BOUNDARY CONDITIONS IN GEOMETRICAL CRITICAL PHENOMENA

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BOUNDARY CONDITIONS AND RENORMALISATION

- Renormalisation group: Flow from one critical point to another upon changing the bulk coupling constant(s)

- Right at bulk critical point: Flow from one RG invariant boundary condition to another upon changing the surface coupling constant(s)

- In two dimensions: Conformal field theory (CFT)

> How to characterise conformally invariant boundary conditions in two dimensional statistical models?

CLASSIFICATION OF CIBC IN TWO DIMENSIONS

- Many statistical models (Ising, 3-state Potts,...) are described by unitary minimal CFT
  - Finite number of fundamental local operators (primaries)
  - All critical exponents are known (Kac table)
- CIBC means no energy-momentum flow across boundary

1-to-1 correspondence between CIBC and primaries [Cardy]

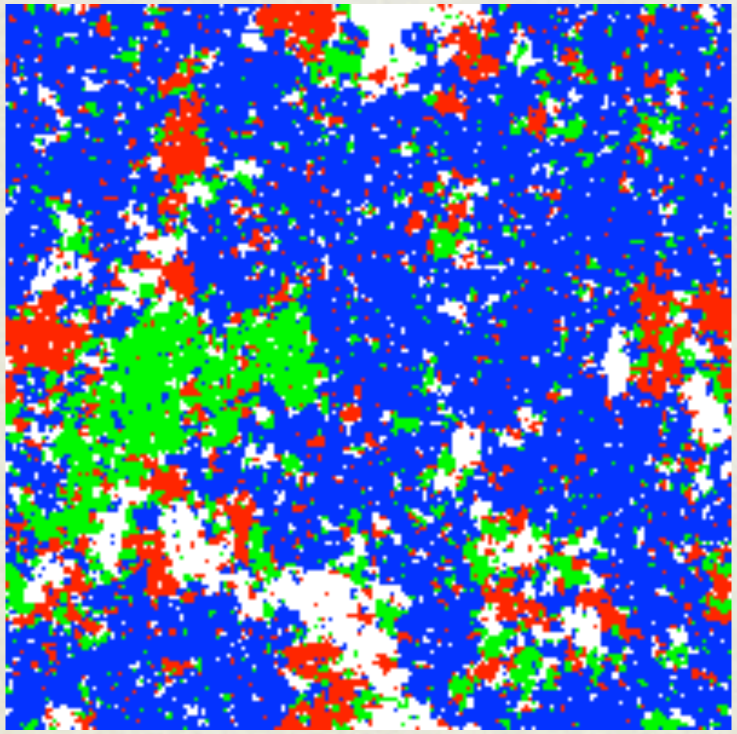
OUTLINE OF RESULTS

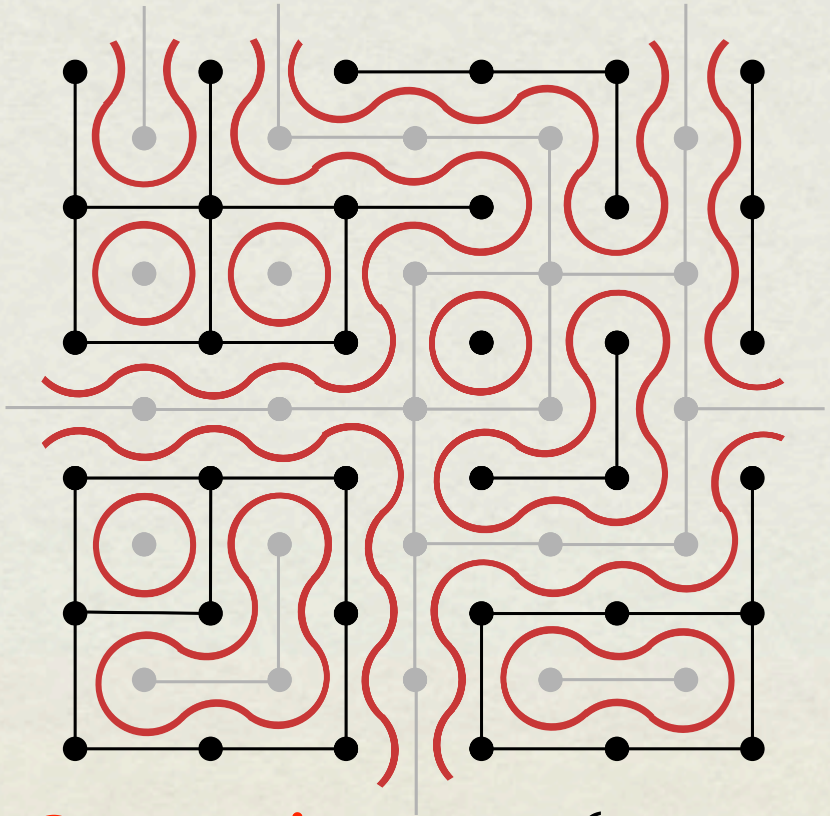
- Continuous families of CIBC in non-unitary CFT
  - Simple and well-known models: Potts and O(n)
  - The CIBC have a clear geometrical meaning, in terms of loops, clusters, and domain walls
  - Exactly known critical exponents and partition functions
  - Geometrical applications: fractal dimensions, crossing probabilities

