Logarithmic CFT in a disordered electronic system

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Paris, October, 2011

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OUTLINE

- Free Dirac electrons subject to random-gauge and mass disorder
- formulate quenched disorder-average via super-symmetry method: N electrons and N super-partners (bosons)
- package the fermions and bosons together into super-currents

$$J^A = J^{\binom{a}{\bar{a}}} = \Psi^\dagger_{\bar{a}} \Psi^a$$

- The currents are elements of $gl(N|N)$
- Analyse tensor product of two currents,

$$O^{AB} = J^A \bar{J}^B$$

- Indecomposable representations of $gl(N|N) \Rightarrow$ logarithmic operators
- Logarithmic pair in a disordered system: (M^A, L^A)

$$\langle M^A(z) M^B(0) \rangle = 0$$

$$\langle L^A(z) M^B(0) \rangle = \kappa^{AB}\, k^2 (1 - \frac{\lambda^2 k^2}{2} + \ldots)/|z|^4$$

$$\langle L^A(z) L^B(0) \rangle = \kappa^{AB}\, 8k^2 \lambda \log|z/a|/|z|^4$$

- 4-point functions of currents

Disordered electrons

- Dirac-electrons, given by N Grassmanian fields $\psi^a(x)$, and $\bar{\psi}^a(x)$, and their conjugate fields $\psi^\dagger_a(x)$, and $\bar{\psi}^\dagger_a(x)$ with action

$$\mathscr{S}_{mA} = \frac{1}{2}\sum_{a=1}^{N}\int_z \psi_a^\dagger\left(\bar{\partial}+\bar{A}\right)\psi^a + \bar{\psi}_a^\dagger\left(\partial + A\right)\bar{\psi}^a + m\bar{\psi}_a^\dagger\psi^a + m^*\psi_a^\dagger\bar{\psi}^a$$

- The expectation value of an observable $\mathscr{O}$ is defined by the Grassmanian path integral

$$\langle \mathscr{O} \rangle := \int \mathscr{D}[\psi]\,\mathscr{D}[\bar{\psi}]\,\mathscr{D}[\psi^\dagger]\,\mathscr{D}[\bar{\psi}^\dagger]\,\mathscr{O}\,\mathrm{e}^{-\mathscr{S}_{mA}}\ ,$$

- disorder averages over random $m(x)$, $A(x)$

$$\overline{(\cdots)}^m := \int \mathscr{D}[m]\,\mathscr{D}[m^*]\,\mathrm{e}^{-\frac{1}{4\lambda}\int_z mm^*}\ , \quad \overline{(\cdots)}^A := \int \mathscr{D}[m]\,\mathscr{D}[m^*]\,\mathrm{e}^{-\frac{1}{4\tilde{\lambda}}\int_z A\bar{A}}$$

- but one has to average:

$$\overline{\left[\frac{\langle O\rangle}{\mathscr{Z}}\right]}$$

- What to do with the denominator?

The super-symmetry method

- introduce N bosonic fields β, “super-partners” of the ψ’s

$$\mathscr{S}_{mA} = \frac{1}{2}\sum_{a=1}^{N}\int_z \psi_a^{\dagger}\left(\bar{\partial}+\bar{A}\right)\psi^a + \bar{\psi}_a^{\dagger}\left(\partial + A\right)\bar{\psi}^a + m\bar{\psi}_a^{\dagger}\psi^a + m^*\psi_a^{\dagger}\bar{\psi}^a$$

$$+\frac{1}{2}\sum_{a=1}^{N}\int_z \beta_a^{\dagger}\left(\bar{\partial}+\bar{A}\right)\beta^a + \bar{\beta}_a^{\dagger}\left(\partial + A\right)\bar{\beta}^a + m\bar{\beta}_a^{\dagger}\beta^a + m^*\beta_a^{\dagger}\bar{\beta}^a$$

