IS THE TUTTE 5-FLOW CONJECTURE ALMOST FALSE?

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ANY PLANAR MAP IS 4-COLOURABLE

VERTEX COLOURING

- Every planar graph admits a 4-vertex colouring
  - Conjectured by Augustus de Morgan (23 Oct. 1852)
  - Proved by K.I. Appel and W. Haken (July 1976)
- Quantitative version [Birkhoff 1912]
  - Chromatic polynomial $\chi_G(Q)$ = number of colourings
  - Polynomial in Q, hence can consider $Q \in \mathbb{R}$ or $Q \in \mathbb{C}$

CHROMATIC POLYNOMIAL

- Let $G = (V, E)$ be any planar graph
- [Birkhoff-Lewis 1946]: $\chi_G(Q) \neq 0$ for $Q \in [5, \infty)$
- Conjecture [Birkhoff-Lewis]: $\chi_G(Q) \neq 0$ for $Q \in [4, \infty)$
- [Beraha-Kahane 1979]: Complex roots accumulate at $Q = 4$
    - "Is the four-colour conjecture almost false?"
- [Royle 2008]: Real roots accumulate at $Q = 4$

NOWHERE ZERO FLOW

- Let $G = (V, E)$ be an *arbitrary* bridgeless graph
- Nowhere zero Q-flow: Map $\phi : E \mapsto \mathbb{Z}_Q \setminus \{0\}$
  - The flow variables are conserved "mod Q" at vertices
  - Edge orientation is assumed, but is immaterial
- Flow polynomial $\Phi_G(Q)$ = number of nowhere zero Q-flows
- In planar case, obviously $\chi_{G^*}(Q) = Q\,\Phi_G(Q)$ [bijection!]

RELATION TO THE POTTS MODEL

- Potts model partition function: $Z_G(Q,v) = \sum_{A\subseteq E} v^{|A|} Q^{k(A)}$
  - Here $v = \mathrm{e}^K - 1$, and $k(A)$ = #connected components
- $\Phi_G(Q) = \sum_{A\subseteq E} (-1)^{|E|-|A|} Q^{c(A)}$
  - Follows since total # Q-flows = $Q^{c(E)}$, with $c(E)$ the cyclomatic number, then exclusion-inclusion of zero edges. Then use $k(A) = |V| - |A| + c(A)$ to obtain:

$$\boxed{\Phi_G(Q) = (-1)^{|E|} Q^{-|V|} Z_G(Q, v=-Q)}$$

$$\chi_G(Q) = Z_G(Q,-1)$$

RESULTS ON THE FLOW POLYNOMIAL

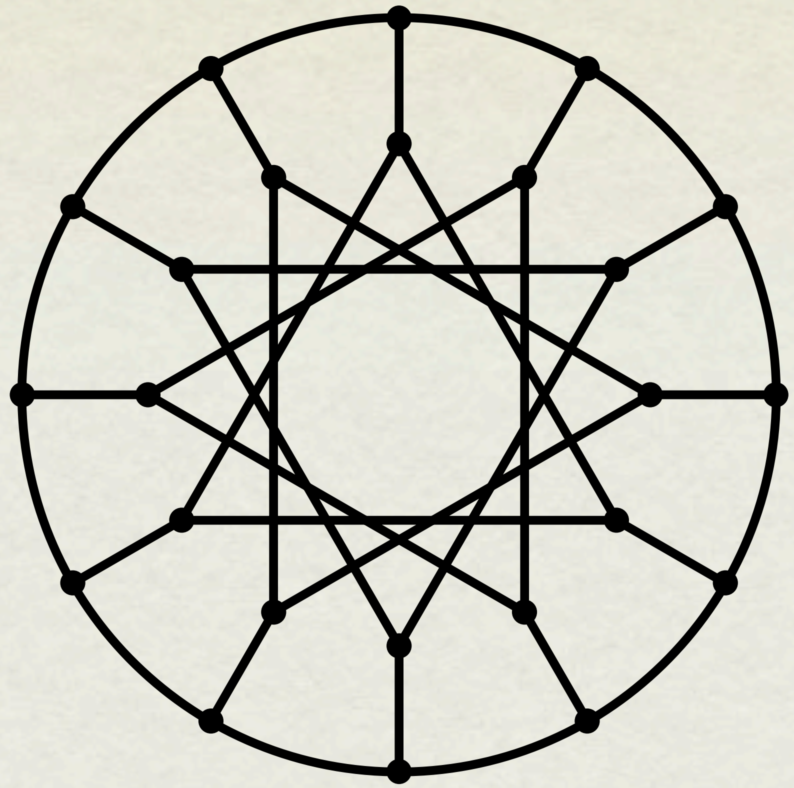
- Let G be any bridgeless graph
- [Seymour 1981]: $\Phi_G(6) > 0$.
- Conjecture [Tutte 1954]: $\Phi_G(5) > 0$.
- Conjecture [Welsh]: $\Phi_G(Q) \neq 0$ for $Q \in (4, \infty)$. (False!)
- Conjecture [Haggard-Pearce-Royle]:

  $\Phi_G(Q) \neq 0$ for $Q \in (5, \infty)$. (False!)
- Q: Is there any upper limit on real flow roots?