Q-STATE POTTS MODEL

$$H = -J \sum_{\langle ij \rangle} \delta_{\sigma_i,\sigma_j} \;\text{ with } \sigma_i = 1, 2, \dots, Q$$

- Two geometrical interpretations: spins and FK clusters/loops





- Both make sense also for Q non-integer (non-unitary case)

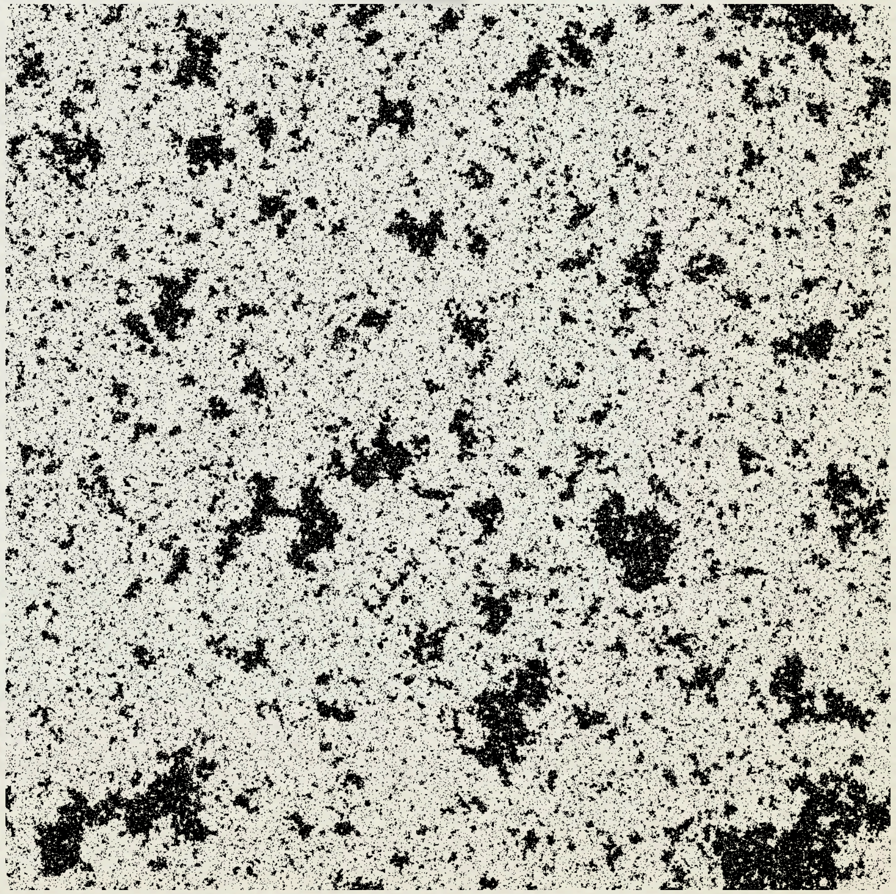
FORTUIN-KASTELEYN CLUSTER EXPANSION

- We have $\mathrm{e}^{J\delta(\sigma_i,\sigma_j)} = 1 + v\delta(\sigma_i,\sigma_j)$ with $v = \mathrm{e}^J - 1$
- To compute Z, expand out: $\prod_{\langle ij \rangle} \left(1 + v\delta(\sigma_i,\sigma_j)\right)$
- Let the v-terms define an edge subset $A \subseteq \langle ij \rangle$
- This gives $Z(Q,v) = \sum_{A \subseteq \langle ij \rangle} Q^{k(A)} v^{|A|}$
- Here $k(A)$ is the number of connected components
- Indeed Q is only a parameter, hence can take $Q \in \mathbb{R}$

