Logarithmic correlations in critical percolation

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Introduction

Logarithms in critical phenomeana

- Scale invariance $\Rightarrow$ correlations are power-law or logarithmic
- Two possibilities for logarithms:
  1. Marginally irrelevant operator:
     Gives logs upon approach to fixed point theory.
  2. Dilatation operator not diagonalisable:
     Logs directly in the fixed point theory. Subject of this talk.

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Where do such logarithms appear

- CFT with $c = 0$ [Gurarie, Gurarie-Ludwig, Cardy, ...]
  - Percolation, self-avoiding polymers ($c \to 0$ catastrophe)
  - Quenched random systems (replica limit catastrophe)
- Logarithmic minimal models [Read-Saleur, Pearce-Rasmussen-Zuber]
- For any $d \leq$ upper critical dimension

Logarithms and non-unitarity [Cardy 1999]

Standard unitary CFT

- Expand local density $\Phi(r)$ on sum of scaling operators $\varphi(r)$

$$\langle \Phi(r)\Phi(0)\rangle \sim \sum_{ij} \frac{A_{ij}}{r^{\Delta_i+\Delta_j}}$$

- $A_{ij} \propto \delta_{ij}$ by conformal symmetry [Polyakov 1970]
- $A_{ii} \geq 0$ by reflection positivity
- Hence only power laws appear

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The non-unitary case

- Cancellations may occur
- Suppose $A_{ii} \sim -A_{jj} \to \infty$ with $A_{ii}(\Delta_i - \Delta_j)$ finite
- Then leading term is $r^{-2\Delta_i} \log r$

Jordan cells and indecomposability parameters

Logarithmic pair ($\phi(z)$,$\psi(z)$) with conformal weight h

- Dilatation op. $L_0 = \begin{pmatrix} h & 1 \\ 0 & h \end{pmatrix}$ in basis (ϕ,ψ) is indecomposable
- Global conformal invariance fixes [Gurarie 1993]

$$\langle \phi(z)\phi(0)\rangle = 0\,, \quad \langle \phi(z)\psi(0)\rangle = \frac{\beta}{z^{2h}}\,, \quad \langle \psi(z)\psi(0)\rangle = \frac{\theta - 2\beta \log z}{z^{2h}}$$

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Indecomposability parameter β

- $\theta = 0$ by change of basis, but $\beta = \langle\psi|\phi\rangle$ is fundamental quantity
- $\psi(z)$ is the logarithmic partner of the null-field $\phi(z)$
- Indecomposability appears already in Temperley-Lieb algebra [Read-Saleur, Pearce-Rasmussen-Zuber]
  - Measure β by numerics on lattice models [Dubail-JJ-Saleur, Vasseur-JJ-Saleur]

Algebraic structure

Staggered (projective) modules

- Reducible yet indecomposable representation of Virasoro
- Staggered module [Rohsiepe 1996; Nahm, Gaberdiel, Kausch, Mathieu, Ridout, Kytölä,...]

$$\mathcal{P} = \begin{array}{ccccc} & & \psi & & \\ & \swarrow & & \searrow & \\ \xi & & & & \rho \\ & \searrow & & \swarrow & \\ & & \phi & & \end{array}$$

- $\phi(z)$ is a descendent of $\xi(z)$ at level $n \geq 0$

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Hidden treasures

- In general, several $\mathcal{P}$ glued to form more complex structures
- In general, a theory is characterised by infinitely many β

Computing β for 2D percolation (boundary case)

Colliding fields

- Boundary 4-leg operator $\Phi_{1,5}(z)$ and $T(z)$ collide when $c \to 0$
- Let $\Phi_h(z)$ be any field containing I in its OPE with itself:

$$\Phi_h(z)\Phi_h(0) \sim \frac{a_\Phi}{z^{2h}} \left[1 + \frac{2h}{c} z^2 T(0) + z^{h_t} \Phi_{1,5}(0) + \dots \right]$$

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Defining the logarithmic partner field $t(z)$