- path integral now runs over fermions and bosons.
- partition function becomes $\mathscr{Z} = \langle 1 \rangle = 1$.
- disorder average simplifies $\overline{\left[\frac{\langle O \rangle}{\mathscr{Z}}\right]} = \overline{\langle O \rangle}$.
- Package together into a super-field $\Psi^a(x) = (\beta^a(x), \psi^a(x))$, etc., with double index $(a, g(a))$, with *grade* $g(a) = \begin{cases} 0 & \text{boson} \\ 1 & \text{fermion} \end{cases}$

$$\mathscr{S}_{mA} = \frac{1}{2}\sum_{a=1}^{N}\sum_{g(a)=0}^{1}\int_z \Psi_a^{\dagger}\left(\bar{\partial}+\bar{A}\right)\Psi^a + \bar{\Psi}_a^{\dagger}\left(\partial + A\right)\bar{\Psi}^a + m\bar{\Psi}_a^{\dagger}\Psi^a + m^*\Psi_a^{\dagger}\bar{\Psi}^a$$

The disorder-averaged action

- Action

$$\mathscr{S} = \mathscr{S}_0 + \mathscr{S}_m + \mathscr{S}_A$$

$$\mathscr{S}_0 = \frac{1}{2}\int_z \Psi^\dagger_a \bar{\partial}\Psi^a + \bar{\Psi}^\dagger_a \partial \bar{\Psi}^a$$

$$\mathscr{S}^m = -\lambda \int_z \left(\bar{\Psi}^\dagger_a \Psi^a\right)\left(\Psi^\dagger_b \bar{\Psi}^b\right) = -\lambda \int_z \underbrace{\left(\bar{\Psi}^\dagger_a \bar{\Psi}^b\right)}_{\bar{J}^{\binom{b}{a}}}\underbrace{\left(\Psi^\dagger_b \Psi^a\right)}_{J^{\binom{a}{b}}}(-1)^{g(b)}$$

$$= -\lambda \int_z \bar{J}^B J^A \kappa_{AB}$$

$$\mathscr{S}^A = -\tilde{\lambda} \int_z \left(\bar{\Psi}^\dagger_a \bar{\Psi}^a\right)\left(\Psi^\dagger_b \Psi^b\right) = -\tilde{\lambda} \int_z \bar{J}^B J^A \tilde{\kappa}_{AB}$$

- Free theory expectation values: $\left\langle \Psi^a(z)\Psi^\dagger_b(w)\right\rangle = \dfrac{\delta^a_b \delta^{g(a)}_{g(b)}}{z-w}$.
- defines current algebra

$$J^{\binom{a}{\bar{a}}}(z_1)\;\; J^{\binom{b}{\bar{b}}}(z_2) = -k\frac{\kappa^{\binom{a}{\bar{a}}\binom{b}{\bar{b}}}}{z_{12}^2} + \frac{1}{z_{12}}\, J^{\binom{c}{\bar{c}}}(z_2) f_{\binom{\bar{c}}{c}}{}^{\binom{a}{\bar{a}}\binom{b}{\bar{b}}}$$

$$f_C{}^{AB} = f_{\binom{\bar{c}}{c}}{}^{\binom{a}{\bar{a}}\binom{b}{\bar{b}}} = \delta^a_{\bar{b}}\delta^b_c\delta^{\bar{c}}_{\bar{a}} - (-1)^{g(A)g(B)}\;\; \delta^b_{\bar{a}}\delta^a_c\delta^{\bar{c}}_{\bar{b}}$$

$$\kappa^{AB} = \kappa^{\binom{a}{\bar{a}}\binom{b}{\bar{b}}} = (-1)^{g(\bar{a})}\;\delta^b_{\bar{a}}\delta^a_{\bar{b}}$$

Theory, β-functions

The theory is defined by the two $gl(N|N)$ invariant interactions

$$\mathscr{S}_{\text{int}} = -\lambda \int_z \Phi(z,\bar{z}) - \tilde{\lambda} \int_z \tilde{\Phi}(z,\bar{z})$$

$$\Phi(z,\bar{z}) = J^A(z)\bar{J}^B(\bar{z})\kappa_{BA} \,, \qquad \kappa_{BA} = (-1)^{g(\bar{b})g(a)}\ \delta_b^{\bar{a}}\delta_a^{\bar{b}}$$

$$\tilde{\Phi}(z,\bar{z}) = J^A(z)\bar{J}^B(\bar{z})\tilde{\kappa}_{BA} \,, \qquad \tilde{\kappa}_{BA} = -\frac{1}{2} f_B{}^{CD} f_{DCA} = \delta_b^{\bar{b}}\delta_a^{\bar{a}}$$