O(N) LOOP MODEL

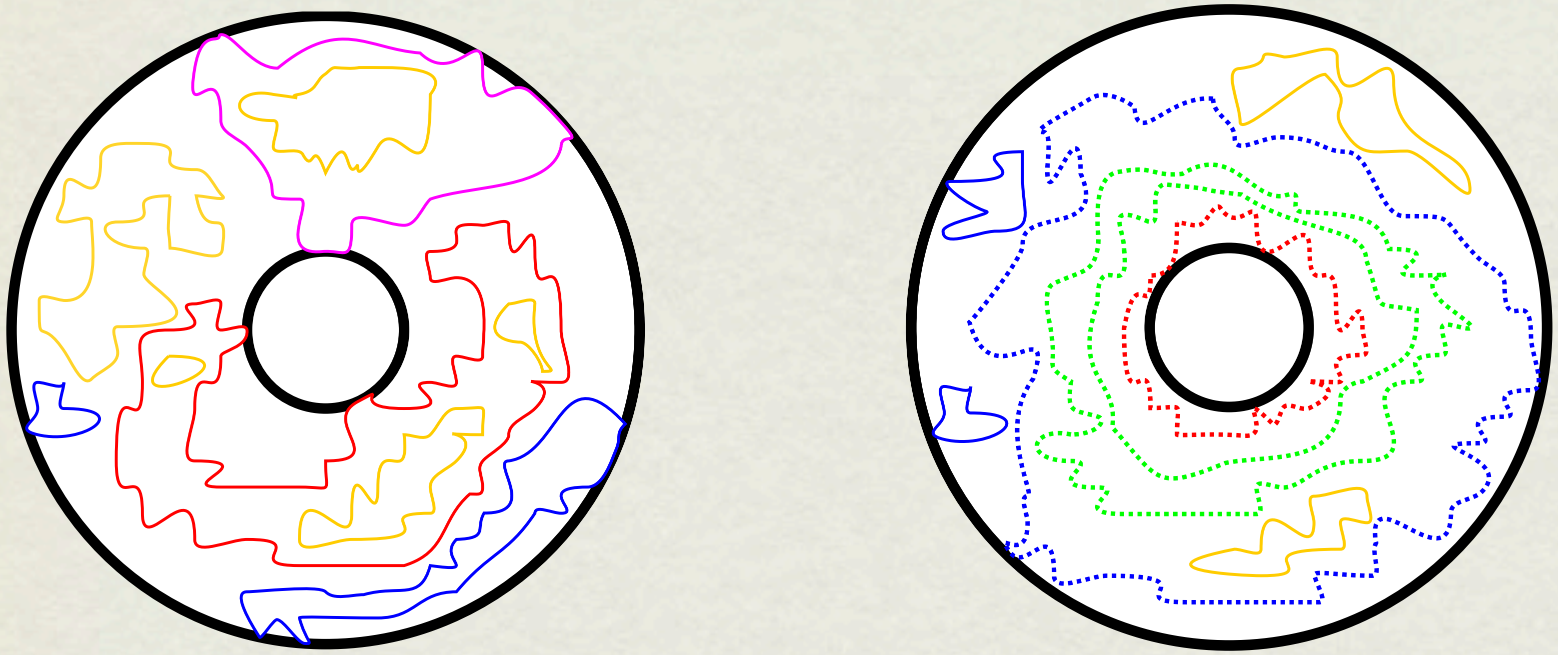
$$Z = \mathrm{tr} \left[\prod_{\langle ij \rangle} (1 + x S_i^\alpha S_j^\alpha) \right] \quad \text{with } \alpha = 1, 2, \ldots, n$$

- Expands as $Z = \sum_{\text{loops}} x^{\text{length}} n^{\#loops}$
  - $n = 1$: Ising model
  - $n \to 0$: Self-avoiding walk
- "Dilute" critical point at $x = x_{\mathrm{c}}$



THE NEW CIBC

- Define models on annulus, giving modified weight to loops touching one or the other boundary, or both:



- Weight also depends on whether the loop encircles the hole
- CIBC for any real value of these 7 different boundary weights

LINK WITH LATTICE ALGEBRAS

- Loops obtained from the Temperley-Lieb (TL) algebra
- Distinguishing boundary-touching loops is natural within the one- and two-boundary extensions of the TL algebra [Martin-Saleur, de Gier-Nichols,...]
- The correct parameterisation of the loop weights follow from representation theory for these algebras

PARAMETERISATION OF THE RESULTS (CFT)

- Bulk loops: $n = 2\cos\gamma$
- Corresponding Coulomb gas coupling: $g = 1 \pm \gamma/\pi$
- Central charge: $c = 1 - \frac{1}{6}\left(\sqrt{g} - 1/\sqrt{g}\right)^2$
- Critical exponents from Kac formula:

$$h_{r,s} = \begin{cases} \frac{(gr-s)^2-(g-1)^2}{4g}, & g \geq 1 \text{ (Dilute O}(n)) \\ \frac{(r-gs)^2-(1-g)^2}{4g}, & g < 1 \text{ (Dense O}(n) \text{ or Potts)} \end{cases}$$

TWO-BOUNDARY PARTITION FUNCTION

- On $T \times L$ annulus, setting $q = \exp(-\pi T/L)$
- Loops touching one boundary: $n_1 = \dfrac{\sin\left((r_1+1)\gamma\right)}{\sin(r_1\gamma)}$
- ...and both: $n_{12} = \dfrac{\sin\left((r_1+r_2+1-r_{12})\frac{\gamma}{2}\right)\sin\left((r_1+r_2+1+r_{12})\frac{\gamma}{2}\right)}{\sin(r_1\gamma)\sin(r_2\gamma)}$
- Continuum-limit partition function in dense/Potts case:

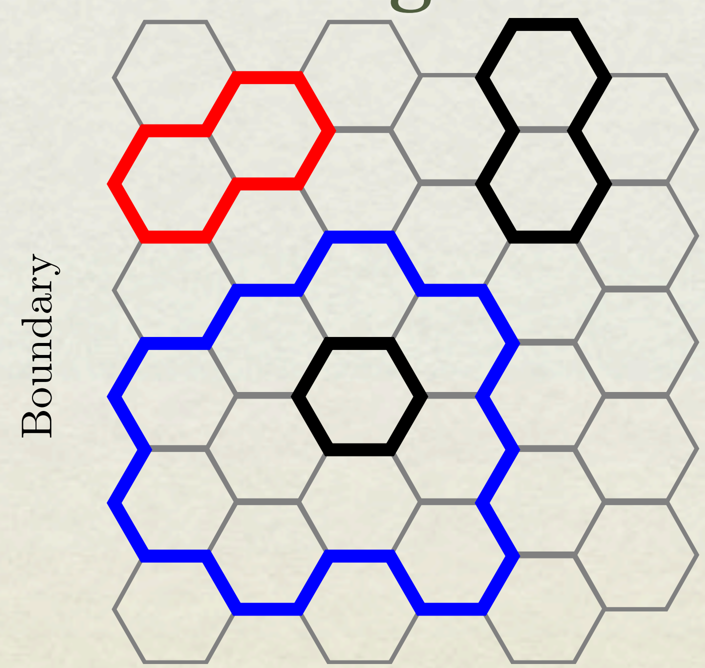
$$Z = \frac{q^{-c/24}}{\prod_{p=1}^{\infty}(1-q^p)}\left(\sum_{n=-\infty}^{\infty} q^{h_{r_{12}-2n,r_{12}}} + \sum_{\epsilon_1,\epsilon_2=\pm 1}\sum_{k=1}^{\infty} D_k^{(\epsilon_1,\epsilon_2)}\sum_{n=0}^{\infty} q^{h_{\epsilon_1 r_1+\epsilon_2 r_2-1-2n,\epsilon_1 r_1+\epsilon_2 r_2-1+k}}\right)$$

DILUTE MODEL WITH SURFACE ANISOTROPY

- Two types of boundary loops, with weights n_1 and $n - n_1$
- Surface interaction with anisotropy parameter Δ :