- Unacceptable divergence when $c \to 0$. Define a new field $t(z)$ by

$$\Phi_{1,5}(z) = \frac{2h\langle T|T\rangle}{c\beta(c)} t(z) - \frac{2h}{c} T(z)$$

where

$$\beta(c) = -\frac{\langle \phi|\phi\rangle}{h_\psi - h_\xi - n} = -\frac{\langle T|T\rangle}{h_t - 2} = -\frac{c/2}{h_{1,5} - 2}$$

Avoiding the $c \to 0$ catastrophe

- Now the OPE is finite:

$$\Phi_h(z)\Phi_h(0) \sim \frac{a_\Phi}{z^{2h}} \left[1 + \frac{h}{\beta} z^2 (T(0) \log z + t(0)) + \ldots \right]$$

- The logarithmic pair $\{T, t\}$ have the expected OPEs:

$$\langle T(z) T(0) \rangle = 0 \,, \quad \langle T(z) t(0) \rangle = \frac{\beta}{z^4} \quad \langle t(z) t(0) \rangle = \frac{\theta - 2\beta \log z}{z^4}$$

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Computing the indecomposability parameter

- Recall that $\beta = \langle t | T \rangle$ so that

$$\beta = \lim_{c \to 0} \beta(c) = -\frac{5}{8}$$

Correlators in bulk percolation in any dimension

Reminders

- Two and three-point functions fixed in any d by global conformal invariance
- This is supposing only conformal invariance!
- Extra discrete symmetries must be taken into account as well
- Physical operators are irreducible under such symmetries [Cardy 1999]
  - $O(n)$ symmetry for polymers ($n \to 0$)
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Remainder of this talk

- Two and three-point functions in bulk percolation
- Limit $Q \to 1$ of Potts model with S_Q symmetry
- Structure for any d; but universal prefactors only for $d = 2$

Potts model

- Hamiltonian $H = J \sum_{\langle ij \rangle} \delta(\sigma_i, \sigma_j)$ with $\sigma_i = 1, 2, \ldots, Q$
- Operators must be irreducible under the S_Q symmetry

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Operators acting on one spin

- Most general one-spin operator: $\mathcal{O}(r_i) \equiv \mathcal{O}(\sigma_i) = \sum_{a=1}^{Q} \mathcal{O}_a \delta_{a,\sigma_i}$

$$\underbrace{\delta_{a,\sigma_i}}_{\text{reducible}} = \underbrace{\frac{1}{Q}}_{\text{invariant}} + \underbrace{\left(\delta_{a,\sigma_i} - \frac{1}{Q} \right)}_{\varphi_a(\sigma_i)}$$

- Dimensions of representations: $(Q) = (1) \oplus (Q-1)$
  - Identity operator $1 = \sum_a \delta_{a,\sigma_i}$
  - Order parameter $\varphi_a(\sigma_i)$ satisfies the constraint $\sum_a \varphi_a(\sigma_i) = 0$

Operators acting on two spins

- $Q \times Q$ matrices $\mathcal{O}(r_i) \equiv \mathcal{O}(\sigma_i, \sigma_j) = \sum_{a=1}^{Q} \sum_{b=1}^{Q} \mathcal{O}_{ab} \delta_{a,\sigma_i} \delta_{b,\sigma_j}$
- The Q operators with $\sigma_i = \sigma_j$ decompose as before: $(1) \oplus (Q-1)$
- Other $\frac{Q(Q-1)}{2}$ operators with $\sigma_i \neq \sigma_j$: $(1) + (Q-1) + \left(\frac{Q(Q-3)}{2}\right)$

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Easy representation theory exercise

$$E = \delta_{\sigma_i \neq \sigma_j} = 1 - \delta_{\sigma_i,\sigma_j}$$

$$\phi_a = \delta_{\sigma_i \neq \sigma_j} \left(\varphi_a(\sigma_i) + \varphi_a(\sigma_j)\right)$$

$$\hat{\psi}_{ab} = \delta_{\sigma_i,a}\delta_{\sigma_j,b} + \delta_{\sigma_i,b}\delta_{\sigma_j,a} - \frac{1}{Q-2}(\phi_a + \phi_b) - \frac{2}{Q(Q-1)} E$$