WILLIAM THOMAS TUTTE (1917-2002)

- Broke the German army's "Fish" code at Bletchley Park (1941)
- A contribution to the theory of chromatic polynomials (1954)
- A census of planar maps (1963)

GENERALISED PETERSEN GRAPHS

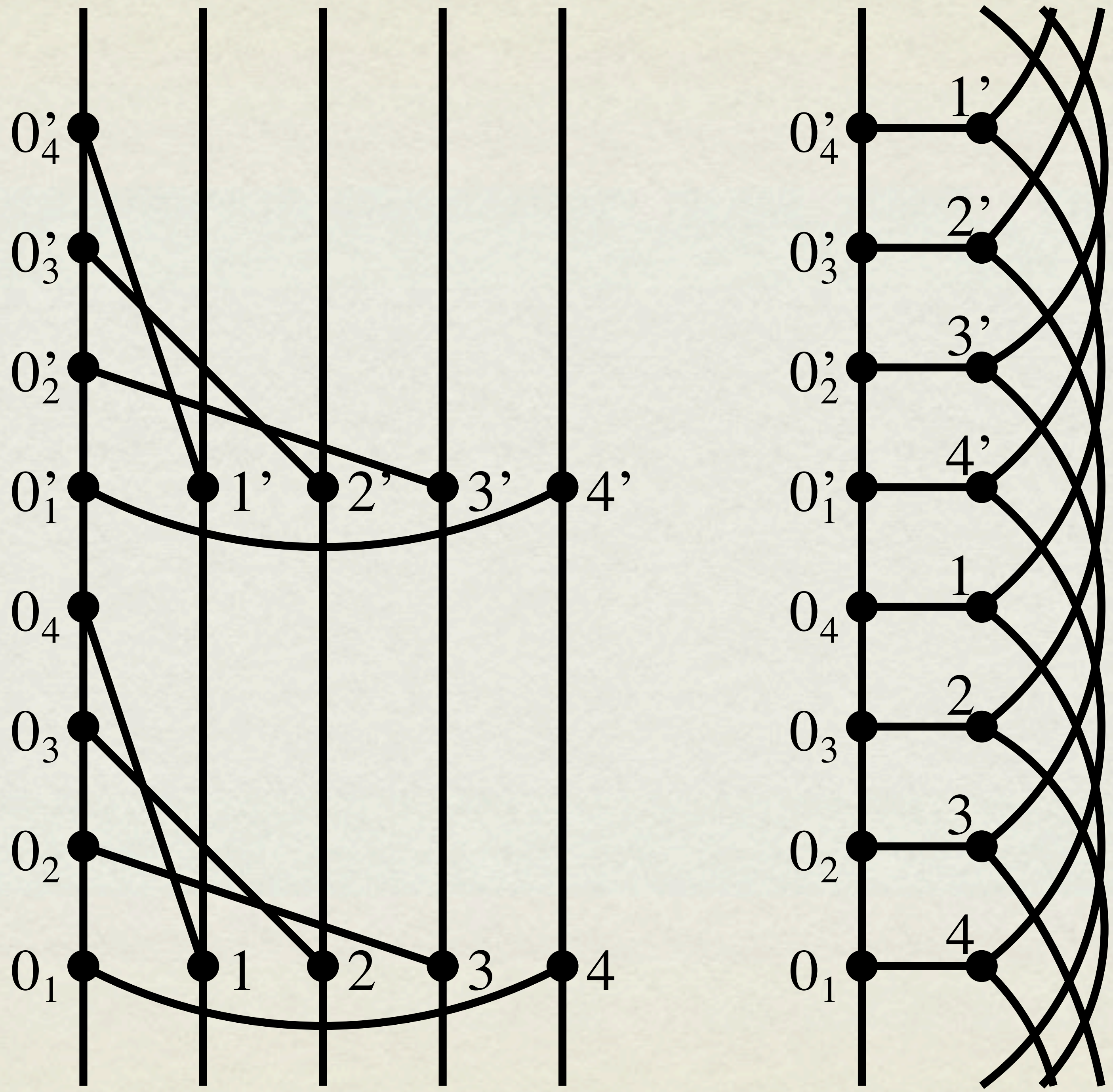


$G(12, 4)$

- Large real flow roots have been found in the family $G(n, k)$.
- $G(16, 6)$ gives roots $Q_1 \approx 4.0252$ and $Q_2 \approx 4.2331$
  - Welsh conjecture proved wrong [Royle et al, 2010]

REPRESENTATION AS LAYERED STRUCTURE

- G(n,4)