The operator-product expansions are

$$\Phi(z,\bar{z})\Phi(0,0) = -\frac{2}{z\bar{z}}\tilde{\Phi}(0,0)$$

$$\Phi(z,\bar{z})\tilde{\Phi}(0,0) = 0$$

$$\tilde{\Phi}(z,\bar{z})\tilde{\Phi}(0,0) = 0$$

The β-functions are

$$\beta_\lambda(\lambda,\tilde{\lambda}) := \frac{L\partial}{\partial L}\lambda = 0 \qquad \text{(to all orders)}$$

$$\beta_{\tilde{\lambda}}(\lambda,\tilde{\lambda}) := \frac{L\partial}{\partial L}\tilde{\lambda} = 2\lambda^2 - 4k\lambda^2 + \dots$$

λ is an exactly marginal perturbation: Conformal sector

WARNING: Use traceless observables $\Leftrightarrow$ connected expectations.

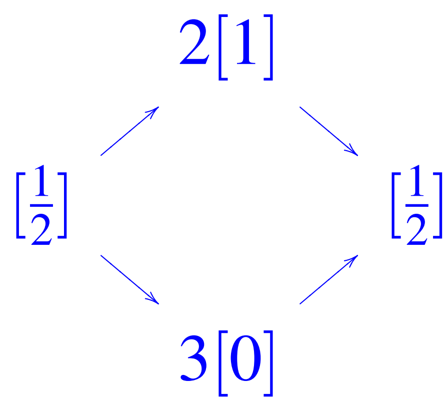
Composite operators and representation theory

- consider operators of the form $J^A \bar{J}^B$
- group theoretically: tensor product of two adjoints (Götz, Quella, Schomerus, 2005)

$$\left[\tfrac{1}{2}\right] \otimes \left[\tfrac{1}{2}\right] \; = \; \pi_g\left(\left(\tfrac{1}{2},\tfrac{1}{2}\right) \otimes \left(\tfrac{1}{2},\tfrac{1}{2}\right)\right) \; \oplus \; [0] \; \oplus \; \pi^{\text{indec}}_{\frac{1}{2}\otimes\frac{1}{2}}$$

- $\pi_g\left(\left(\tfrac{1}{2},\tfrac{1}{2}\right)\right)$ is the symmetric part
- the antisymmetric part has indecomposable representation

$\pi^{\text{indec}}_{\frac{1}{2}\otimes\frac{1}{2}}$:



- we will see later: the renormalization group operator $\mathscr{R}$ maps the left $\left[\tfrac{1}{2}\right]$ to the right $\left[\tfrac{1}{2}\right]$:

$$\mathscr{R} \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

- $\mathscr{R}$ has Block-Jordan form

Indecomposable representations = Block-Jordan form $\Rightarrow$ logarithms

Suppose the renormalization group acts as follows

$$\mathscr{R}\begin{pmatrix} L^A \\ M^A \end{pmatrix} = -4\lambda \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} L^A \\ M^A \end{pmatrix} .$$

This implies the following correlation functions

$$\langle M^A(z) M^B(0) \rangle = \kappa^{AB} \frac{d}{|z|^4}$$

$$\langle L^A(z) M^B(0) \rangle = \kappa^{AB} \frac{b}{|z|^4}$$

$$\langle L^A(z) L^B(0) \rangle = \kappa^{AB} \frac{c + 8b\lambda \log|z/a|}{|z|^4}$$

An Excursion: Conserved Currents

- Current conservation in conformal coordinates reads

$$\partial \bar{J}^A + \bar{\partial} J^A = 0 \ .$$

This remains correct for the interacting theory.

- However the curl of the current gets corrected,

$$\bar{\partial} J^A - \partial \bar{J}^A = \frac{2\lambda}{1+\lambda k} M^A$$
$$M^A := \bar{J}^E J^D f_{DE}{}^A \ .$$

This is proven by contracting $\bar{\partial} J^A - \partial \bar{J}^A$ versus $\mathrm{e}^{\lambda \int_z \Phi(z)}$, and noting that one gets back either $\bar{\partial} J^A - \partial \bar{J}^A$ or M^A times $\mathrm{e}^{\lambda \int_z \Phi(z)}$.

- M^A is sometimes termed the Maurer-Cartan form.