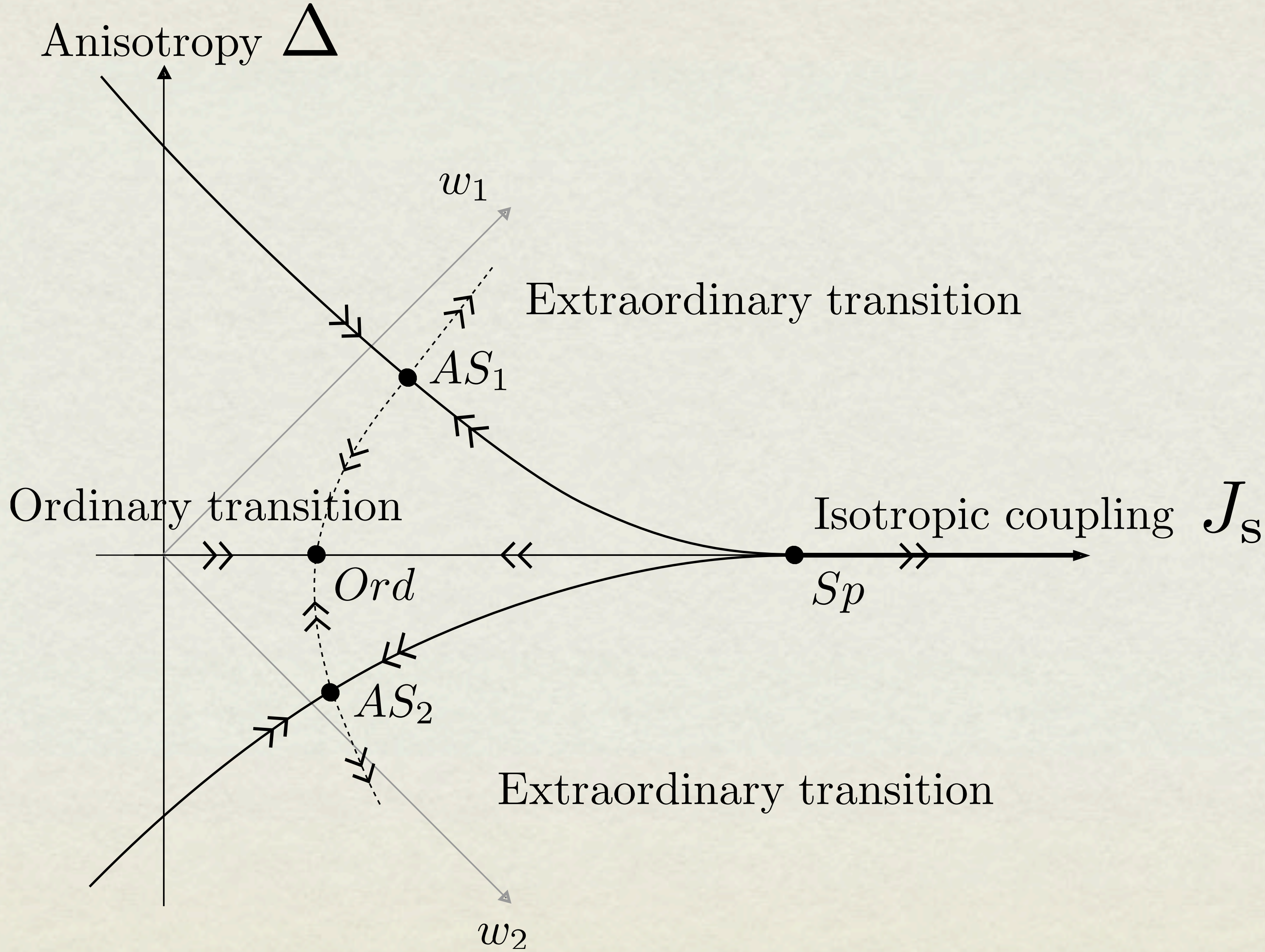
$$H_{\mathrm{s}} = -J_{\mathrm{s}} \sum_{\langle ij \rangle_s} \left((1+\Delta) \sum_{\alpha=1}^{n_1} S_i^\alpha S_j^\alpha + (1-\Delta) \sum_{\beta=n_1+1}^{n} S_i^\beta S_j^\beta \right)$$

- Surface monomers must have type-dependent weight
  - Integrable choice leading to anisotropic special transition