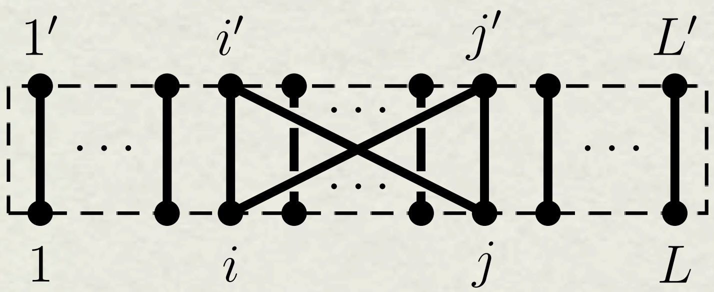


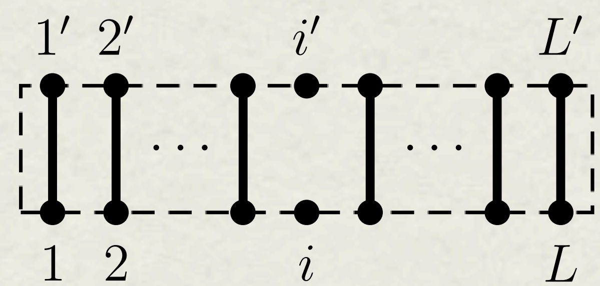
TRANSFER MATRIX CONSTRUCTION (1)

- Transfer matrices for "horizontal" and "vertical" edges:

$$\mathsf{H}_{ij} = I + v\mathsf{J}_{ij} \text{ and } \mathsf{V}_i = vI + \mathsf{D}_i$$

- Join and detach operators: act on set partitions of 2L points

$\mathsf{J}_{ij} =$ 

$\mathsf{D}_i =$ 

  - Generators of associative partition monoid $A_L(Q)$
  - Give weight Q per components: partition algebra $\mathbb{C}A_L(Q)$
  - TM for $G(nk,k)$: $\mathsf{T}_L = \mathsf{H}_{01}\left(\prod_{i=k}^{2}\mathsf{V}_0\mathsf{H}_{0i}\right)\left(\prod_{i=0}^{k}\mathsf{V}_i\right)$

TRANSFER MATRIX CONSTRUCTION (2)

- Markov trace construction: $\Phi_{G(nk,k)}(Q) = \mathrm{Tr}\ (\mathsf{T}_L)^n$
  - Glues top & bottom of diagrams and counts components
  - Decomposition on ordinary matrix traces

$$\mathrm{Tr}\ \mathsf{O}_L = \sum_{\ell=0}^{L} \sum_{\lambda \in S_\ell} \alpha_{\ell,\lambda}\ \mathrm{tr}_{\ell,\lambda}\, \mathsf{O}_L\,, \qquad Y(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$$

$$\alpha_{\ell,\lambda} = \frac{\dim\ \lambda}{\ell!} \prod_{i=0}^{\ell-1} (Q - i - \lambda_{\ell-i})\,, \qquad \dim\ \lambda = \frac{\ell!}{\prod_{x \in Y(\lambda)} h_x}$$

STATE SPACE

- The factor $\mathsf{P}_L = \prod_{i=0}^{L} \mathsf{V}_i$ in the transfer matrix is a projector that annihilates states with unmarked singletons

- Representations $A_{k+1}^{(\ell)}$ are partitions with ℓ marked distinguishable points and no unmarked singletons

$$\text{card } A_k^{(\ell)} = k![z^k](\mathrm{e}^z - 1)^\ell \exp(\mathrm{e}^z - 1 - z)$$

- Sumrule:

$$\boxed{\sum_{\ell}^{k} \frac{1}{\ell!} \left(\sum_{\lambda \in S_\ell} \alpha_{\ell,\lambda} \dim \lambda \right) \text{card } A_k^{(\ell)} = (Q-1)^k}$$

COMPUTATION OF P(119,7)

- Naive transfer matrix has dimension 10 480 142 147.
- Using the state space decomposition, we have to compute traces of matrices with dimension up to 11 816.

$$\begin{aligned}\Phi_{G(119,7)} &= (Q-1)(Q-2)(Q-3)P_{117}(Q)\\ P_{117}(Q) &= Q^{117} - 351Q^{116} + 61191Q^{115} - \ldots\end{aligned}$$

- $\Phi_{G(119,7)} = 0$ for $Q_1 \approx 5.00002$ and $Q_2 \approx 5.16534$
    - The [Haggard-Pearce-Royle] conjecture is false!

BERAHA-KAHANE-WEISS THEOREM

- Suppose (as is the case here): $Z(Q) = \sum_{k=1}^{M} \alpha_k(Q)\lambda_k(Q)^n$
- Two ways of having limiting points of $Z(Q) = 0$:
  - (a) Two dominant eigenvalues are equimodular
    - Leads to continuous curve of limiting points
  - (b) One dominant eigenvalue whose amplitude vanishes
    - Leads to isolated limiting point

PHASE DIAGRAM AND TUTTE'S CONJECTURE

- When $Q < Q_{\mathrm{c}}$, the ℓ sector is dominant for
  $2(\ell - 1) < Q < 2\ell$
- For $k \geq 6$ we have $Q_{\mathrm{c}} > 5$
  $\alpha_{3,(3)}(Q) = \frac{1}{8} Q(Q-1)(Q-5)$
- Hence isolated limiting point in $Q = 5$
- [JJ-Salas 2010]: Tutte conjecture "almost false".