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- Scalar E (energy), vector φ_a (order parameter) and tensor $\hat{\psi}_{ab}$ (two propagating clusters)
- Constraint $\sum_{a=1}^{Q} \phi_a = 0$ and $\sum_{a(\neq b)} \hat{\psi}_{ab} = 0$

Extension to rank-k tensors for all $k \geq 0$

$$
\begin{aligned}
t_1 &= (3\delta) - \frac{3}{Q}(1\,t_0)\,, \\
t_2 &= (6\delta) - \frac{2}{Q-2}(2t_1) - \frac{6}{Q(Q-1)}(1\,t_0)\,, \\
t_3 &= (6\delta) - \frac{1}{Q-4}(3t_2) - \frac{2}{(Q-2)(Q-3)}(3t_1) - \frac{6}{Q(Q-1)(Q-2)}(1\,t_0
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Physical interpretation

- For $k \geq 2$, operator that makes k clusters propagate
- In 2D equivalent to $2k$-leg watermelon operator ($2k$ through lines in TL algebra)

Continuum limit

Energy operator $\varepsilon_i = E - \langle E \rangle$, with $E = \delta_{\sigma_i \neq \sigma_{i+1}}$ invariant

$$\langle \varepsilon(r)\varepsilon(0) \rangle = (Q-1)\tilde{A}(Q) r^{-2\Delta_\varepsilon(Q)},$$

- All correlators of ε_i vanish at $Q = 1$ (true already on the lattice)
- In 2D: exponent $\Delta_\varepsilon(Q) = d - \nu^{-1}$ known exactly

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Two-cluster operator $\hat{\psi}_{ab}(\sigma_i, \sigma_{i+1})$

$$\langle \hat{\psi}_{ab}(r)\hat{\psi}_{cd}(0) \rangle = \frac{2A(Q)}{Q^2} \left(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} - \frac{1}{Q-2}(\delta_{ac} + \delta_{ad} + \delta_{bc} + \delta_{bd}) + \frac{2}{(Q-1)(Q-2)} \right) \times \underbrace{r^{-2\Delta_2(Q)}}_{\text{CFT part}},$$

- In 2D: exponent $\Delta_2 = \frac{(4+g)(3g-4)}{8g}$ known from Coulomb gas

Percolation limit $Q \to 1$

Avoiding the $Q \to 1$ catastrophe

- The "scalar" part of $\langle \hat{\psi}_{ab}(r) \hat{\psi}_{cd}(0) \rangle$ diverges
- But $\Delta_2 = \Delta_\varepsilon = \frac{5}{4}$ at $Q = 1$ in 2D
  - And actually $\Leftrightarrow$ $d^F_{\rm red\ bonds} = \nu^{-1}$ for all $2 \leq d \leq d_{\rm u.c.}$ [Coniglio 1982]
- So we can cure the divergence by mixing the two operators:

$$\tilde{\psi}_{ab}(r) = \hat{\psi}_{ab}(r) + \frac{2}{Q(Q-1)} \varepsilon(r).$$

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Using $\langle \hat{\psi}_{ab}\varepsilon\rangle = 0$, we find a finite limit at $Q = 1$

$$\langle \tilde{\psi}_{ab}(r)\tilde{\psi}_{cd}(0)\rangle = 2A(1)r^{-5/2}\left(\delta_{ac} + \delta_{ad} + \delta_{bc} + \delta_{bd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right) + 4A(1)\frac{2\sqrt{3}}{\pi}r^{-5/2} \times \log r,$$

where we assumed that $A(1) = \tilde{A}(1)$.