Our explicit example for a logarithmic pair

By explicit calculations, one finds that

- the image of the renormalization-group operator $\mathscr{R}$ is the Maurer-Cartan form

$$M^A := \bar{J}^E J^D f_{DE}{}^A \ .$$

Idea:

- $M^A \sim \partial \bar{J}^A - \bar{\partial} J^A$ is not renormalized, due to the sum over grades in the running index.
- insert a factor of $(-1)^{\text{grade of running index}}$ or restrict to grade 0
- The preimage is an equivalence class. Through experimentation one finds that one representative is

$$L^A := \bar{J}^E J^D b_{DE}{}^A \ .$$

$$b_{CB}{}^A := \delta^a_c \delta^{\bar{b}}_{\bar{a}} \delta^{\bar{c}}_b - (-1)^{[g(A)+1]g(B)} \delta^a_b \delta^{\bar{b}}_c \delta^{\bar{c}}_{\bar{a}}$$

- An alternative is

$$\tilde{b}_{CB}{}^A := 2\left[\delta^0_{g(b)} \delta^a_c \delta^{\bar{b}}_{\bar{a}} \delta^{\bar{c}}_b - \delta^{g(\bar{b})}_0 (-1)^{g(A)g(B)} \delta^a_b \delta^{\bar{b}}_c \delta^{\bar{c}}_{\bar{a}}\right]$$

- The relation between these two definitions is

$$b_{CB}{}^A + f_{CB}{}^A = \tilde{b}_{CB}{}^A$$

Prediction from OPE for 2-point function was

$$\langle M^A(z)M^B(0)\rangle = \kappa^{AB}\,\frac{d}{|z|^4}$$

$$\langle L^A(z)M^B(0)\rangle = \kappa^{AB}\,\frac{b}{|z|^4}$$

$$\langle L^A(z)L^B(0)\rangle = \kappa^{AB}\,\frac{c+8b\lambda\log|z/a|}{|z|^4}$$

Explicit calculation to 2-loop order gives :

$$\langle M^A(z)M^B(0)\rangle = \kappa^{AB}\,\frac{0}{|z|^4}$$

$$\langle L^A(z)M^B(0)\rangle = \kappa^{AB}\,\frac{k^2}{|z|^4}\left(1-\frac{\lambda^2 k^2}{2}+\ldots\right)$$

$$\langle L^A(z)L^B(0)\rangle = \kappa^{AB}\,\frac{0+8k^2\lambda\log|z/a|}{|z|^4}$$

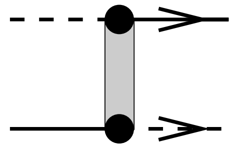
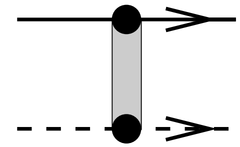
The even objects have correlation functions

$$\langle O_+(z)O_+(0)\rangle = \frac{1}{|z|^{4+2\lambda}}$$

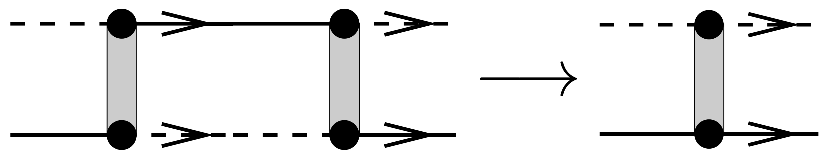
$$\langle O_-(z)O_-(0)\rangle = \frac{1}{|z|^{4-2\lambda}}$$

Re-exponentiation of logs has been checked to higher-loop order.

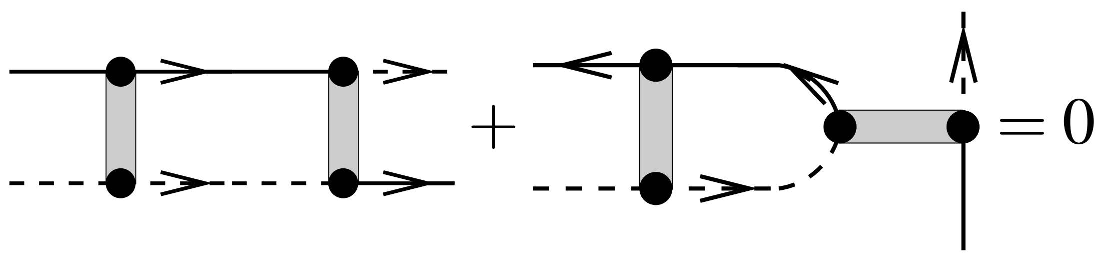
Replica: HOW TO? (1)