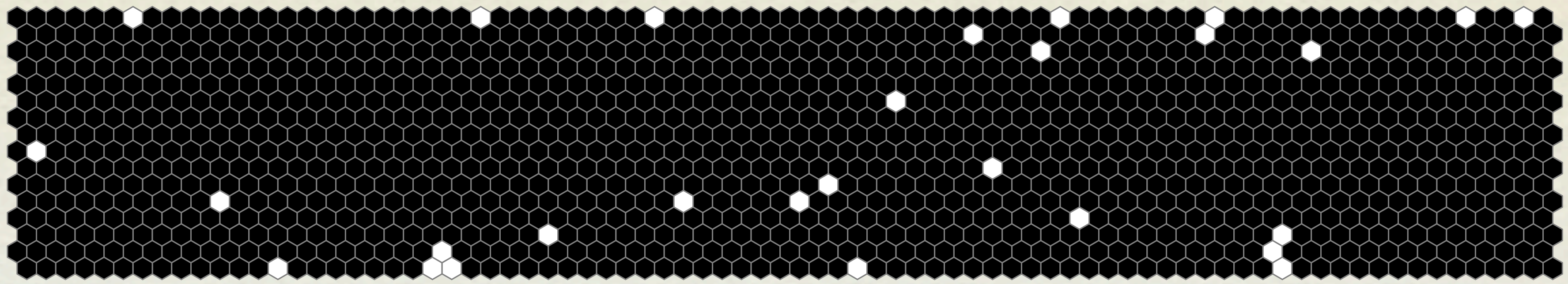
SURFACE ANISOTROPY: THE PHASE DIAGRAM

- First found in epsilon expansion [Diehl-Eisenriegler]



APPLICATION TO ISING CROSSING EVENTS

$J > J_c$:



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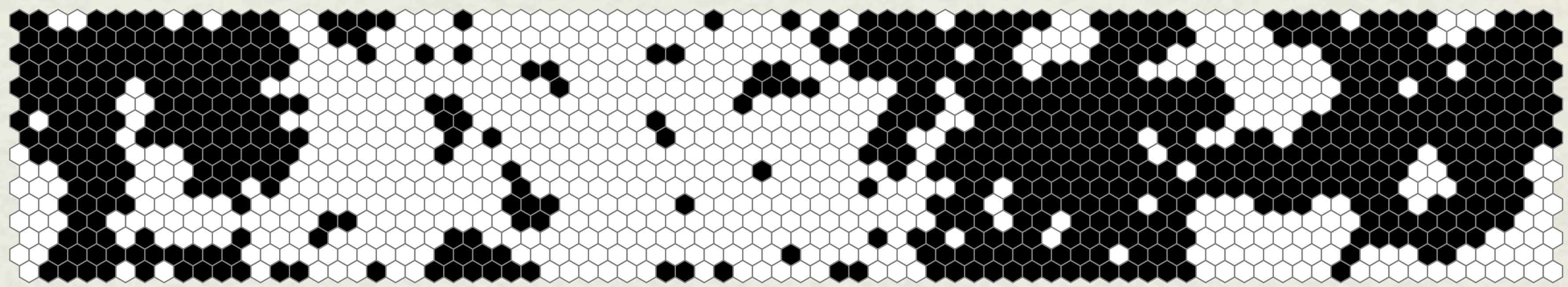

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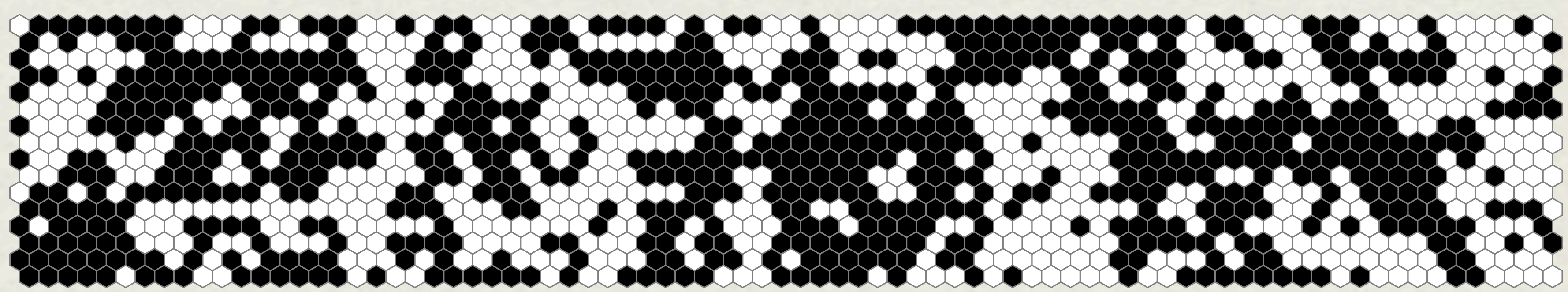