Where does the log come from?

$$\frac{1}{Q-1}\left(r^{-2\Delta_\varepsilon(Q)} - r^{-2\Delta_2(Q)}\right) \sim 2\left.\frac{\mathrm{d}(\Delta_2 - \Delta_\varepsilon)}{\mathrm{d}Q}\right|_{Q=1} r^{-5/2}\log r$$

- We need 2D only to compute this derivative (universal prefactor)

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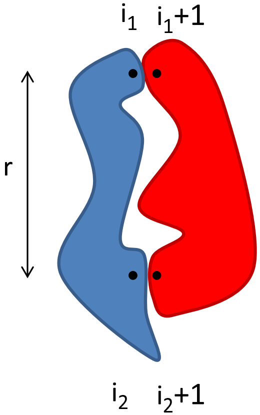
Geometrical interpretation of this logarithmic correlator?

- Idea: Translate the spin expressions into FK cluster formulation
- One can show exactly on the lattice that $\langle \varepsilon \hat{\psi}_{ab} \rangle = \langle \varepsilon \phi_a \rangle = \langle \hat{\psi}_{ab} \phi_c \rangle = 0$, and also $\langle \hat{\psi}_{ab} \rangle = \langle \phi_a \rangle = \langle \varepsilon \rangle = 0$.
- All correlators take a simple form in terms of FK clusters

For example we find:

$$\langle \hat{\psi}_{ab}(\sigma_{i_1}, \sigma_{i_1+1}) \hat{\psi}_{cd}(\sigma_{i_2}, \sigma_{i_2+1}) \rangle \propto \mathbb{P}_2(r = r_1 - r_2).$$



$$\mathbb{P}_2(r_1 - r_2) = \mathbb{P}\left[\begin{array}{l} (i_1, i_1 + 1) \notin \text{ same cluster} \\ (i_2, i_2 + 1) \notin \text{ same cluster} \\ \text{two clusters } 1 \to 2 \end{array} \right].$$

This probability should thus behave as $r^{-2\Delta_2}$

- Just like in the CFT limit, we introduce

$$\tilde{\psi}_{ab}(r_i) \equiv \tilde{\psi}_{ab}(\sigma_i, \sigma_{i+1}) = \hat{\psi}_{ab}(\sigma_i, \sigma_{i+1}) + \frac{2}{Q(Q-1)} \varepsilon(\sigma_i, \sigma_{i+1})$$

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- Exact lattice expression for $\langle \tilde{\psi}_{ab}(r_1)\tilde{\psi}_{cd}(r_2)\rangle$ at $Q = 1$
- Expression in terms of simple percolation probabilities.

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Exact two-point function of $\tilde{\psi}_{ab}$ at $Q = 1$

$$\langle \tilde{\psi}_{ab}(r_1) \tilde{\psi}_{cd}(r_2) \rangle = 2\left(\delta_{ac} + \delta_{ad} + \delta_{bc} + \delta_{bd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right) \times \mathbb{P}_2(r) + 4\left[\mathbb{P}_0(r) + \mathbb{P}_1(r) - 2\mathbb{P}_2(r) - \mathbb{P}_{\neq}^2\right].$$

Putting it all together

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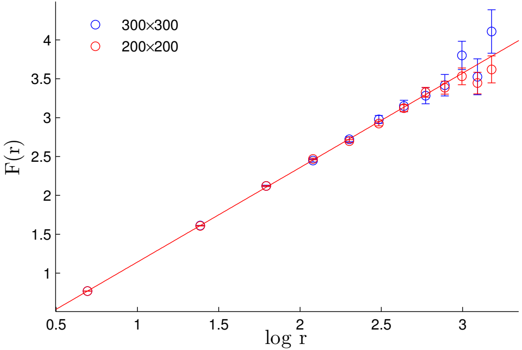
Reminder: CFT Expression

$$\langle \tilde{\psi}_{ab}(r) \tilde{\psi}_{cd}(0) \rangle = 2A(1) r^{-5/2} \left(\delta_{ac} + \delta_{ad} + \delta_{bc} + \delta_{bd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} \right) + 4A(1) \frac{2\sqrt{3}}{\pi} r^{-5/2} \times \log r,$$