$\Phi =$  $\tilde{\Phi} =$ 

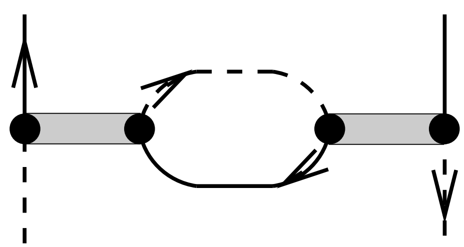
- 1-loop order: $\Phi\Phi$ only generates $\tilde{\Phi}$:



- $\tilde{\Phi}$ leads to cancelations:

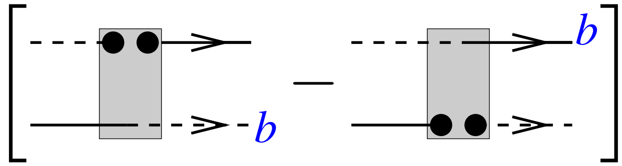


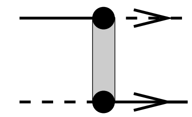
- Supersymmetry (Bosons $-$ fermions):

 $\sim (N - N) = 0$

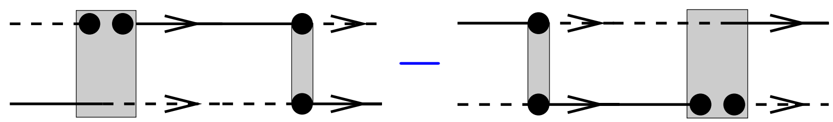
- can also be achieved with $N = 0$ replica.
- But this is not enough structure for M^A and L^A.

Replica: HOW TO? (2)

$M^A = J^E \bar{J}^D f_{DE}{}^A = \sum_b$  ,

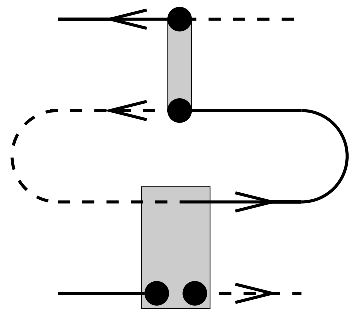
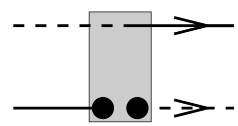
$\Phi =$ 

- First cancelation:

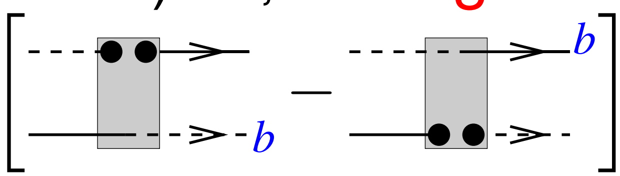
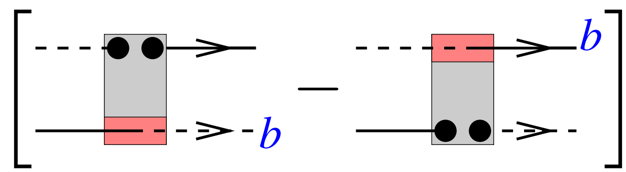


$= 0$

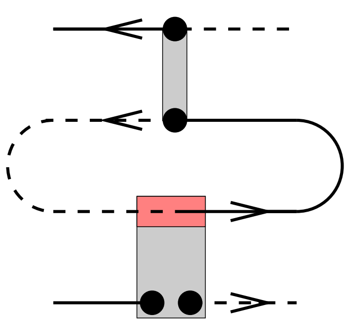
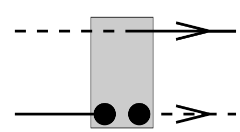
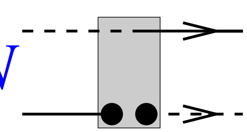
- Second cancelation: loop, N physical and $\overline{N}$ ghost species, $N + \overline{N} \to 0$