- Probability (≥ 1 clusters crossing the $T \times L$ annulus)

$$P_c(\tau) = \frac{\eta(i\tau)\eta(i\tau/12)^2}{\eta(i\tau/2)^2\eta(i\tau/6)} \text{ with } \tau = T/L$$

CROSSING EVENTS IN PERCOLATION

- Probabilities for j clusters wrapping the annulus, refined according to whether they touch no/one/both rims
  - For instance, in a square geometry:

j	$\sum_{\alpha,\beta} P_j^{\alpha\beta}$	P_j^{++}	$P_j^{-+} = P_j^{+-}$	P_j^{--}
0	0.636454001888			
1	0.361591025956	0.277067148156	0.041313949815	0.0018959781702
2	0.001954814340	0.001895978170	0.000029339472	0.0000001572261
3	0.000000157814	0.000000157226	0.000000000294	0.0000000000002

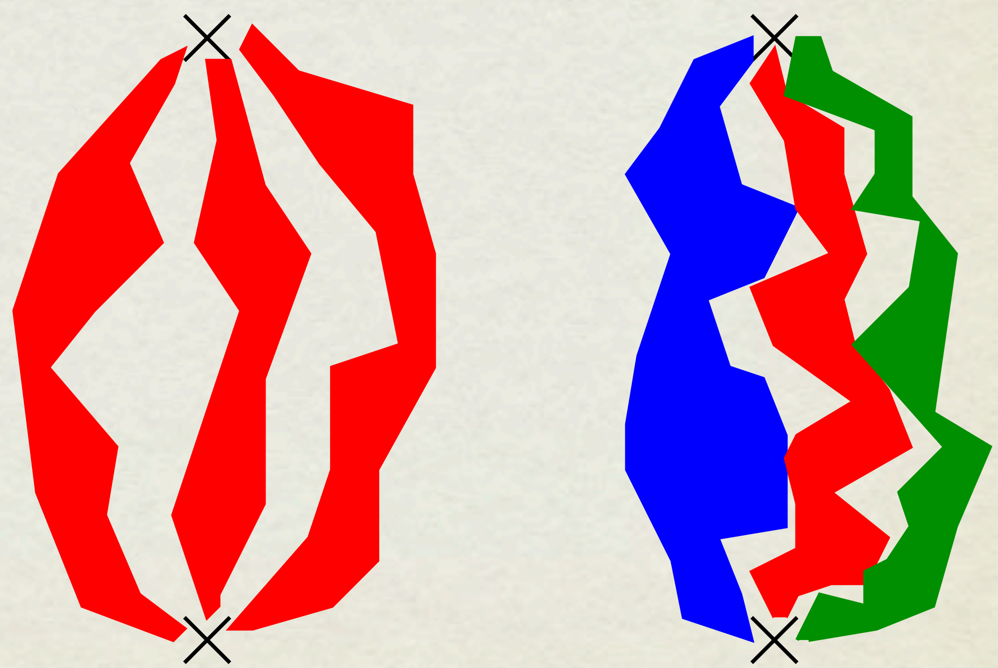
- Result without refinement: [Cardy]

POTTS DOMAIN WALLS

- Expand Z_{Potts} in powers of e^J. Makes sense even for $Q \notin \mathbb{N}$
- Let ℓ clusters propagate in $\mathbb{H}$, starting at O. Write $\ell = \ell_1 + \ell_2$. The first cluster contributes to ℓ_1. Each remaining cluster is in ℓ_1 (resp. in ℓ_2) if the cluster on its left has a different (resp. the same) colour.

Critical exponent:

$$h_{1+2(\ell_1-\ell_2),1+4\ell_1}$$



(a) $(\ell_1, \ell_2) = (1, 2)$ **(b)** $(\ell_1, \ell_2) = (3, 0)$