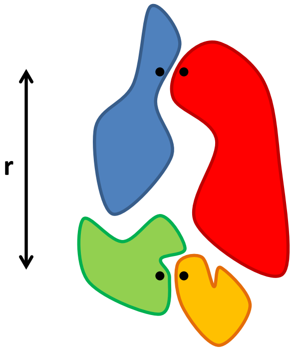
Numerical check

Comparison with the CFT expression yields geometrical interpretation

$$F(r) \equiv \frac{\mathbb{P}_0(r) + \mathbb{P}_1(r) - \mathbb{P}^2_{\neq}}{\mathbb{P}_2(r)} \sim \underbrace{\frac{2\sqrt{3}}{\pi}}_{\text{universal}} \log r,$$



Generalisation



- Log is in the *disconnected* part $\mathbb{P}_0(r)$
- Also true for polymers and disordered systems [Cardy 1999]
- Should hold for $2 \leq d \leq d_{\rm u.c.}$, but prefactor depends on d
- Compute universal prefactor in $\epsilon = 6 - d$ expansion?

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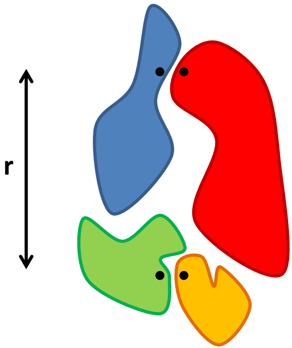

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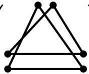
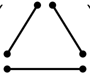
Other interesting logarithmic limits

- $Q \to 0$ (spanning trees, dense polymers, resistor networks . . .)
- $Q \to 2$ (Ising model)
- Logarithms for any integer Q?

Three-point functions on two spins (for $Q = 1$)

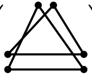
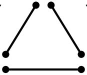
Just example, but we have complete results… $\left(\delta = \lim_{Q\to 1} \frac{\Delta_{\hat{\psi}} - \Delta_{\varepsilon}}{Q-1}\right)$

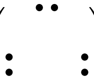
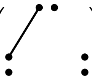
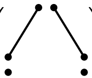
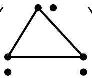
$\mathbb{P}$ () $\sim \frac{F_1(1)}{(r_{12}r_{23}r_{31})^{\Delta_{\hat{\psi}}(1)}}$

$\mathbb{P}$ () $\sim \frac{F_2(1)}{(r_{12}r_{23}r_{31})^{\Delta_{\hat{\psi}}(1)}}$

Three-point functions on two spins (for $Q = 1$)

Just example, but we have complete results… $\left(\delta = \lim_{Q\to 1} \frac{\Delta_{\hat\psi} - \Delta_\varepsilon}{Q-1}\right)$

$\mathbb{P}\Big($  $\Big) \sim \frac{F_1(1)}{(r_{12}r_{23}r_{31})^{\Delta_{\hat\psi}(1)}}$ $\mathbb{P}\Big($  $\Big) \sim \frac{F_2(1)}{(r_{12}r_{23}r_{31})^{\Delta_{\hat\psi}(1)}}$

$\mathbb{P}\Big($  $\Big) + \mathbb{P}\Big($  $\Big) + \mathbb{P}\Big($  $\Big) + \mathbb{P}\Big($  $\Big)$

$- \mathbb{P}_{\neq} \Big[\mathbb{P}\Big($  $\Big) + \mathbb{P}\Big($  $\Big) \Big] + 2\mathbb{P}_{\neq}^3$

$$\sim \frac{F_1(1) - F_2(1)}{(r_{12}r_{23}r_{31})^{\Delta_{\hat\psi}(1)}} \left[\mathrm{cst} - \delta^2 \log\left(\frac{r_{12}r_{23}r_{31}}{a^3}\right)^2 \right]$$

Conclusion

- We found a logarithmic observable specific to percolation ($Q = 1$) $\Rightarrow$ Log CFTs as limits of ordinary CFTs
- We somehow completed [Polyakov 1970] for percolation
- Logarithms tend to appear in *disconnected* observables
- The logarithmic dependency can be checked numerically
- Many possible generalisations (in particular in higher dimensions). Try to connect this to more formal work?
- The universal prefactor in front of the log is closely related to indecomposability parameters β that are crucial in Log CFT