$\Phi\, M^A \to$  $\sim \Big[\underbrace{\sum_{b=1}^{N}}_{N} + \underbrace{\sum_{b=N+1}^{N+\bar{N}}}_{\overline{N}} \Big]$  $\to 0$

- Now define (traceless) L^A, via a graded sum

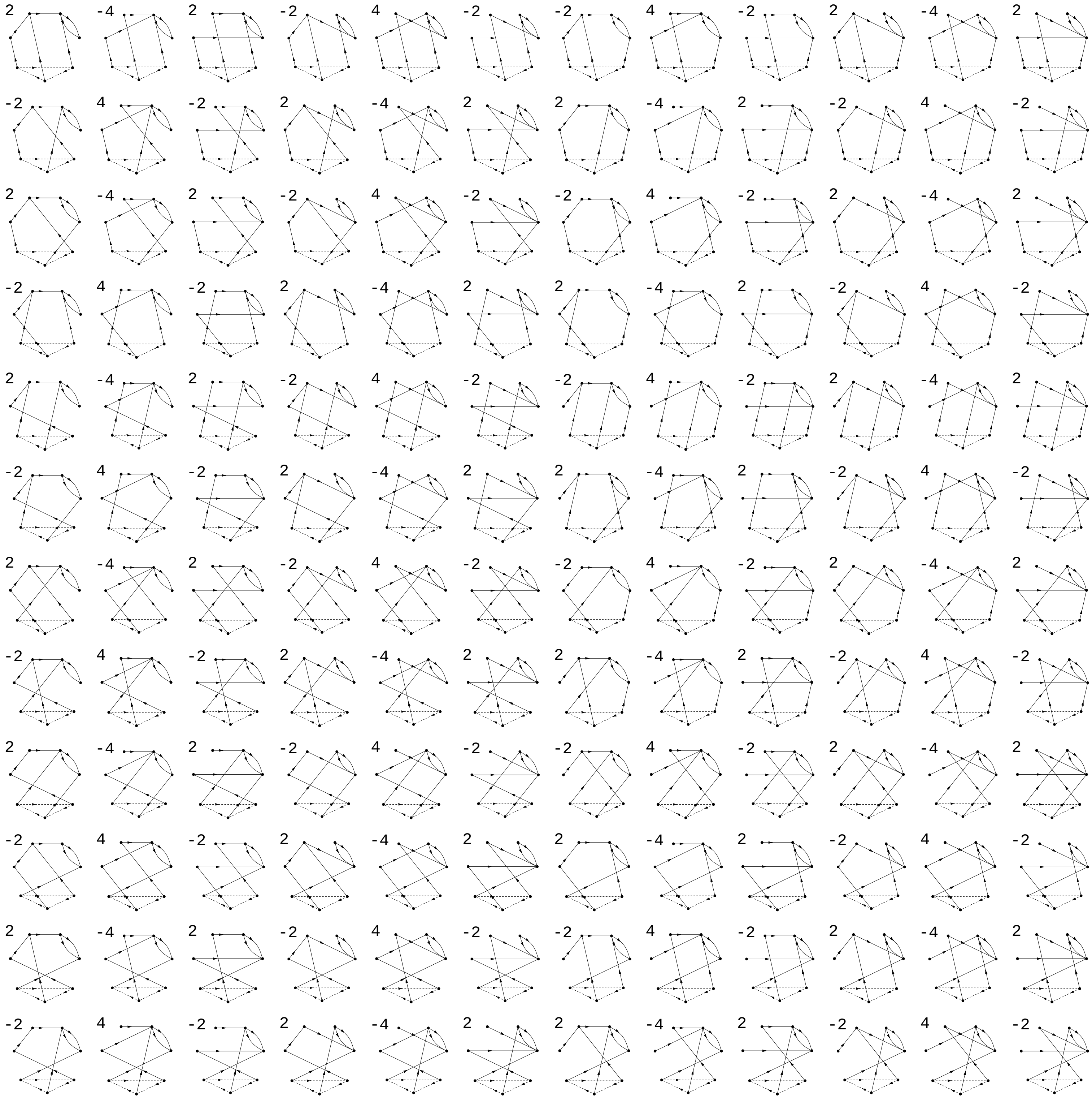
$L^A = \sum_b (-1)^{g(b)}$  $= \sum_b$ 

- loop changes

$\Phi\, L^A \to$  $\sim \Big[\underbrace{\sum_{b=1}^{N}}_{N} + \underbrace{\sum_{b=N+1}^{N+\bar{N}} (-1)}_{-\overline{N}} \Big]$  $\to 2N$ 

