 \simeq -1.4583$

$$\beta_{1,5} = -\lim_{\epsilon \to 0} \frac{\langle \phi^{(2)} | \phi^{(2)} \rangle}{h_{1,5} - h_{1,3} - 2} = -\frac{35}{24}$$

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- Geometrical/physical observables involving these numbers ? ? ?

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Thank you !