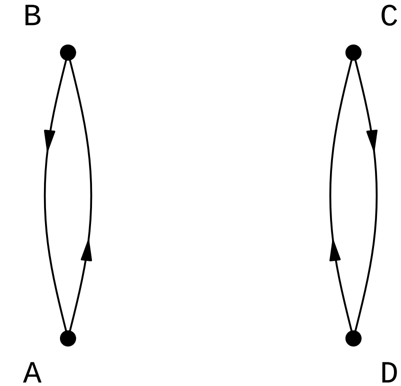
4-point functions in $psl(N|N)$ via SUSY current algebra

$$\langle J_1^A J_2^B J_3^C J_4^D \rangle =$$



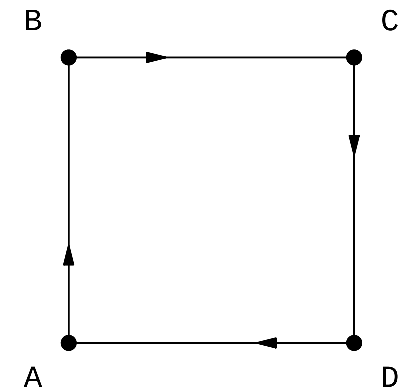
$$+\dots$$

4-point functions in $psl(N|N)$ at 3 loop



- index-choice $\{(1,2),(2,1),(3,4),(4,3)\}$ gives

$$\langle J_1^A J_2^B J_3^C J_4^D \rangle = \frac{1}{z_{12}z_{23}z_{34}z_{41}} \left\{ \frac{k^2\left(1+2\lambda^2 k^2\right)}{1-\theta} - 2\lambda^2 k^2 (1-4\lambda k) \left[\ln|\theta|^2 + \frac{(1-\theta)}{\theta} \ln|1-\theta|^2 \right] + O(\lambda^4) \right\}$$



- index-choice $\{(1,2),(2,3),(3,4),(4,1)\}$

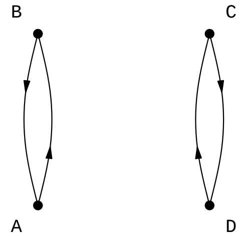
$$\langle J_1^A J_2^B J_3^C J_4^D \rangle = \frac{1}{z_{12}^2 z_{34}^2} \left\{ k\left(1+\lambda^2 k^2\right)(1-\theta) - \lambda^2 k^3 (1-\lambda k) \left[(1-\theta)^2 \ln\left|\frac{\theta}{1-\theta}\right|^2 + \ln|\theta|^2 \right] + 2\lambda^3 k^2 \frac{1-\theta}{\theta}(2-\theta) \left[\ln(\theta\bar{\theta}) \ln\left(\frac{1-\bar{\theta}}{1-\theta}\right) + 2\,\mathrm{Li}_2(\bar{\theta}) - 2\,\mathrm{Li}_2(\theta) \right] + O(\lambda^4) \right\}$$

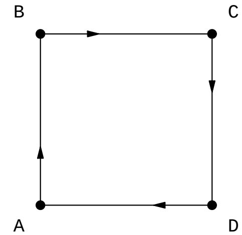
$$\theta := \frac{z_{13}z_{24}}{z_{23}z_{14}}$$

Constraints on the 4-point function

$$\langle J_1^A J_2^B J_3^C J_4^D \rangle^{\text{index choice 1}} =: \frac{1}{z_{12}^2 z_{34}^2} \mathscr{F}^{(1)}(\theta, \bar{\theta})$$

$$\langle J_1^A J_2^B J_3^C J_4^D \rangle^{\text{index choice 2}} =: \frac{1}{z_{12} z_{23} z_{34} z_{41}} \mathscr{F}^{(2)}(\theta, \bar{\theta})$$





Constraints $\mathscr{F}_1$

- Exchange of points 1 and 2: $\mathscr{F}^{(1)}(\theta) = \mathscr{F}^{(1)}(\theta^{-1})$
- For psl$(1|1)$, $f_C^{AB} \to 0$.

$$\left\langle J_1^{\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)} J_2^{\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)} J_3^{\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)} J_4^{\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)} \right\rangle = \frac{k^2}{(1-\lambda^2 k^2)^2} \left[\frac{1}{z_{12}^2 z_{34}^2} - \frac{1}{z_{14}^2 z_{23}^2} \right]$$

- since valid for all N, $\mathscr{F}^{(1)}(\theta) - (\theta-1)^2 \mathscr{F}^{(1)}\left(\frac{\theta-1}{\theta}\right) = \frac{k^2}{(1-\lambda^2 k^2)^2} \theta(2-\theta)$

Constraints $\mathscr{F}_2$

- cyclic rotation invariance: $\mathscr{F}^{(2)}(\theta) = \mathscr{F}^{(2)}\left(\frac{\theta}{\theta-1}\right)$
- group theory $(1-\theta)\mathscr{F}^{(2)}(1-\theta) - \mathscr{F}^{(2)}\left(\frac{1}{\theta}\right) + \theta \mathscr{F}^{(2)}(\theta) = 0$

Beyond 3-loop order: reconstruction procedure

- 4-point function is function of cross-ratio $\zeta = \frac{z_{12}z_{34}}{z_{23}z_{14}}$

$$\langle J_1^A J_2^B J_3^C J_4^D \rangle = \frac{1}{z_{12}z_{23}z_{34}z_{14}} \mathscr{F}(\zeta, \bar{\zeta})$$

- take one anti-holomorphic derivative

$$\partial_{\bar{z}_1} \langle J_1^A J_2^B J_3^C J_4^D \rangle = \frac{1}{z_{12}z_{23}z_{34}z_{14}} \left(\frac{1}{\bar{z}_{12}} - \frac{1}{\bar{z}_{14}} \right) \underbrace{\bar{\zeta} \frac{\partial}{\partial \bar{\zeta}} \mathscr{F}(\zeta, \bar{\zeta})}_{\mathscr{L}(\zeta, \bar{\zeta})}$$

- (partially) reconstruct $\mathscr{F}(\zeta, \bar{\zeta})$

$$\mathscr{F}(\zeta, \bar{\zeta}) = a(\zeta) + \int \mathrm{d}\bar{\zeta} \, \frac{\mathscr{L}(\zeta, \bar{\zeta})}{\bar{\zeta}}$$

- interesting because $\partial_{\bar{z}}$ kills one integration:

$$\partial_{\bar{z}_1} J_1^b(z_1, \bar{z}_1) \Phi(z_2, \bar{z}_2) = \partial_{\bar{z}_1} \frac{1}{z_1 - z_2} \Psi^b(z_2, \bar{z}_2) + \cdots$$
$$= \delta^2(z_1 - z_2) \Psi^b(z_2, \bar{z}_2) + \ldots$$

4-th order

$$\mathscr{F}(\zeta,\bar{\zeta}) = a(\zeta) + b(\zeta)\ln(\zeta\bar{\zeta}) + c(\zeta)\ln\big((1+\bar{\zeta})(1+\zeta)\big)$$

$$+d(\zeta)\left[\ln(1+\zeta)(1+\bar{\zeta})\ln\frac{\bar{\zeta}}{\zeta} + 2\,\mathrm{Li}_2\big(1+\bar{\zeta}\big) - 2\,\mathrm{Li}_2(1+\zeta)\right]$$

$$+\int\frac{1}{\bar{\bar{\zeta}}}\int\frac{1}{(1+\bar{\zeta})^2}\int\frac{1}{\bar{\bar{\zeta}}}\int\frac{\mathscr{K}(\zeta,\bar{\zeta})}{(1+\bar{\zeta})^2}$$

Conclusion

- electrons with random-gauge and random-mass disorder have a conformal sector, whose group-theoretical representation is $psl(N|N)$
- there are logarithmic operators in this sector. We have identified a logarithmic pair L^A and M^A.
- explicit results for the 4-point functions $\langle JJJJ \rangle$ (3-loop order), and $\langle JJJ\bar{J} \rangle$, $\langle JJ\bar{J}\bar{J} \rangle$ (2-loop order)
- this allows to analyze the operator content in the different channels, and to check conjectures on the structure of the 4-point functions.
- calculations can also be done with replicas, even bosonic replicas.